Discussiones Mathematicae
Graph Theory 16 (1996) 143–149

CLIQUE PACKINGS AND CLIQUE PARTITIONS OF GRAPHS WITHOUT ODD CHORDLESS CYCLES

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Abstract

In this paper we consider partitions (resp. packings) of graphs without odd chordless cycles into cliques of order at least 2. We give a structure theorem, min-max results and characterization theorems for this kind of partitions and packings.

Keywords: clique partition, matching, min-max theorems.

1991 Mathematics Subject Classification: 05C70.

The problems of packing and partitioning of graphs into cliques are among most commonly studied problems in graph theory.

Hell and Kirkpatrick [2] considered the problems of packing and partitioning of a graph (or more precisely its vertex set) into cliques with prescribed orders. Let $\mathcal{K}$ be any family of cliques, i.e. $\mathcal{K} \subseteq \{K_1, K_2, K_3, ...\}$, where K_i stands for the complete i-vertex graph. By a $\mathcal{K}$-*packing* of a graph we mean a subgraph H of G whose every component is isomorphic to some member of $\mathcal{K}$. A $\mathcal{K}$-packing of G is called a $\mathcal{K}$-*partition* if it is a spanning subgraph of G.

Hell and Kirkpatrick [2] proved that for any fixed family $\mathcal{K} \subseteq \{K_3, K_4, ...\}$ the decision problem if the instance graph G admits a $\mathcal{K}$-partition is NP-complete. When $K_1 \in \mathcal{K}$ or $K_2 \in \mathcal{K}$ it is polynomial. In [3] the authors examined the case $\mathcal{K} = \{K_2, K_3, ...\}$ and obtained structure and min-max theorems for $\mathcal{K}$-packings in this case. They also gave a good characterization of graphs admitting a $\{K_2, K_3, ...\}$-partition. Similar results were obtained independently by Cornuéjols, Hartvigsen and Pulleyblank [1]. Obviously the problem of existence of a $\mathcal{K}$-partition of a graph is a natural generalization of the problem of existence of a perfect matching. Similarly, the problem of finding a $\mathcal{K}$-packing of maximum order is a generalization of the problem of finding a maximum-sized matching in a graph.

Lonc strengthened in [4] the results of Hell and Kirkpatrick [3] by showing that for $\mathcal{K} \subseteq \{K_3, K_4, ...\}$ the decision problem if the instance graph G admits a $\mathcal{K}$-partition is NP-complete even for comparability graphs.

This paper is devoted to the case of $\mathcal{K} = \{K_2, K_3, ...\}$. The results we obtain concern the class $\mathcal{W}$ of graphs without odd chordless cycles. The class of perfect graphs is obviously a proper subclass of this class. We give a structure theorem, min-max results and characterization theorems for these packings and partitions of graphs belonging to $\mathcal{W}$. They are analogous to the results obtained by Hell and Kirkpatrick [3]. It turned out that if we restrict ourselves to the class $\mathcal{W}$, then we can simplify the theorems greatly using a strong machinery of the matching theory (c.f. Lovász and Plummer [5]). We also show that the problems we consider in this section are, from the algorithmic point of view, essentially as hard as the maximum matching problem for bipartite graphs.

Let us call a $\{K_2, K_3, ...\}$-packing (respectively $\{K_2, K_3, ...\}$-partition) of a graph G a 2-*packing* (respectively a 2-*partition*), for simplicity.

The following lemma sheds much light on the problems we deal with in this paper.

