Discussiones Mathematicae
Differential Inclusions, Control and Optimization 21 (2001) 191–206

EXTREMAL SOLUTIONS FOR NONLINEAR NEUMANN PROBLEMS

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Abstract

In this paper, we study a nonlinear Neumann problem. Assuming the existence of an upper and a lower solution, we prove the existence of a least and a greatest solution between them. Our approach uses the theory of operators of monotone type together with truncation and penalization techniques.

Keywords and phrases: upper solution, lower solution, order interval, truncation function, penalty function, pseudomonotone operator, coercive operator, extremal solution.

2000 Mathematics Subject Classification: 35J60, 35J65.

1. Introduction

In this paper, we prove the existence of a least and a greatest solution to the nonlinear Neumann problem, involving an elliptic equation. We obtain the existence of extremal solutions assuming the existence of an upper and a lower solution for this problem. Our approach uses the theory of operators of monotone type as this was developed by Browder-Hess in [4] together with truncation and penalization techniques to prove the existence of a solution in the order interval K determined by the fixed upper and lower solutions. Then we show that the set of such solutions is directed and finally the existence of extremal solutions is established via Zorn's lemma.

Deuel-Hess in [8] use the method of upper and lower solutions in order to prove the existence of a solution for a Dirichlet problem with a more general nonlinear differential operator than the p-Laplacian that we have in our problem. But these authors do not address the existence of extremal solutions.

In [7] Dancer-Sweers obtain the existence of a maximal and a minimal solution in an ordered interval for a Dirichelet problem in which is present the semilinear version ($p = 2$) of our elliptic equation. However, their approach is different from ours although they too end up using Zorn's lemma.

Recently, in 1992, Nieto-Cabada in [15] examined the one-dimensional case. These authors, using the method of upper and lower solutions and the monotone iterative technique, obtained the existence of solutions for a Sturm-Liouville boundary-value problem involving a semilinear second order ordinary differential equation, which is a particular version of our equation. The one dimensional case of our problem was studied by Cardinali-Papageorgiou-Servadei in [5]: they obtained, using the method of upper and lower solutions, the existence of C^1-extremal solutions to their problem.

2. Preliminaries

Let X be a reflexive Banach space and X^* its topological dual. In what follows, by $(.,.)$ we denote the duality brackets of the pair (X, X^*). A map $A : X \to 2^{X^*}$ is said to be 'monotone', if for all $[x_1, x_1^*], [x_2, x_2^*] \in GrA$, we have $(x_2^* - x_1^*, x_2 - x_1) \geqslant 0$. The set $D = \{x \in X \mid A(x) \neq \varnothing\}$ is called the 'domain of A'. We say that $A(\cdot)$ is maximal monotone, if its graph is maximal monotone with respect to inclusion among the graphs of all monotone maps from X into X^*. It follows from this definition that $A(\cdot)$ is maximal monotone if and only if $(v^* - x^*, v - x) \geqslant 0$ for all $[x, x^*] \in GrA$, implies $[v, v^*] \in GrA$. For a maximal monotone map $A(\cdot)$, for every $x \in D, A(x)$ is nonempty, closed and convex. A single valued operator $A : X \to X^*$ is said to be 'demicontinuous' at $x \in D$, if for every $\{x_n\}_{n\geqslant 1} \subseteq D$ with $x_n \to x$ in X, we have $A(x_n) \xrightarrow{w^*} A(x)$ in X^*. A monotone demicontinuous everywhere defined operator is maximal monotone (see Hu-Papageorgiou [12]). A map $A : X \to 2^{X^*}$ is said to be 'pseudomonotone', if for all $x \in X$, $A(x)$ is nonempty, closed and convex, for every sequence $\{[x_n, x_n^*]\}_{n\geqslant 1} \subseteq GrA$ such that $x_n \xrightarrow{w} x$ in X, and $\limsup(x_n^*, x_n - x) \leq 0$, we have that for each $y \in X$, there corresponds a $y^*(y) \in A(x)$ such that $(y^*(y), x-y) \leq \liminf(x_n^*, x_n-y)$ and finally A is upper semicontinuous (as a set-valued map) from every finite dimensional subspace of X into X^* endowed with the weak topology. Note that this requirement is automatically satisfied if $A(\cdot)$ is bounded, i.e., maps bounded sets into bounded sets. A map $A : X \to 2^{X^*}$ with nonempty, closed and convex values, is said to be 'generalized pseudomonotone' if for any sequence $\{[x_n, x_n^*]\}_{n\geqslant 1} \subseteq GrA$ such that $x_n \xrightarrow{w} x$ in X, $x_n^* \xrightarrow{w} x^*$ in X^* and $\limsup(x_n^*, x_n - x) \leq 0$, we have $[x, x^*] \in GrA$ and $(x_n^*, x_n) \to (x^*, x)$. The sum of two pseudomonotone maps is pseudomonotone. A pseudomonotone map which is also coercive (i.e. $\frac{\inf[(x^*,x)|x^*\in A(x)]}{\|x\|} \to \infty$ as $\| x \| \to \infty,\ x \in D$) is surjective.

