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ASSOCIATIVE GRAPH PRODUCTS AND THEIR INDEPENDENCE, DOMINATION AND COLORING NUMBERS

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Abstract

Associative products are defined using a scheme of Imrich & Izbicki [18]. These include the Cartesian, categorical, strong and lexicographic products, as well as others. We examine which product $\otimes$ and parameter p pairs are multiplicative, that is, $p(G \otimes H) \geq p(G)p(H)$ for all graphs G and H or $p(G \otimes H) \leq p(G)p(H)$ for all graphs G and H. The parameters are related to independence, domination and irredundance. This includes Vizing's conjecture directly, and indirectly the Shannon capacity of a graph and Hedetniemi's coloring conjecture.

Keywords: graph products, independence, domination, irredundance, coloring.

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1. Product Definitions

We consider products of finite simple graphs. A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$. We use $\otimes$ as the symbol for an arbitrary product where, for the purposes of this paper, the product graph is defined by $V(G \otimes H) = \{ax | a \in V(G), x \in V(H)\}$ and whether two vertices in the product are adjacent depends solely on the adjacency relations in the factors. This can be represented by a 3×3 matrix, called the *edge* matrix. The rows (columns) are labeled by E which denotes adjacency of the vertices of

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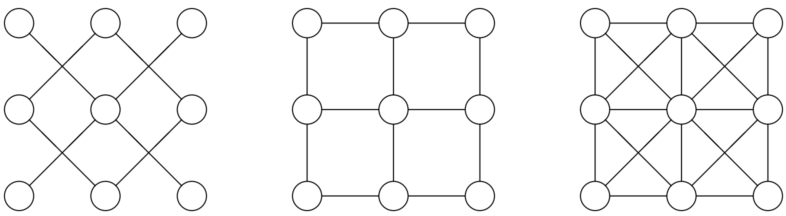
the first (second) factor; N nonadjacency; and Δ the case where the vertex is the same. An E in the matrix indicates there is an edge between the vertices of the product; an N nonadjacency; and in the case where the relationship in both factors is Δ then the two vertices are the same and so the entry is Δ.

$$\begin{array}{c c} & \begin{array}{ccc} E & \Delta & N \end{array} \\ \begin{array}{c} E \\ \Delta \\ N \end{array} & \begin{pmatrix} - & - & - \\ - & \Delta & - \\ - & - & - \end{pmatrix} \end{array}$$

Since the rows and columns will always be labeled in this fashion we drop the labels in the sequel.

This scheme was first introduced by Imrich & Izbicki [18]. They showed that out of the 256 possible products there are 20 associative products but only 10 of these depend on the edge structure of both factors (that is, these products do not have all E's or all N's in the 1st and 3rd rows or in the 1st and 3rd columns). Further, 8 of these are also commutative. (See also Harary & Wilcox [9]).

Since a graph can be defined in terms of non-edges, there is the notion of a complementary product. Specifically, if $\overline{G}$ is the complementary graph to G then the *complementary* product $\otimes^c$ to a product $\otimes$ is given by $G \otimes^c H = \overline{(\overline{G} \otimes \overline{H})}$. The only two of these ten products which are not commutative are self-complementary. They are the lexicographic product and the product whose edge matrix is the transpose of that of the lexicographic product. We do not consider this latter product. Below are the definitions of these 9 associative products. Examples can be found in Figure 1.



Categorical Cartesian Strong

Figure 1a. $P_3 \times P_3$, $P_3 \,\Box\, P_3$ & $P_3 \boxtimes P_3$.

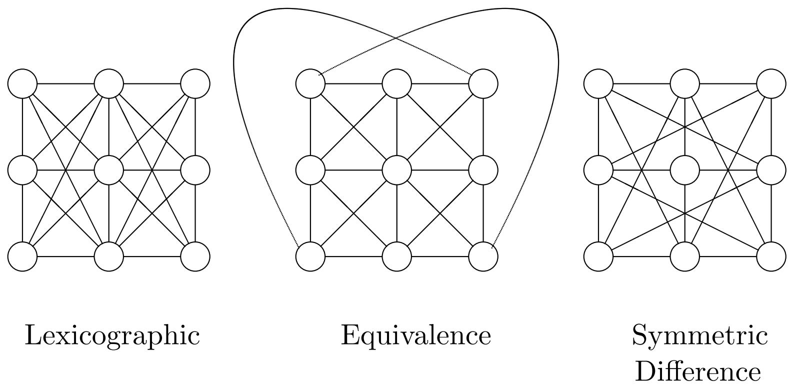


Figure 1b. $P_3 \bullet P_3,\ P_3 \,\textcircled{$\cong$}\, P_3$ & $P_3 \nabla P_3$.