Lemma 1. *Every Hamiltonian graph $G \in \mathcal{W}$ admits a 2-partition.*

Proof. If the order of G is even then G obviously has a perfect matching (being a 2-partition). Thus assume that the order of G is odd. In this case we proceed by induction on $|V(G)|$. For $|V(G)| = 3$ our lemma holds. Let $|V(G)| > 3$. Since $G \in \mathcal{W}$, the Hamiltonian cycle in G has a chord dividing the cycle into odd and even cycles. Thus the set of vertices of G can be partitioned into a part inducing an odd cycle and a part inducing a path with an even number of vertices. Our assertion follows now from the induction hypothesis. ■

The above lemma implies that for graphs $G \in \mathcal{W}$ the existence of a 2-partition is equivalent to existence of a 2-matching. By a 2-*matching* we mean a subgraph of G whose every component is either a single edge or an odd cycle. Moreover, it is not hard to show that for $G \in \mathcal{W}$ the maximum order of a 2-partition is equal to the maximum order of a 2-matching. Thus we can apply the strong matching theory methods (c.f. [5]) to our partitions and packings.

A 2-packing in G is called *maximum* if its order is maximal.

For a graph G define a bipartite graph H in the following way. Replace each vertex v in G by two vertices v' and v'' in H and each edge uv by two edges $u'v''$ and $u''v'$.

Lemma 2. *Let $G \in \mathcal{W}$. The number of edges in a maximum matching in H is equal to the number of vertices covered by a maximum 2-packing in G.*

Proof. Let F be a maximum 2-packing in G and denote by $C_1, C_2, ..., C_t$ its connected components. If $u_1, ..., u_p$ are the vertices of some C_i then the edges $u_1'u_2'', u_2'u_3'', ..., u_{p-1}'u_p'', u_p'u_1''$ form a matching of size $|V(C_i)|$ in H. The union of such edges for every C_i is a matching of size $|F|$ in H.

On the other hand, let M be a collection of edges in H forming a maximum matching. Denote $N = \{uv \in E(G) : u'v'' \in M \text{ or } u''v' \in M\}$. Clearly the edges of N induce a subgraph I of G whose every component is either a path or a cycle. Let C be a component of I which is a path with at least 2 edges. Notice that the set of edges in M corresponding to C is either of the form $u_1'u_2'', u_2'u_3'', ..., u_q'u_{q+1}''$ or of the form $u_1''u_2', u_2''u_3', ...u_q''u_{q+1}'$. Since both cases are analogous, consider the former only. Replace the above edges by the edges $u_1'u_2'', u_2'u_1'', u_3'u_4'', u_4'u_3'', ...$. The resulting set of edges forms a matching in H of size $|M|$ when q is even and $|M|+1$ when q is odd. By the definition of M the latter case is not possible, so q must be even. Repeat this procedure for every path in I. The resulting set of edges M' forms a maximum matching in H and the corresponding set of edges N' induces a subgraph in G whose every component is either a 2-vertex clique or a cycle. This subgraph covers $|M'| = |M|$ vertices of G and by Lemma 1 it admits a 2-partition. ■

The following structure theorem for maximum 2-packings is true. Let $D(G)$ be the set of all vertices in a graph G which are not covered by at least one maximum 2-packing. Denote by $A(G)$ the set of vertices in $V(G) - D(G)$ adjacent to at least one vertex in $D(G)$. Finally, let $C(G) = V(G) - A(G) - D(G)$.

Theorem 1. *Let $G \in \mathcal{W}$. Then*

(1) *$D(G)$ is an independent set,*
(2) *$C(G)$ has a 2-partition,*
(3) *the bipartite graph obtained from G by deletion of the vertices in $C(G)$ and the edges spanned by $A(G)$ has a matching covering $A(G)$,*
(4) *every maximum 2-packing contains a matching described in* (3),
(5) *maximum 2-packings have orders $|V(G)| - |D(G)| + |A(G)|$.*

***Proof of Theorem* 1.** In the proof we shall use the structure theorem on maximum matching in bipartite graphs (c.f. [5, pp. 99–100]). Notice that in the case of bipartite graphs 2-packing are just matchings.