3. Existence result

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with C^1 boundary Γ. In this section, we study the following nonlinear Neumann problem:

$$(1) \qquad \left\{ \begin{array}{ll} -div(\| Dx \|^{p-2} Dx(z)) = f(z, x(z), Dx(z)) & \text{a.e. on } Z \\ \dfrac{\partial x}{\partial n_p} = 0 & \text{a.e. on } \Gamma,\ 2 \leq p \end{array} \right\}$$

Here $\frac{\partial x}{\partial n_p}$ is defined by $\frac{\partial x}{\partial n_p} = \| Dx \|^{p-2} (Dx, n)_{\Re^N}$, with $n(z)$ denoting the exterior normal at $z \in \Gamma$.

Let us start by introducing the hypotheses on the right hand side function $f(z, x, \xi)$.

H(f): $f : Z \times \Re \times \Re^N \to \Re$ is a function such that

(i) for every $(x, \xi) \in \Re \times \Re^N$, $z \mapsto f(z, x, \xi)$ is measurable;

(ii) for almost all $z \in Z$, $(x, \xi) \mapsto f(z, x, \xi)$ is continuous;

(iii) for almost all $z \in Z$, all $x \in \Re$ and all $\xi \in \Re^N$, we have

$$| f(z, x, \xi) | \leq a(z) + c \mid \xi \mid^{p-1}$$

with $a \in L^q(Z)$, $c > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Following Kenmochi [13], we introduce the following space

$$E^q(Z) = \{v = (v_k)_{k=1}^N \mid v_k \in L^q(Z),\ k = 1, \ldots, N,\ div\ v \in L^q(Z)\}.$$

This space furnished with the norm

$$\| v \|_{E^q} = \| div\ v \|_q + \sum_{k=1}^N \| v_k \|_q$$

is a Banach space. Using this space we can define the notion of solution to problem (1).

Definition 1. By a solution to (1) we mean a function $x \in W^{1,p}(Z)$ such that $\| Dx \|^{p-2} Dx \in E^q(Z)$ and it satisfies (1).

We also introduce the notions of upper and lower solution, which will be our basic analytical tools.

Definition 2. A function $\varphi \in W^{1,p}(Z)$ is an 'upper solution' to (1) if and only if

$$\int_Z \| D\varphi \|^{p-2} (D\varphi(z), Dy(z))_{\Re^N} dz \geq \int_Z f(z, \varphi(z), D\varphi(z)) y(z) dz$$

for all $y \in W^{1,p}(Z) \cap L^p(Z)_+$.

Definition 3. A function $\psi \in W^{1,p}(Z)$ is a 'lower solution' to (1) if and only if

$$\int_Z \| D\psi \|^{p-2} (D\psi(z), Dy(z))_{\Re^N} dz \leq \int_Z f(z, \psi(z), D\psi(z)) y(z) dz$$

for all $y \in W^{1,p}(Z) \cap L^p(Z)_+$.

We will assume the existence of an upper and a lower solution. More precisely we make the following hypothesis:

$\mathbf{H}_0$: There exist an upper solution φ and a lower solution ψ such that $\psi(z) \leq \varphi(z)$ a.e. on Z.

Let $K = [\psi, \varphi] = \{x \in W^{1,p}(Z) \mid \psi(z) \leq x(z) \leq \varphi(z) \text{ a.e. on } Z\}$.

First we prove the existence of a solution in the order interval K. Our approach will be based on the use of truncation and penalization techniques (see Deuel-Hess [8]) coupled with results from the general theory of operators of monotone type. So we introduce the truncation map $\tau : W^{1,p}(Z) \to W^{1,p}(Z)$

$$\tau(x)(z) = \begin{cases} \varphi(z) & \text{if } \varphi(z) \leq x(z) \\ x(z) & \text{if } \psi(z) \leq x(z) \leq \varphi(z) \\ \psi(z) & \text{if } x(z) \leq \psi(z). \end{cases}$$

We see that $\tau(\cdot)$ has values in $W^{1,p}(Z)$ and we check easily that $\tau(\cdot)$ is continuous.