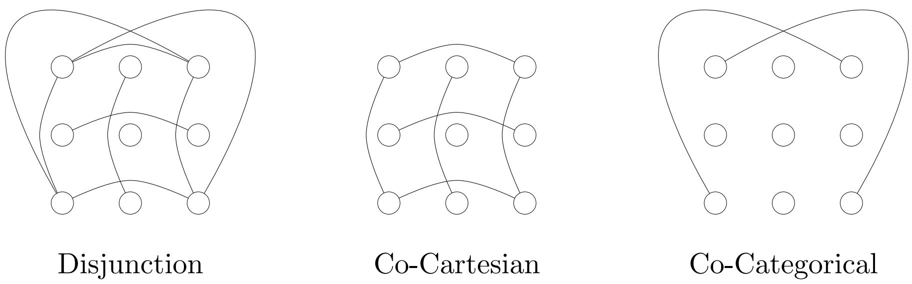


Figure 1c. The nonedges of $P_3 \boxtimes^c P_3,\ P_3 \square^c P_3$ & $P_3 \times^c P_3$.

The symbols used to denote products are based mainly on those found in [21]. Some of these products are also known by other names (for more details see [22]):

$$\text{Categorical: } G \times H \begin{pmatrix} E & N & N \\ N & \Delta & N \\ N & N & N \end{pmatrix}; \text{ Co-Categorical: } G \times^c H \begin{pmatrix} E & E & E \\ E & \Delta & E \\ E & E & N \end{pmatrix};$$

$$\text{Cartesian: } G \square H \begin{pmatrix} N & E & N \\ E & \Delta & N \\ N & N & N \end{pmatrix}; \quad \text{Co-Cartesian: } G \square^c H \begin{pmatrix} E & E & E \\ E & \Delta & N \\ E & N & E \end{pmatrix};$$

$$\text{Strong: } G \boxtimes H \begin{pmatrix} E & E & N \\ E & \Delta & N \\ N & N & N \end{pmatrix}; \text{ Disjunction: } G \boxtimes^c H \begin{pmatrix} E & E & E \\ E & \Delta & N \\ E & N & N \end{pmatrix};$$

$$\text{Equivalence: } G \textcircled{\cong} H \begin{pmatrix} E & E & N \\ E & \Delta & N \\ N & N & E \end{pmatrix}; \text{ Lexicographic: } G \bullet H \begin{pmatrix} E & E & E \\ E & \Delta & N \\ N & N & N \end{pmatrix};$$

$$\text{Symmetric Difference: } G \nabla H \begin{pmatrix} N & E & E \\ E & \Delta & N \\ E & N & N \end{pmatrix}.$$

All 256 products can be ordered by 'inclusion': that is, $\oplus \leq \otimes$ if for each pair of graphs G and H, $E(G \oplus H) \subset E(G \otimes H)$. The suborder for the products of interest in this paper is shown in Figure 2.

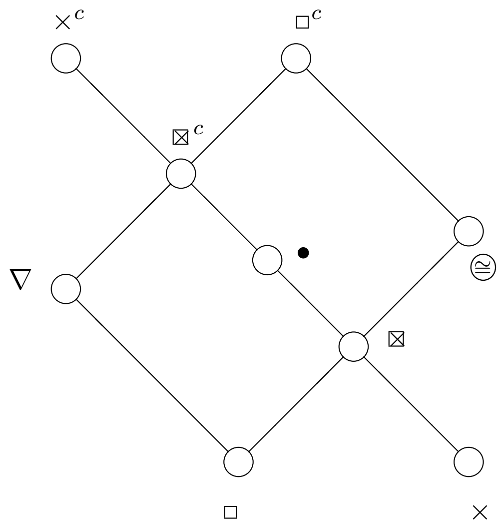


Figure 2. Partial Ordering of the Products.

We are mainly concerned with the question whether a parameter p with respect to a product $\otimes$ has one of the following properties for all graphs G and H:

(i) $p(G \otimes H) \geq p(G)p(H)$ or

(ii) $p(G \otimes H) \leq p(G)p(H)$ or

(iii) $p(G \otimes H) = p(G)p(H)$.

Given p and $\otimes$, if (iii) is not true then a fourth question is to characterize *universal* graphs: that is, those graphs G where $p(G \otimes H) = p(G)p(H)$ for all graphs H. Some of our results preclude the existence of universal graphs for certain parameter/product pairings. We have not pursued the question in this paper.

There are many questions in the literature about independence-type parameters and products. Most take a form like our (i), (ii) and (iii). Indeed they motivated us to try a more systematic approach to these questions. Some of these we list below.

Vizing's conjecture [31] (see [10] and [11] for a survey of some of the results) — *for all G and H is $\gamma(G \,\square\, H) \geq \gamma(G)\gamma(H)$?*

Hedetniemi's coloring conjecture [12] (see [6] for a survey of some of the results) — *for any indexed set of graphs $\{G_i | i \in I\}$ is $\chi(\times_{i\in I} G_i) = \min_{i\in I}\{\chi(G_i)\}$?*

The *Shannon capacity* of a graph G is $\Theta(G) = \lim_{n\to\infty} \alpha(G^n)^{1/n}$ where $\alpha(G^n)$ is the maximum number of nonconfusable codewords of length n taken from an alphabet G. Let the letters be the vertices with two vertices being adjacent if they could be confused. The maximum number of nonconfusable codewords of length 1 is therefore $\beta(G)$ and thus $\Theta(G) = \lim_{n\to\infty} \beta(\boxtimes_{i=1}^n G)^{1/n}$.