Theorem A. *Let H be a bipartite graph with vertex classes X_1 and X_2, $D_i = D(H) \cap X_i$, $A_i = A(H) \cap X_i$ and $C_i = C(H) \cap X_i$, $i = 1, 2$. Then*

(i) *$D(H)$ is an independent set of vertices,*

(ii) *the subgraph of H induced by the vertices in $C(H)$ has a perfect matching,*

(iii) *there are matchings in H that match A_1 into D_2 and A_2 into D_1,*

(iv) *every maximum matching on H consists of a perfect matching covering $C(H)$, a matching of A_1 into D_2 and a matching of A_2 into D_1.*

Let H be the bipartite graph defined before Lemma 2. Let $D = \{v', v'' \in V(H) : v \in D(G)\}$ and $A = \{v', v'' \in V(H) : v \in A(G)\}$. We shall show that $D = D(H)$.

Let, for some $v \in V(G)$, $v' \in D(H)$ (the case $v'' \in D(H)$ is analogous). It means that there is a maximum matching M in H not covering v'. Suppose M covers v''. Then there is a sequence of edges $v''u_1', u_1''u_2', u_2''u_3', ..., u_{p-1}''u_p'$ in M which does not cover u_p''. If p were odd, then M would not be maximum for we could replace the above p edges by $p+1$ edges $v''u_1', u_1''v'', u_2''u_3', u_3''u_2', ..., u_{p-1}''u_p', u_p''u_{p-1}'$. The resulting set of edges is a matching in H contradicting the maximality of M. Thus p is even. Replacing the above mentioned edges by the edges $u_1''u_2', u_2''u_1', u_3''u_4', u_4''u_3', ..., u_{p-1}''u_p', u_p''u_{p-1}'$ we get a maximum matching in H not covering v''. Proceeding like in the proof of Lemma 2 we construct a maximum 2-packing in G corresponding to M which does not cover v. Thus $v \in D(G)$ so $v' \in D$.

Conversely, let $v' \in D$ (again the case $v'' \in D$ is analogous). Then $v \in D(G)$. Let P be a maximum 2-packing in G which does not cover v. Denote connected components of P by $C_1, C_2, ..., C_t$. Let $u_1, u_2, ..., u_p$ be the vertices of some C_i. Then the edges $u_1'u_2'', u_2'u_3'', ..., u_{p-1}'u_p'', u_p'u_1''$ form a matching M_i in H. The set $M = \bigcup_{i=1}^{t} M_i$ is matching of size $|P|$ in H. By Lemma 2 M is a maximum matching. Clearly M does not cover v' so $v' \in D(H)$.

Suppose now that for some $u, v \in D(G)$, uv is an edge in G. Then $u'v'', v'u'' \in D = D(H)$ so $u'v'', v'u''$ are the edges in H contradicting Theorem A (i). Thus (1) holds.

By the definition of the graph H and the equality $D = D(H)$, $A =$

$A(H)$. Consequently, $C = C(H)$, where $C = V(H) - A - D = \{v', v'' \in V(H) : v \in C(G)\}$. By Theorem A (ii), C can be covered by a perfect matching. This matching corresponds in G to a family of cycles. By Lemma 1, $C(G)$ admits a 2-partition, which proves (2).

To show (3) note that, by Theorem A (iii), there is a matching in H covering A and such that its edges have one vertex in D and the other one in A. This matching corresponds in G to a subgraph F whose vertices are contained in $A(G) \cup D(G)$. The components of F are either even cycles, paths of even size or 2-vertex cliques. Vertices of $A(G)$ have degrees 2 or they belong to 2-vertex cliques in F. Clearly, F contains a matching covering $A(G)$. This completes the proof of (3).

Consider a maximum 2-packing P in G and let M be any matching of size $|P|$ corresponding to P in H. Clearly, M has a maximum size so by Theorem A (iv), M contains a submatching M' covering A such that one endvertex of each its edge is in A and the other one in D. This submatching M' coresponds in G to a subgraph F' covering $A(G)$ whose vertices are contained in $A(G) \cup D(G)$. The components of F' are either even cycles whose vertices belong alternatively to A and D or 2-vertex cliques with one endvertex in A and the other one in D. Vertex sets of each of the components are vertex sets of components in P. Since D is an independent set, the only possibility is that each component of P is a 2-vertex clique with one endvertex in $A(G)$ and the other one in $D(G)$. Thus (4) holds.