The penalty function $\beta : Z \times \Re \to \Re$ is defined by

$$\beta(z,x) = \begin{cases} (x - \varphi(z))^{p-1} & \text{if } \varphi(z) \leq x \\ 0 & \text{if } \psi(z) \leq x \leq \varphi(z) \\ -(\psi(z) - x)^{p-1} & \text{if } x \leq \psi(z). \end{cases}$$

This too is a Carathéodory function such that

$$\mid \beta(z,x) \mid \leq a_1(z) + c_1 \mid x \mid^{p-1} \text{ a.e. on } Z$$

and

$$\int_Z \beta(z, x(z))x(z)dz \geq \| x \|_p^p - c_2 \| x \|_p^{p-1} \text{ for all } x \in L^p(Z)$$

with $a_1 \in L^q(Z)$ and $c_1, c_2 > 0$.

Proposition 1. *If hypotheses H_0 and $H(f)$ hold, then problem* (1) *has at least one solution $x \in K$.*

Proof. Our approach will be based on the use of truncation and penalization techniques coupled with results from the general theory of operators of monotone type. Using the truncation and the penalty function, we introduce the following auxiliary Neumann problem

$$(2)\qquad \left\{\begin{array}{ll} -div(\| Dx \|^{p-2} Dx(z)) & \\ = f(z,\tau(x)(z), D\tau(x)(z)) - \lambda\beta(z,x(z)) & \text{a.e. on } Z \\ \dfrac{\partial x}{\partial n_p} = 0 \quad \text{a.e. on } \Gamma, & 2 \le p,\ \lambda > 0 \end{array}\right\}.$$

Let $A : W^{1,p}(Z) \to W^{1,p}(Z)^*$ be defined by

$$\langle A(x), y\rangle = \int_Z \| Dx \|^{p-2} (Dx(z), Dy(z))_{\Re^N} dz$$

for all $y \in W^{1,p}(Z)$.

Claim 1. $A(\cdot)$ is monotone, demicontinuous, hence maximal monotone.

First we show that $A(\cdot)$ is monotone. So let $x, y \in W^{1,p}(Z)$. We have:

$$\begin{aligned} &\langle A(x) - A(y), x - y\rangle \\ &\quad = \int_Z \| Dx(z) \|^p \, dz - \int_Z \| Dx(z) \|^{p-2} (Dx(z), Dy(z))_{\Re^N} dz \\ &\qquad - \int_Z \| Dy(z) \|^{p-2} (Dx(z), Dy(z))_{\Re^N} dz + \int_Z \| Dy(z) \|^p \, dz \\ &\quad \ge \| Dx \|_p^p + \| Dy \|_p^p - \| Dx \|_p^{p-1} \| Dy \|_p - \| Dy \|_p^{p-1} \| Dx \|_p \\ &\quad = \| Dx \|_p^{p-1} (\| Dx \|_p - \| Dy \|_p) - \| Dy \|_p^{p-1} (\| Dx \|_p - \| Dy \|_p) \\ &\quad = (\| Dx \|_p^{p-1} - \| Dy \|_p^{p-1})(\| Dx \|_p - \| Dy \|_p) \ge 0. \end{aligned}$$

Then $A(\cdot)$ is monotone.

Next we prove that $A(\cdot)$ is demicontinuous. To this end, let $x_n \to x$ in $W^{1,p}(Z)$ as $n \to \infty$. Then for every $y \in W^{1,p}(Z)$, we have

$$\begin{aligned} &| \langle A(x_n) - A(x), y\rangle | \\ &\quad = \left| \int_Z (\| Dx_n \|^{p-2} (Dx_n, Dy)_{\Re^N} - \| Dx \|^{p-2} (Dx, Dy)_{\Re^N}) dz \right|. \end{aligned}$$

Note that since $x_n \to x$ in $W^{1,p}(Z)$, we have $Dx_n \to Dx$ in $L^p(Z, \Re^N)$ and, by passing to a subsequence if necessary, we may also assume that $Dx_n(z) \to Dx(z)$ a.e. on Z as $n \to \infty$. Invoking the generalized Lebesgue convergence theorem (see Ash [2]), we have that

$$\int_Z \| Dx_n \|^{p-2} (Dx_n, Dy)_{\Re^N} dz \to \int_Z \| Dx \|^{p-2} (Dx, Dy)_{\Re^N} dz \text{ as } n \to \infty.$$

So $| \langle A(x_n) - A(x), y \rangle | \to 0$ as $n \to \infty$. Since $y \in W^{1,p}(Z)$ was arbitrary, we conclude that $A(x_n) \xrightarrow{w} A(x)$ in $W^{1,p}(Z)^*$ as $n \to \infty$. Thus we have proved that $A(\cdot)$ is demicontinuous. Finally, recall that a monotone, demicontinuous everywhere defined operator is maximal monotone.