The problem is to find ways to determine $\Theta(G)$. It is easy to see that $\beta(G) \leq \Theta(G)$. This problem was introduced by Shannon [30] (see also Ore [23], Berge [2] and Roberts [26]). Rosenfeld [27] found a characterization of universal graphs for β and $\boxtimes$ using linear programming techniques. A new approach was introduced by Lovász [20] who used eigenvalue techniques.

The *ultimate chromatic number* of a graph G is $\chi_u(G) = \lim_{n\to\infty} (\chi(\square_{i=1}^n G))^{1/n}$. This was introduced by Hilton, Rado & Scott [15] and is related to the problem of assigning radio frequencies to vehicles operating in zones (see Gilbert [8] and also Roberts [25]). The determination of the ultimate chromatic number can be solved using linear programming techniques; see Hell & Roberts [14].

In addition there have been several other conjectures. V. Pus [24] answered in the negative a question of C. Thomassen: Is there any product $\otimes$ such that for all graphs G and H, $\chi(G \otimes H) = \chi(G)\chi(H)$.

The following question is attributed to Lovász (see Hsu [16]). Let $h_H(G)$ be the number of homomorphisms of G to H. An *increasing multiplicative* graph function f is a function from graphs to the real numbers with the properties that $f(G \times H) = f(G)f(H)$ and if $G \subset H$ then $f(G) \leq f(H)$. Are all increasing multiplicative functions generated by functions of the type h_H? Hsu answers this negatively for both the categorical product [16] and the strong product [17]. It still leaves unanswered the question of characterizing such functions.

The parameters that we chose to consider reflect the content of these questions: that is, those involving notions of independence, domination and coloring.

Most of our results apply to just one product-parameter pair. However, some are tied to the product order and apply to many pairs. These are to be found in Section 2.2. The other results are to be found in Section 2.3. We were not able to settle all the cases. Some conjectures are to be found in Section 2.4. In some cases, better results can be found by using a mix of parameters, these are given in Section 3.

1.1. Terminology

Let $G = (V(G), E(G))$ be a graph. We will always assume that graphs are finite, simple and have at least two vertices. The latter assumption is made so as to avoid listing many exceptions in our results. We write, $a \simeq b$ if a is either equal or adjacent to b, $a \sim b$ if a is adjacent to but not equal to b, and $a \perp b$ if a is neither adjacent nor equal to b. Edges we denote by ordered pairs of vertices, e.g. $(a, b) \in E(G)$; we use ax to denote a vertex in the product $G \otimes H$ where $a \in V(G)$ and $x \in V(H)$.

Let $S \subset V(G)$ and $v \in V(G)$. $\langle S \rangle$ is the *induced subgraph* on the vertices of S. The *neighborhood* of $v \in V(G)$ is $N(v) = \{y|y \sim v\}$ and $N(S) = \cup_{v\in S} N(v)$; $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v and $N[S] = \cup_{v\in S} N[v]$; $I(S, v) = N[v] - N[S - \{v\}]$ is the *private neighborhood* of v (with respect to S). The set S is called: *independent* if $\langle S \rangle$ contains no edges; *dominating* if $N[S] = V(G)$; and *irredundant* if $I(S, v) \neq \emptyset$ for each $v \in S$. If $N(S) = V(G)$ then S is a *total-dominating* set. Note that only graphs without isolated vertices have total-dominating sets.

The minimum (maximum) cardinality of a minimal dominating set is denoted by $\gamma(G)$ ($\Gamma(G)$), and, if G has no isolated vertices, the minimum (maximum) cardinality of a minimal total-dominating set by $\gamma_t(G)$ ($\Gamma_t(G)$). Note that a minimal dominating set is also irredundant.

The maximum cardinality of an independent set is denoted by $\beta(G)$, and $i(G)$ denotes the minimum cardinality of an independent, dominating set; $IR(G)$ is used to denote the maximum cardinality of an irredundant set and $ir(G)$ the minimum cardinality of a maximal irredundant set.

A set $S \subset V(G)$ is a *two-packing* if $N[x] \cap N[y] = \emptyset$ for every pair $x, y \in S$, $x \neq y$. The maximum number of vertices in a two-packing is denoted by$P_2(G)$. 2-packings correspond to independent sets in the *square* of G: that is, the graph on $V(G)$ but where a is adjacent to b just if they are at distance 1 or 2 in G. Therefore the results for 2-packings can be obtained from the results concerning independence. However, this parameter is useful in obtaining lower bounds in Section 3.

The following inequalities are easy consequences of the preceding definitions (see [5]).

Lemma 1.1. *For any graph G, $ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G)$. If G contains no isolated vertex, then $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$.*

The vertices of a graph can be decomposed into subsets with specified properties. An *independence partition* of G is a partition of $V(G)$ into nonempty, independent subsets where the union of any two subsets is not independent. The *chromatic* number is the fewest number of subsets in an independence partition and is denoted by $\chi(G)$. The *achromatic* number $\psi(G)$ is the greatest number of subsets in an independence partition.