The statement (5) follows easily from (4). ∎

Theorem 1 has many important consequences. We shall state some of them as corollaries.

For $A \subseteq V(G)$, denote by $\Gamma(A)$ the set of neighbors of A and by $c_2(G)$ the maximum number of vertices in a graph G that can be covered by a 2-packing.

Corollary 1. *Let $G \in \mathcal{W}$. Then*

$$c_2(G) = \min\{|V(G)| - |A| + |\Gamma(A)|\},$$

where the minimum is taken over all independent sets $A \subseteq V(G)$.

Proof. Let $A_0 = D(G)$. Then $\Gamma(A_0) = A(G)$ so

$$\begin{aligned} c_2(G) &= |V(G)| - |D(G)| + |A(G)| = |V(G)| - |A_0| + |\Gamma(A_0)| \\ &\geq \min\{|V(G)| - |A| + |\Gamma(A)|\}. \end{aligned}$$

Conversely, consider any maximum 2-packing F in G. Let $A \subseteq V(G)$ be independent. Clearly, F covers at most $|\Gamma(A)|$ vertices in A leaving at least $|A|-|\Gamma(A)|$ uncovered. Thus, F covers at most $|V(G)|-|A|+|\Gamma(A)|$ vertices of G. Consequently

$$c_2(G) = |F| \leq |V(G)| - |A| + |\Gamma(A)|$$

so $c_2(G) = \min\{|V(G)| - |A| + |\Gamma(A)|\}$. ■

The proofs of the remaining three corollaries are analogous to the above one or trivial so we omit them. Let, for a subset $S \subseteq V(G)$, $t(S)$ denote the number of singletons (i.e. one-vertex components) in the graph $G - S$.

Corollary 2. *Let $G \in \mathcal{W}$. Then*

$$c_2(G) = \min\{|V(G)| + |S| - |t(S)|\},$$

where the minimum is taken over all subsets $S \subseteq V(G)$. ■

Corollary 3. *Let $G \in \mathcal{W}$. Then G admits a 2-partition if and only if $|\Gamma(A)| \geq |A|$, for every independent set $A \subseteq V(G)$.* ■

Corollary 4. *Let $G \in \mathcal{W}$. Then G admits a 2-partition if and only if $t(S) \leq |S|$, for every subset $S \subseteq V(G)$.* ■

It was mentioned earlier that $c_2(G)$ is, for $G \in \mathcal{W}$, equal to the maximum order of a 2-matching in G. From the algorithmic point of view, the latter problem is known to be essentially as hard as the maximum matching problem for bipartite graphs. Similarly, constructing a 2-partition of order $c_2(G)$ is also as hard as constructing a maximum matching in a bipartite graph. It follows from the following proposition.

Proposition 2. *Let $G \in \mathcal{W}$ be an odd order graph with a Hamiltonian cycle C. Then there is a 2-partition of G whose one component is a triangle and the other ones are edges contained in C.* ■

The proof of the above proposition is similar to the proof of Lemma 1 and we leave it to the reader.

To construct a 2-partition of order $c_2(G)$, we first construct a 2-matching of order $c_2(G)$ (this is essentially a construction of a maximum matching in a bipartite graph). Denote the odd components of this 2-matching by $C_1, C_2, ..., C_p$. What remains, is to find 2-partitions of the Hamiltonian graphs C_i. In view of Proposition 2 it suffices to find, for each i, a triangle T_i in C_i such that after the deletion of T_i the Hamiltonian cycle of C_i splits into even order parts. This can be easily done by a linear breadth first algorithm.

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Received 15 April 1996