Let $B : L^p(Z) \to L^q(Z)$ be the Nemytskii operator corresponding to β, i.e.

$$B(x(\cdot)) = \beta(\cdot, x(\cdot)).$$

Evidently, B is continuous and monotone (check the definition of $\beta(z, x)$).

Finally, let $F : W^{1,p}(Z) \to L^q(Z)$ be defined by

$$F(x)(\cdot) = f(\cdot, \tau(x)(\cdot), D\tau(x)(\cdot)).$$

Using hypotheses $H(f)$ and continuity of the truncation map, we check easily that F is continuous. Set $R = A + \lambda B - F$.

Claim 2. $R : W^{1,p}(Z) \to W^{1,p}(Z)^*$ is pseudomonotone and coercive.

Let $x_n \xrightarrow{w} x$ in $W^{1,p}(Z)$ and suppose that $\limsup \langle R(x_n), x_n - x \rangle \leq 0$. We have

$$\langle R(x_n), x_n - x \rangle = \langle A(x_n) + \lambda B(x_n) - F(x_n), x_n - x \rangle$$

$$= \langle A(x_n), x_n - x \rangle + \lambda (B(x_n), x_n - x)_{pq} - (F(x_n), x_n - x)_{pq}.$$

Since $W^{1,p}(Z)$ is compactly embedded in $L^p(Z)$, we have $x_n \to x$ in $L^p(Z)$. So

$$(B(x_n), x_n - x)_{pq} \to 0$$

and

$$(F(x_n), x_n - x)_{pq} \to 0.$$

Hence we have

$$\limsup \langle A(x_n), x_n - x\rangle \leq 0$$

$\Rightarrow \limsup\langle A(x_n), x_n\rangle \leq \limsup\langle A(x_n), x\rangle$

$\Rightarrow \limsup \| Dx_n \|_p^p \leq \limsup \| Dx_n \|_p^{p-1} \| Dx \|_p.$

Recall that $\limsup \| Dx_n \|_p{}^\theta = (\limsup \| Dx_n \|_p)^\theta$ for every $0 \leq \theta < \infty$. So, if we set $\xi = \limsup \| Dx_n \|_p$, we have

$$\xi^p \leq \xi^{p-1} \| Dx \|_p$$

$\Rightarrow \xi \leq \| Dx \|_p$

$\Rightarrow \limsup \| Dx_n \|_p \leq \| Dx \|_p.$

On the other hand, from the weak lower semicontinuity of the norm functional, we have

$$\| Dx \|_p \leq \liminf \| Dx_n \|_p \, .$$

So we have $\| Dx_n \|_p \to \| Dx \|_p$ and $Dx_n \stackrel{w}{\to} Dx$ in $L^p(Z, \Re^N)$ as $n \to \infty$. The space $L^p(Z, \Re^N)$, being uniformly convex, has the Kadec-Klee property. So $Dx_n \to Dx$ in $L^p(Z, \Re^N)$ and so $x_n \to x$ in $W^{1,p}(Z)$. Thus $A(x_n) \stackrel{w}{\to} A(x)$, $B(x_n) \to B(x)$ and $F(x_n) \to F(x)$ as $n \to \infty$. Hence $R(x_n) \stackrel{w}{\to} R(x)$ and $\langle R(x_n), x_n\rangle \to \langle R(x), x\rangle$. This shows that $R(\cdot)$ is generalized pseudomonotone. But since $R(\cdot)$ is everywhere defined and bounded, from Proposition 4 of Browder-Hess [4], we have that $R(\cdot)$ is pseudomonotone.

Next we show that $R(\cdot)$ is coercive. We have

$$\begin{aligned}
\langle R(x), x\rangle &\geq \langle A(x), x\rangle + (B(x), x)_{pq} - \| F(x) \|_q \| x \|_p \\
&\geq \| Dx \|_p^p + \lambda \| x \|_p^p - \lambda c_2 \| x \|_p^{p-1} - \frac{\epsilon}{q} \| F(x) \|_q^q - \frac{1}{\epsilon p} \| x \|_p^p \\
&= \| Dx \|_p^p + \lambda \| x \|_p^p - \lambda c_2 \| x \|_p^{p-1} - \frac{2^{q-1}\epsilon}{q} \| a \|_q^q \\
&\quad - \frac{2^{q-1}\epsilon}{q} \| Dx \|_p^p - \frac{1}{\epsilon p} \| x \|_p^p - \delta
\end{aligned}$$

for some $\delta > 0$ (note that $\| D\tau(x) \|_p^p \leq \delta + \| Dx \|_p^p$). First, let $\epsilon > 0$ be such that $\frac{2^{q-1}\epsilon}{q} < 1$ and then let $\lambda > 0$ be such that $\lambda - \frac{1}{\epsilon p} > 0$. So we conclude that $R(\cdot)$ is coercive.