A *domination decomposition* of G is a partition of $V(G)$ into subsets each of which is dominating where none of these subsets can be further partitioned into two dominating subsets. The *domatic* number, $d(G)$, is the largest number of dominating sets in a domination decomposition of G; and the smallest number is the *adomatic* number, denoted by $ad(G)$. For any other notation, please see [1].

2. Product Results

2.1. Partial Product Results

In this section we consider what properties are required of a product to allow the cartesian product of independent (dominating, *etc*) sets in the factors to be independent (dominating, *etc*) in the product graph.

We call a graph product, $\otimes$, *independent* (*dominating, total-dominating, irredundant, two-packing*) *multiplicative* if for any two graphs G and H and any two independent (dominating, total-dominating, irredundant, two-packing) sets $A \subset V(G)$ and $B \subset V(H)$, the set $A \times B$ is an independent (dominating, total-dominating, irredundant, two-packing) subset of $G \otimes H$. The graph product is *color* (*domatic*) multiplicative if for any two graphs G and H, $\{A_i \times B_j : i = 1, \ldots, p;\ j = 1. \ldots, q\}$ is an independence partition (domination decomposition) of $G \otimes H$ whenever $\{A_i\}_{i=1}^{p}$ and $\{B_j\}_{j=1}^{q}$ are independence partitions (domination decompositions) of G and H, respectively.

Lemma 2.1. *Let $\otimes$ be a graph product.*

1. *If* $\otimes \leq \boxtimes^c$*, then* $\otimes$ *is independent multiplicative.*
2. *If* $\otimes \leq \boxtimes$*, then* $\otimes$ *is irredundant and two-packing multiplicative.*
3. *If* $\times \leq \otimes$*, then* $\otimes$ *is total-dominating multiplicative.*
4. *If* $\boxtimes \leq \otimes$*, then* $\otimes$ *is dominating multiplicative.*
5. *If* $\boxtimes \leq \otimes \leq \boxtimes^c$*, then* $\otimes$ *is color multiplicative.*

Proof. 1. Let G and H be graphs and suppose that $a \perp b$ in G and $x \perp y$ in H. Suppose that $\otimes \leq \boxtimes^c$. Then, it follows that in $G \otimes H$ the vertices ax, ay, bx and by are mutually nonadjacent. Therefore, the product of two independent sets is independent in $G \otimes H$.

2. Let A and B be irredundant sets of graphs G and H respectively. Suppose that $\otimes \leq \boxtimes$. Let $a \in A$ and $x \in B$. In each case $I(A, a) \neq \phi$ and $I(B, x) \neq \phi$.

If $a \in I(A, a)$ and $x \in I(B, x)$ then $ax \in I(A \times B, ax)$.

Suppose that $x \in I(B, x)$ and that $a \notin I(A, a)$ then there exists $b \neq a$, $b \in I(A, a)$. In the edge matrix of $\otimes$ if (E,Δ) = E, then $bx \in I(A \times B, ax)$. If (E,Δ) = N, then $ax \in I(A \times B, ax)$. The case where $x \notin I(B, x)$ and $a \in I(A, a)$ is similar.

Lastly, suppose that $x \notin I(B, x)$ and $a \notin I(A, a)$ then there exist $b \in I(A, a)$ and $y \in I(B, x)$, where $a \neq b$ and $x \neq y$. If, in the edge matrix of $\otimes$, (E,E) = E, then $by \in I(A \times B, ax)$. If (E,E) = N and (Δ,E) = E then $ay \in I(A \times B, ax)$ and if (E,Δ) = E then $bx \in I(A \times B, ax)$. If (Δ,E) = (E,Δ) = N then there are no edges in the product graph and trivially $ax \in I(A \times B, ax)$.

Let A and B be 2-packings of $V(G)$ and $V(H)$ respectively. Since $G \otimes H$ is a spanning subgraph of $G \boxtimes H$, the distance from vertex ax to by is at least $\max\{dist_G(a, b), dist_H(x, y)\}$ and so $A \times B$ is a 2-packing of $G \otimes H$.

3. Let A and B be total-dominating sets of graphs G and H respectively and suppose that $\times \leq \otimes$. Let $a \in V(G)$ and $x \in V(H)$. There exist $b \in A$, $b \sim a$ and $y \in B$, $y \sim x$. In $G \otimes H$, it follows that $by \sim ax$ and so $A \times B$ is a total-dominating set in $G \otimes H$.

4. Let A and B be dominating sets of graphs G and H respectively. Let $a \in V(G)$ and $x \in V(H)$. There exist $b \in A$ and $y \in B$ with $b \simeq a$ and $y \simeq x$. Then, since $\boxtimes \leq \otimes$, $ax \simeq by$ and therefore $A \times B$ is a dominating set of $G \otimes H$.