Now recall that a pseudomonotone coercive operator is surjective. So we can find $x \in W^{1,p}(Z)$ such that $R(x) = 0$.

Claim 3. The solution $x \in W^{1,p}(Z)$ to the operator equation $R(x) = 0$ also solves the boundary value problem (2).

Let $g(z) = f(z, \tau(x)(z), D\tau(x)(z)) - \lambda\beta(z, x(z))$. Then $g \in L^q(Z)$. From the representation theorem for functions in $W^{-1,q}(Z)$, we have that $div(\| Dx \|^{p-2} Dx) \in W^{-1,q}(Z)$. From the fact that $R(x) = 0$, for every $\theta \in D(Z)$ (i.e. the space of the restrictions of all C_c^∞-functions on $\Re^N$ to Z), we have

$$(g, \theta)_{pq} = \langle g, \theta \rangle = \langle A(x), \theta \rangle$$

$$\Rightarrow (g, \theta)_{pq} = \int_Z \| Dx \|^{p-2} (Dx(z), D\theta(z))_{\Re^N} dz = \langle -div(\| Dx \|^{p-2} Dx), \theta \rangle.$$

Since $\theta \in D(Z)$ was arbitrary, we conclude that

$$-div(\| Dx \|^{p-2} Dx(z)) = g(z) \quad \text{a.e. on } Z.$$

Note that $div(\| Dx \|^{p-2} Dx) \in L^q(Z)$ and $\| Dx \|^{p-2} Dx \in L^q(Z, \Re^N)$. From Proposition 1.4 of Kenmochi [13], we know that

$$\frac{\partial x}{\partial n_p} \in W^{-\frac{1}{q},q}(\Gamma) = W^{\frac{1}{q},p}(\Gamma)^*$$

and

$$\begin{aligned} &\int_Z div(\| Dx \|^{p-2} Dx(z)) y(z) dz + \int_Z \| Dx \|^{p-2} (Dx(z), Dy(z))_{\Re^N} dz \\ &= \left\langle \frac{\partial x}{\partial n_p}, \gamma(y) \right\rangle_\Gamma \end{aligned} \tag{3}$$

for all $y \in W^{1,p}(Z)$, where by $\langle \cdot, \cdot \rangle_\Gamma$ we denote the duality brackets for the pair $(W^{-\frac{1}{q},q}(\Gamma), W^{\frac{1}{q},p}(\Gamma))$ and $\gamma(\cdot)$ is the trace operator on $W^{1,p}(Z)$. From the first part of the proof of this claim, we have that the left hand side of (3), is equal to zero. So

$$\left\langle \frac{\partial x}{\partial n_p}, \gamma(y) \right\rangle_\Gamma = 0$$

for all $y \in W^{1,p}(Z)$

$$\Rightarrow \quad \frac{\partial x}{\partial n_p} = 0 \text{ a.e. on } \Gamma.$$

Therefore the solution $x \in W^{1,p}(Z)$ to the operator equation $R(x) = 0$ also solves the boundary value problem (2).

Since by hypothesis ψ is a lower solution, by definition we have

$$\int_Z \| D\psi \|^{p-2} (D\psi(z), Dy(z))_{\Re^N} dz \leq \int_Z f(z, \psi(z), D\psi(z)) y(z) dz$$

for all $y \in W^{1,p}(Z) \cap L^p(Z)_+$.
So if we take $y = (\psi - x)^+ \in W^{1,p}(Z) \cap L^p(Z)_+$ (see Gilbarg-Trudinger [10], Lemma 7.6, p. 145), we have

$$\begin{aligned} &\int_Z (\| D\psi \|^{p-2} (D\psi(z), D(\psi - x)^+(z))_{\Re^N} \\ &\quad - \| Dx \|^{p-2} (Dx(z), D(\psi - x)^+(z))_{\Re^N}) dz \\ &\leq \int_Z (f(z, \psi(z), D\psi(z)) - f(z, \tau(x)(z), D\tau(x)(z)))(\psi - x)^+(z) dz \\ &\quad + \int_Z \beta(z, x(z))(\psi - x)^+(z) dz. \end{aligned}$$

Recall that

$$D(\psi - x)^+(z) \quad = \quad \begin{cases} D(\psi - x)(z) & \text{if } \ x(z) < \psi(z) \\ 0 & \text{if } \ \psi(z) \leq x(z). \end{cases}$$

So we have

$$\begin{aligned} &\int_Z (\| D\psi \|^{p-2} (D\psi(z), D(\psi - x)^+(z))_{\Re^N} \\ &\quad - \| Dx \|^{p-2} (Dx(z), D(\psi - x)^+(z))_{\Re^N}) dz \\ &= \int_{\{x<\psi\}} (\| D\psi \|^p - \| D\psi \|^{p-2} (D\psi(z), Dx(z))_{\Re^N}) dz \\ &\quad - \int_{\{x<\psi\}} (\| Dx \|^{p-2} (Dx(z), D\psi(z))_{\Re^N} - \| Dx \|^p) dz \end{aligned}$$

$$
\begin{aligned}
&\geq \int_{\{x<\psi\}} (\| D\psi \|^p - \| D\psi \|^{p-1} \| Dx \| - \| Dx \|^{p-1} \| D\psi \| + \| Dx \|^p) dz \\
&= \int_{\{x<\psi\}} (\| D\psi \|^{p-1} (\| D\psi \| - \| Dx \|) - \| Dx \|^{p-1} (\| D\psi \| - \| Dx \|)) dz \\
&= \int_{\{x<\psi\}} (\| D\psi \|^{p-1} - \| Dx \|^{p-1})(\| D\psi \| - \| Dx \|) dz \geq 0.
\end{aligned}
$$

Also we have

$$
\begin{aligned}
&\int_Z (f(z,\psi(z),D\psi(z)) - f(z,\tau(x)(z),D\tau(x)(z)))(\psi - x)^+(z)dz \\
&= \int_{\{x<\psi\}} (f(z,\psi(z),D\psi(z)) - f(z,\psi(z),D\psi(z)))(\psi - x)(z)dz = 0.
\end{aligned}
$$

Finally we obtain

$$
\begin{aligned}
0 &\leq \int_Z \beta(z,x(z))(\psi - x)^+(z)dz \\
&= \int_{\{x<\psi\}} -(\psi - x)^{p-1}(z)(\psi - x)(z)dz \\
&= -\int_Z \left((\psi - x)^+\right)^p (z)dz \\
&\Rightarrow \psi(z) \leq x(z) \text{ a.e. on } Z.
\end{aligned}
$$

Similarly, we show that $x(z) \leq \varphi(z)$ a.e. on Z. Therefore $x \in K$. ∎

Now we show that the set S of solutions to (1) in the order interval K is directed, i.e. if $x_1, x_2 \in S$, then there exists $x_3 \in S$ such that $x_1 \vee x_2 \leq x_3$.

Proposition 2. *If hypotheses H_0 and $H(f)$ hold, then S is directed.*

Proof. Let $x_1, x_2 \in S$ and let $u = x_1 \vee x_2$.

Claim. $u \in W^{1,p}(Z)$ is a lower solution to (1).

We need to show that for every $\theta \in W^{1,p}(Z) \cap L^p(Z)_+$, we have

$$
\int_Z \| Du \|^{p-2} (Du(z), D\theta(z))_{\Re^N} dz \leq \int_Z f(z,u(z),Du(z))\theta(z)dz.
$$

Note that

$$Du(z) = \begin{cases} Dx_1(z) & \text{if } x_2(z) \le x_1(z) \\ Dx_2(z) & \text{if } x_1(z) \le x_2(z) \end{cases}$$

and $-div(\| Dx_i(z) \|^{p-2} Dx_i(z)) = f(z, x_i(z), Dx_i(z))$ a.e. on Z, $\frac{\partial x_i}{\partial n_p} = 0$ a.e. on Γ for $i = 1, 2$.

Given $\epsilon > 0$, we introduce the function $\gamma_\epsilon : \Re \to \Re$ defined by

$$\gamma_\epsilon(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \frac{t}{\epsilon} & \text{if } 0 \le t \le \epsilon \\ 1 & \text{if } \epsilon \le t. \end{cases}$$

Evidently, $\gamma_\epsilon(t)$ is Lipschitz-continuous and differentiable everywhere except $t = 0$, $t = \epsilon$. Moreover, the derivative for $t \ne 0$, $t \ne \epsilon$ is given by

$$\gamma_\epsilon^{'}(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{\epsilon} & \text{if } 0 < t < \epsilon \\ 0 & \text{if } \epsilon < t. \end{cases}$$