5. Let $\{A_i\}_{i=1}^p$ and $\{B_j\}_{j=1}^q$ be independence partitions of the graphs G and H respectively. Color classes are independent sets and so $\boxtimes \leq \otimes \leq \boxtimes^c$ implies (from 1) that each $A_i \times B_j$ is an independent set of $G \otimes H$. For any

two distinct sets $A_i \times B_j$ and $A_r \times B_s$ there are vertices $a \in A_i$, $b \in A_r$, $x \in B_j$ and $y \in B_s$ (else the partitions would not be independence partitions) where $a \simeq b$ and $x \simeq y$. Then $ax \sim by$ and consequently $\{A_i \times B_j : i = 1, \dots, p; j = 1, \dots, q\}$ is an independence partition. ■

An immediate consequence of this lemma is.

Corollary 2.2. *Let G and H be finite graphs.*

1. *If $\otimes \leq \boxtimes^c$, then $\beta(G)\beta(H) \leq \beta(G \otimes H)$; if also $\boxtimes \leq \otimes$, then $i(G \otimes H) \leq i(G)i(H)$.*
2. *If $\boxtimes \leq \otimes$, then $\gamma(G \otimes H) \leq \gamma(G)\gamma(H)$; if also $\otimes \leq \boxtimes$, then $\Gamma(G \otimes H) \geq \Gamma(G)\Gamma(H)$.*
3. *If $\otimes \leq \boxtimes$, then $IR(G)IR(H) \leq IR(G \otimes H)$.*
4. *If $\times \leq \otimes$, then $\gamma_t(G \otimes H) \leq \gamma_t(G)\gamma_t(H)$.*
5. *If $\otimes \leq \boxtimes^c$, then $\chi(G \otimes H) \leq \chi(G)\chi(H)$; if also $\boxtimes \leq \otimes$, then $\psi(G)\psi(H) \leq \psi(G \otimes H)$.*
6. *If $\boxtimes \leq \otimes$, then $d(G \otimes H) \geq d(G)d(H)$.*

The following products are helpful when considering the projections of sets down to one or both of the factors.

$$Y = \begin{pmatrix} E & E & E \\ E & \Delta & E \\ N & N & N \end{pmatrix}; \ Z = \begin{pmatrix} E & E & E \\ N & \Delta & N \\ N & N & N \end{pmatrix}.$$

The proofs of the first two of the following lemmas are straightforward so we omit them.

Lemma 2.3. *Let D be a dominating set of $G \otimes H$.*

1. *If $\otimes \leq \times^c$, then either $\prod_G(D)$ is a dominating set of G or $\prod_H(D)$ is a dominating set of H.*
2. *If $\otimes \leq \times^c$, then $d(G \otimes H) \leq |V(G) \times V(H)| / \min\{\gamma(G), \gamma(H)\}$.*
3. *If $\otimes \leq Y$, then $\prod_G(D)$ is a dominating set of G.*

Lemma 2.4. *Let I be an independent set of $G \otimes H$. If $Z \leq \otimes$, then $\prod_G(I)$ is an independent set of G.*

The next result is an extension of Corollary 2.2.(4).

Lemma 2.5. *If $\bullet \leq \otimes$, then $\gamma_t(G \otimes H) \leq \gamma_t(G)$. If $\boxtimes^c \leq \otimes$ then $\gamma(G \otimes H) \leq min\{\gamma_t(G), \gamma_t(H)\}$.*

Proof. Suppose that $\bullet \leq \otimes$. If A is any total-dominating set of G and $x \in V(H)$ is fixed, then $\{ax|a \in A\}$ is a total-dominating set of $G \otimes H$. The second part follows since the lexicographic and the product whose edge matrix is the transpose of the lexicographic edge matrix are both less than $\boxtimes^c$. ■

2.2. Parameters which Respect the Product Order

Some of the parameters behave nicely with respect to the inclusion order of the products. The next lemma extends part of Corollary 2.2.

Lemma 2.6. *For a graph G, let $p(G)$ be one of $\beta(G)$, $\gamma(G)$, $\gamma_t(G)$ and $q(G)$ one of $\chi(G)$, $d(G)$. Let $\oplus$, $\otimes$ be two products such that $\oplus \leq \otimes$. Then, for every pair of graphs G and H, $p(G \oplus H) \geq p(G \otimes H)$ and $q(G \oplus H) \leq q(G \otimes H)$.*

Proof. Let G and H be an arbitrary pair of graphs. An independent set in $G \otimes H$ is also independent set in $G \oplus H$. Therefore, $\beta(G \oplus H) \geq \beta(G \otimes H)$. In addition, any coloring of $G \otimes H$ is also a partition of $V(G \oplus H)$ into independent sets so that $\chi(G \oplus H) \leq \chi(G \otimes H)$.

A dominating (total-dominating) set of $G \oplus H$ is also a dominating (total-dominating) set of $G \otimes H$. Therefore, $\gamma(G \oplus H) \geq \gamma(G \otimes H)$, $\gamma_t(G \oplus H) \geq \gamma_t(G \otimes H)$. Also any domination decomposition of $G \oplus H$ is a partition of $V(G \otimes H)$ into dominating sets and so $d(G \otimes H) \geq d(G \oplus H)$. ■

As the next corollary shows, this result allows us to search just for extremal cases (with respect to the product ordering).

Corollary 2.7. *For a graph G, let $p(G)$ be one of $\beta(G)$, $\gamma(G)$, $\gamma_t(G)$ and $q(G)$ one of $\chi(G)$, $d(G)$. Let $\oplus$, $\otimes$ be two products such that $\oplus \leq \otimes$. Then,*

1. *If for every pair of graphs G and H, $p(G \oplus H) \leq p(G)p(H)$, then $p(G \otimes H) \leq p(G)p(H)$; and if $q(G \oplus H) \geq q(G)q(H)$, then $q(G \otimes H) \geq q(G)q(H)$.*
2. *If for every pair of graphs G and H, $p(G \otimes H) \geq p(G)p(H)$, then $p(G \oplus H) \geq p(G)p(H)$; and if $q(G \otimes H) \leq q(G)q(H)$, then $q(G \oplus H) \leq q(G)q(H)$.*
3. *If for some pair G and H, $p(G \oplus H) < p(G)p(H)$, then $p(G \otimes H) \not\geq p(G)p(H)$; and if $q(G \oplus H) > q(G)q(H)$, then $q(G \otimes H) \not\leq q(G)q(H)$.*
4. *If for some pair G and H, $p(G \otimes H) > p(G)p(H)$, then $p(G \oplus H) \not\leq p(G)p(H)$; and if $q(G \otimes H) < q(G)q(H)$, then $q(G \oplus H) \not\geq q(G)q(H)$.*

In the following table, the columns are labelled by the products and the rows are labelled by the inequalities. Here, $p \geq p^2$ is shorthand for: *for all graphs* G *and* H, $p(G \otimes H) \geq p(G)p(H)$. A '+' entry indicates that the inequality listed for that row and column is true for all pairs of graphs. A '—' entry indicates that there is a pair of graphs for which the inequality is false. From Corollary 2.7, it follows that to prove the correctness of the entries, it suffices to exhibit proofs or counterexamples for the extremal products. The requisite counterexamples immediately follow the table. A '!' indicates an extremal product, where '!x' ('x!') denotes that every product above (below) this product in inclusion order has the same entry 'x'. A '?' indicates we do not know the status of the problem (see Section 2.4).

All the '+' entries follow directly from Corollaries 2.2 and 2.7, except for the lexicographic and disjunctive products with respect to $\beta \leq \beta^2$. The proof of these entries is given in Lemmas 2.8 and 2.9 immediately after the counterexamples. Some stronger results are known. These are given in Section 3.

Table 1

$\otimes =$	$\times$	$\square$	$\boxtimes$	$\bullet$	∇	$\circledcirc$	$\boxtimes^c$	$\square^c$	$\times^c$
1. $\beta \geq \beta^2$	+	+	+	+	+	!–	+!	–	!–
2. $\beta \leq \beta^2$	–	–	–	!+	–!	–!	+	+	+
3. $\chi \geq \chi^2$	–	–	–	–	–	–!	–!	?	?
4. $\chi \leq \chi^2$	+	+	+	+	+	!–	+!	–	!–
5. $\gamma \geq \gamma^2$	!–	$?^1$	–	–	!–	–	–	–	–
6. $\gamma \leq \gamma^2$	–!	–	!+	+	–!	+	+	+	+
7. $\gamma_t \geq \gamma_t^2$	!–	!–	–	–	–	–	–	–	–
8. $\gamma_t \leq \gamma_t^2$	!+	–	+	+	–!	+	+	+	+
9. $d \geq d^2$	–!	–	!+	+	–!	+	+	+	+
10. $d \leq d^2$	!–	!–	–	–	–	–	–	–	–

The graphs G_1 and G_2 can be found in Figure 3. The counterexamples are:

1.	$\beta(P_3 \circledcirc P_3) = 3 < 2 \times 2$;	$\beta(P_3 \times^c P_3) = 2 < 2 \times 2$.
2.	$\beta(C_5 \circledcirc C_5) = 5 > 2 \times 2$;	$\beta(P_3 \nabla P_3) = 5 > 2 \times 2$.
3.	$\chi(C_7 \boxtimes^c C_5) \leq 8 < 3 \times 3$;	$\chi(C_5 \circledcirc C_5) = 5 < 3 \times 3$.
4.	$\chi(P_4 \circledcirc P_4) \geq 5 > 2 \times 2$;	$\chi(P_3 \times^c P_3) = 7 > 2 \times 2$.
5.	$\gamma(G_1 \times G_1) = 3 < 2 \times 2$;	$\gamma(G_2 \nabla C_4) = 2 < 2 \times 2$.
6.	$\gamma(P_3 \times P_3) = 3 > 1 \times 1$;	$\gamma(K_n \nabla K_n) = n > 1 \times 1$.
7.	$\gamma_t(K_3 \times K_3) = 3 < 2 \times 2$;	$\gamma_t(P_3 \square P_3) = 3 < 2 \times 2$.
8.	$\gamma_t(K_n \nabla K_n) = n > 2 \times 2 \ (n > 4)$.	
9.	$d(P_3 \times P_3) = 2 < 2 \times 2$;	$d(K_n \nabla K_n) = n < n \times n$.
10.	$d(C_5 \times C_5) = 5 > 2 \times 2$;	$d(C_5 \square C_5) = 5 > 2 \times 2$.