We remark that $\gamma_\epsilon \to \chi_{\{t>0\}}$ as $\epsilon \downarrow 0$. Let $k \in D(Z)_+$ and set

$$\theta_1(\cdot) = (1 - \gamma_\epsilon((x_2 - x_1)(\cdot)))\, k(\cdot) \quad \text{and} \quad \theta_2(\cdot) = \gamma_\epsilon((x_2 - x_1)(\cdot))\, k(\cdot).$$

We have

$$\int_Z \| Dx_1 \|^{p-2} (Dx_1(z), D\theta_1(z))_{\Re^N} dz + \int_Z \| Dx_2 \|^{p-2} (Dx_2(z), D\theta_2(z))_{\Re^N} dz$$

$$= \int_Z f(z, x_1(z), Dx_1(z))\theta_1(z) dz + \int_Z f(z, x_2(z), Dx_2(z))\theta_2(z) dz.$$

By Stampacchia's chain rule, we have

$$D\theta_1(\cdot) = Dk(\cdot) - \gamma_\epsilon^{'}((x_2 - x_1)(\cdot))\, D(x_2 - x_1)(\cdot) k(\cdot)$$

and

$$D\theta_2(\cdot) = \gamma_\epsilon^{'}((x_2 - x_1)(\cdot))\, D(x_2 - x_1)(\cdot) k(\cdot) + \gamma_\epsilon((x_2 - x_1)(\cdot))\, Dk(\cdot).$$

So we obtain

$$
\begin{aligned}
&\int_Z \| Dx_1 \|^{p-2} (Dx_1(z), Dk(z))_{\Re^N} dz \\
&\quad - \int_Z \| Dx_1 \|^{p-2} \gamma_\epsilon'((x_2 - x_1)(z))k(z))(Dx_1(z), D(x_2 - x_1)(z))_{\Re^N} dz \\
&\quad - \int_Z \| Dx_1 \|^{p-2} \gamma_\epsilon((x_2 - x_1)(z))(Dx_1(z), Dk(z))_{\Re^N} dz \\
&\quad + \int_Z \| Dx_2 \|^{p-2} \gamma_\epsilon((x_2 - x_1)(z))(Dx_2(z), Dk(z))_{\Re^N} dz \\
&\quad + \int_Z \| Dx_2 \|^{p-2} \gamma_\epsilon'((x_2 - x_1)(z))k(z)(Dx_2(z), D(x_2 - x_1)(z))_{\Re^N} dz \\
&\geq \int_Z \| Dx_1 \|^{p-2} (Dx_1(z), Dk(z))_{\Re^N} dz \\
&\quad - \int_Z \gamma_\epsilon ((x_2 - x_1)(z)) [\| Dx_1 \|^{p-2} (Dx_1(z), Dk(z))_{\Re^N} \\
&\quad - \| Dx_2 \|^{p-2} (Dx_2(z), Dk(z))_{\Re^N}] dz \\
&\quad - \int_Z (\| Dx_1 \|^{p-2} + \| Dx_2 \|^{p-2}) \gamma_\epsilon' ((x_2 - x_1)(z)) \, k(z)(Dx_1(z), Dx_2(z))_{\Re^N} dz
\end{aligned}
$$

since $\gamma_\epsilon' ((x_2 - x_1)(\cdot)) \, k(\cdot) \geq 0$.

Now recall that

$$\gamma_\epsilon(x_2 - x_1) \to \chi_{\{x_2 > x_1\}}$$

and

$$\int_Z f(z, x_1(z), Dx_1(z))\theta_1(z)dz + \int_Z f(z, x_2(z), Dx_2(z))\theta_2(z)dz$$

$$\to \int_Z f(z, u(z), Du(z))k(z)dz$$

as $\epsilon \downarrow 0$.

So in the limit as $\epsilon \downarrow 0$, we have

$$\begin{aligned}\int_Z f(z,u(z),Du(z))k(z)dz \geq & \int_{\{x_2\leq x_1\}} \| Dx_1 \|^{p-2} (Dx_1(z),Dk(z))_{\Re^N} dz \\ & + \int_{\{x_2>x_1\}} \| Dx_2 \|^{p-2} (Dx_2(z),Dk(z))_{\Re^N} dz \\ = & \int_Z \| Du \|^{p-2} (Du(z),Dk(z))_{\Re^N} dz.\end{aligned}$$

Since $k \in D(Z)_+$ is arbitrary and the latter is dense in $W^{1,p}(Z) \cap L^p(Z)_+$, we infer that the above inequality holds for every $k \in W^{1,p}(Z) \cap L^p(Z)_+$ and so, by definition, u is a lower solution to problem (1).

Then working with the upper solution φ and the lower solution $u \geq \psi$, from Proposition 1, we obtain a solution of (1) in $K_1 = [u,\varphi]$. This proves that S is directed. ■

Now we are ready to establish the existence of extremal solutions in K.