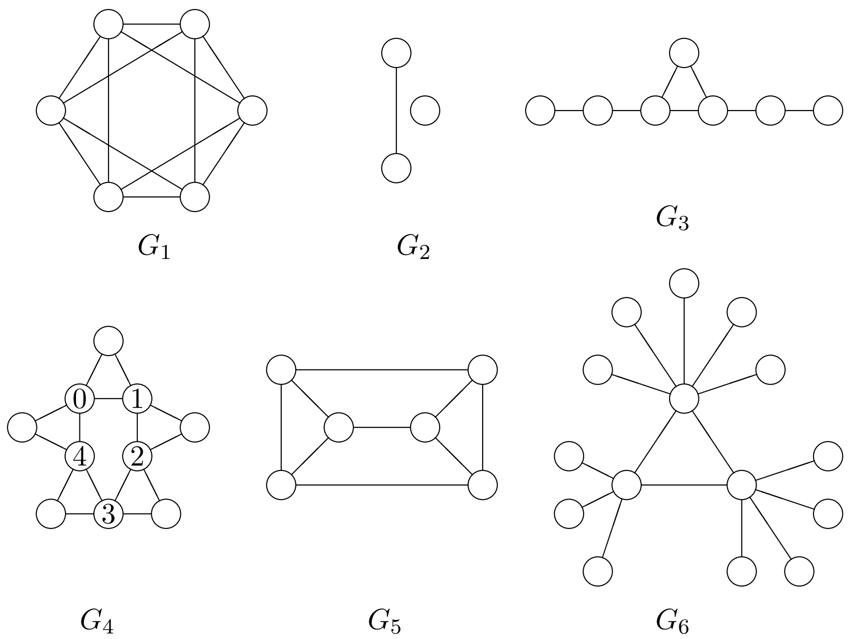


Figure 3. Some Counterexample Graphs.

Part of the next result, $\beta(G \bullet H) = \beta(G)\beta(H)$, can also be found in [7], see also [14].

Lemma 2.8. *For all graphs G and H, $i(G \bullet H) = i(G)i(H)$ and $\beta(G \bullet H) = \beta(G)\beta(H)$.*

Proof. From Corollary 2.2 (1) we have $\beta(G)\beta(H) \leq \beta(G \bullet H)$; $i(G \bullet H) \leq i(G)i(H)$.

Let I be a maximal independent set of $G \bullet H$. Then, by Lemmas 2.4 and 2.3(3), $\prod_G(I)$ is a maximal independent set of G. Also, for any $a \in \prod_G(I)$, $I \cap (\{a\} \times H)$ is a maximal independent set of $\{a\} \times H$ and so the result follows. ∎

The second part of the next result is also to be found in [14].

Lemma 2.9. *For all graphs G and H, $i(G \boxtimes^c H) = i(G)i(H)$ and $\beta(G \boxtimes^c H) = \beta(G)\beta(H)$.*

Proof. Let I be an independent set of $G \boxtimes^c H$. By Lemma 2.4, $\prod_G(I)$ and $\prod_H(I)$ are independent. By Lemma 2.1, the product of two independent sets is an independent set of $G \boxtimes^c H$. Hence, $\beta(G \boxtimes^c H) = \beta(G)\beta(H)$ and $i(G \boxtimes^c H) = i(G)i(H)$. ∎

2.3. Other Parameters

As in Table 1, a '–' entry means that there is a counterexample which is given immediately after the table. A '+' indicates that the inequality is true and the associated number is the number of the result that gives the proof; (3T) refers to the table in Section 3 where a construction is indicated; otherwise the number of the appropriate Theorem, Lemma or Corollary is given. A '?' indicates that we know of no proof nor of a counterexample.

Table 2

$\otimes =$	$\times$	$\square$	$\boxtimes$	$\bullet$	∇	$(\cong)$	$\boxtimes^c$	$\square^c$	$\times^c$
1. $ir \geq ir^2$	–	?	–	–	–	–	–	–	–
2. $ir \leq ir^2$	–	–	–	–	–	+ (3T)	+(3T)	+(3T)	+(3T)
3. $i \geq i^2$	?	–	–	+(2.8)	–	–	+(2.9)	–	–
4. $i \leq i^2$	–	–	+(2.2)	+(2.8)	–	–	+(2.9)	+(3.6)	+(3.10)
5. $\Gamma \geq \Gamma^2$	?	?	+(2.2)	–	–	–	–	?	?
6. $\Gamma \leq \Gamma^2$	–	–	–	+(2.10)	–	–	+(2.12)	+(3.7)	+(3.11)
7. $IR \geq IR^2$	+(2.2)	+(2.2)	+(2.2)	–	–	–	–	–	–
8. $IR \leq IR^2$	–	–	–	+(2.11)	–	–	?	+(3.6)	?
9. $\psi \geq \psi^2$	–	–	+(2.2)	+(2.2)	–	?	+(2.2)	+(3.8)	?
10. $\psi \leq \psi^2$	–	–	–	–	?	–	?	–	–
11. $ad \geq ad^2$	–	–	–	–	–	–	–	–	?
12. $ad \leq ad^2$	?	?	?	?	?	–	?	–	–

The graphs G_1, G_2, G_3, G_4, G_5 and G_6 are to be found in Figure 3.