Theorem 1. *If hypotheses H_0 and $H(f)$ hold, then problem* (1) *has a least solution x_* and a greatest solution x^* in K.*

Proof. Let C be a chain in S. Let $x = \sup C$. By virtue of Corollary 7, p. 336 of Dunford-Schwartz [9], we can find a non decreasing sequence $\{x_n\}_{n\geq 1} \subseteq C$ such that $x_n \to x$ in $L^p(Z)$. Also for every $n \geq 1$ we have

$$\int_Z \| Dx_n \|^p \, dz = \int_Z f(z,x_n(z),Dx_n(z))x_n(z)dz.$$

Using hypothesis $H(f)$(iii), we obtain

$$\begin{aligned}\| Dx_n \|_p^p & \leq (\| a \|_q + c \| Dx_n \|_p^{p-1}) \| x_n \|_p \\ & \leq (\| a \|_q + c \| Dx_n \|_p^{p-1}) 2^{\frac{1}{p}} M ,\end{aligned}$$

where $M = \max\{\| \varphi \|_p, \ \| \psi \|_p\}$.

So $\{x_n\}_{n\geq 1}$ is bounded in $W^{1,p}(Z)$ and we have $x_n \xrightarrow{w} x$ in $W^{1,p}(Z)$ as $n \to \infty$. Moreover, since $A(x_n) - F(x_n) = 0$, working as in Claim 2, in the proof of Proposition 1, we have $x_n \to x$ in $W^{1,p}(Z)$ as $n \to \infty$ and so $\langle A(x),\theta\rangle = \langle F(x),\theta\rangle = (F(x),\theta)_{pq}$ for all $\theta \in W^{1,p}(Z)$. Hence as in Claim 2, in the proof of Proposition 1, we have that $x \in S$. Therefore, every chain in S has an upper bound in S. We can apply Zorn's lemma and produce a maximal element x^* of (1). Since S is directed, x^* is unique and it is the greatest solution to (1) in K. Similarly, we produce the least solution x_* of (1) in K (see Cardinali-Papageorgiou-Servadei [5]). ■

References

[1] R. Adams, Sobolev Spaces, Academic Press, New York 1975.

[2] R.B. Ash, Real Analysis and Probability, Academic Press, New York, San Francisco, London 1972.

[3] H. Brézis, Analyse Functionelle: Théorie et Applications, Masson, Paris 1983.

[4] F.E. Browder and P. Hess, *Nonlinear Mappings of Monotone Type in Banach Spaces*, J. Funct. Anal. **11** (1972), 251–294.

[5] T. Cardinali, N.S. Papageorgiou and R. Servadei, The Neumann Problem for Quasilinear Differential Equations, preprint.

[6] E. Casas and L.A. Fernández, *A Green's Formula for Quasilinear Elliptic Operators*, J. Math. Anal. Appl. **142** (1989), 62–73.

[7] E. Dancer and G. Sweers, *On the Existence of a Maximal Weak Solution for a Semilinear Elliptic Equation*, Diff. Integral Eqns **2** (1989), 533–540.

[8] J. Deuel and P. Hess, *A Criterion for the Existence of Solutions of Nonlinear Elliptic Boundary Value Problems*, Proc. Royal Soc. Edinburgh (A) **74** (1974–75), 49–54.

[9] N. Dunford and J.T. Schwartz, Linear Operators. Part I: General Theory, Interscience Publishers, New York 1958–1971.

[10] D. Gilbarg and N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin 1983.

[11] E. Hewitt and K. Stromberg, Real and Abstract Analysis, Springer, New York 1975.

[12] S. Hu and N.S. Papageorgiou, Handbook of Multivalued Analysis. Volume I: Theory, Kluwert, Dordrecht, The Netherlands 1997.

[13] N. Kenmochi, *Pseudomonotone Operators and Nonlinear Elliptic Boundary Value Problems*, J. Math. Soc. Japan **27** (1975), 121–149.

[14] J. Leray and J.L. Lions, *Quelques Resultats de Visik sur les Problems Elliptiques Nonlinearities par Methodes de Minty-Browder*, Bull. Soc. Math. France **93** (1965), 97–107.

[15] J.J. Nieto and A. Cabada, *A Generalized Upper and Lower Solutions Method for Nonlinear Second Order Ordinary Differential Equations*, J. Appl. Math. Stochastic Anal. **5** (2) (1992), 157–165.

[16] E. Zeidler, Nonlinear Functional Analysis and its Applications II, Springer-Verlag, New York 1990.

Received 7 July 2001