1. $ir(G_1 \times G_1) \leq 3 < 2 \times 2$; $ir(C_4 \boxtimes C_4) = 3 < 2 \times 2$;
   $ir(C_4 \bullet C_4) = 2 < 2 \times 2$; $ir(C_4 \nabla C_4) = 2 < 2 \times 2$;
   $ir(P_n \circledcong P_n) \leq 4n - 3 < n^2/9$, ($n$ large); $ir(C_4 \boxtimes^c C_4) = 2 < 2 \times 2$;
   $ir(C_4 \,\Box^c C_4) = 2 < 2 \times 2$; $ir(C_4 \times^c C_4) = 2 < 2 \times 2$.

2. $ir(K_3 \times K_3) = 3 > 1 \times 1$; $ir(K_3 \,\Box\, K_3) = 3 > 1 \times 1$;
   $ir(G_3 \boxtimes P_3) = 3 > 2 \times 1$; $ir(G_3 \bullet P_3) = 3 > 2 \times 1$;
   $ir(P_3 \nabla P_3) = 2 > 1 \times 1$.

3. $i(G_6 \,\Box\, \overline{G_6}) = 15 < 8 \times 2$; $i(G_4 \boxtimes C_5) = 5 < 3 \times 2$;
   $i(C_5 \nabla G_2) = 3 < 2 \times 2$; $i(\overline{K}_2 \circledcong \overline{K}_2) = 2 < 2 \times 2$;
   $i(\overline{K}_2 \,\Box^c \overline{K}_2) = 2 < 2 \times 2$; $i(C_4 \times^c C_4) = 2 < 2 \times 2$.

4. $i(P_3 \times P_3) = 3 > 1 \times 1$; $i(P_3 \,\Box\, K_2) = 2 > 1 \times 1$;
   $i(P_3 \nabla P_3) = 4 > 1 \times 1$; $i(K_{2,5} \circledcong K_{2,5}) \geq 5 > 2 \times 2$.

5. $\Gamma(G_2 \bullet P_3) = 4 < 3 \times 2$; $\Gamma(G_5 \nabla G_5) < 3 \times 3$;
   $\Gamma(P_3 \circledcong P_4) = 3 < 2 \times 2$; $\Gamma(G_5 \boxtimes^c P_3) = 4 < 3 \times 2$.

6. $\Gamma(K_n \times K_2) = n > 1 \times 1$, $n > 1$; $\Gamma(K_n \,\Box\, K_2) = n > 1 \times 1$, $n > 1$;
   $\Gamma(C_5 \boxtimes C_5) = 5 > 2 \times 2$; $\Gamma(K_n \nabla K_2) = n > 1 \times 1$, $n > 1$;
   $\Gamma(P_4 \circledcong P_4) \geq 5 > 2 \times 2$.

7. $IR(G_5 \bullet P_3) = 4 < 3 \times 2$; $IR(G_5 \nabla G_5) < 3 \times 3$;
   $IR(P_3 \circledcong P_3) = 3 < 2 \times 2$; $IR(G_5 \boxtimes^c P_3) = 4 < 3 \times 2$;
   $IR(C_4 \,\Box^c C_4) = 2 < 2 \times 2$; $IR(P_3 \times^c P_3) = 2 < 2 \times 2$.

8. $IR(K_2 \times K_2) = 2 > 1 \times 1$; $IR(K_2 \,\Box\, K_2) = 2 > 1 \times 1$;
   $IR(C_5 \boxtimes C_5) = 5 > 2 \times 2$; $IR(P_3 \nabla P_3) = 5 > 2 \times 2$;
   $IR(C_5 \circledcong C_5) = 5 > 2 \times 2$.

9. $\psi(K_2 \times K_2) = 2 < 2 \times 2$; $\psi(K_2 \,\Box\, K_2) = 2 < 2 \times 2$;
   $\psi(K_2 \nabla K_2) = 2 < 2 \times 2$.

10. $\psi(P_7 \times P_6) \geq 10 > 3 \times 3$; $\psi(P_7 \,\Box\, P_3) \geq 7 > 3 \times 2$;
    $\psi(P_3 \boxtimes P_3) \geq 6 > 2 \times 2$; $\psi(P_3 \bullet P_5) \geq 7 > 2 \times 3$;
    $\psi(P_3 \circledcong P_3) \geq 5 > 2 \times 2$; $\psi(P_3 \,\Box^c P_3) \geq 5 > 2 \times 2$;
    $\psi(P_3 \times^c P_3) \geq 7 > 2 \times 2$.

11. $ad(K_2 \times P_3) = 2 < 2 \times 2$; $ad(K_2 \,\Box\, P_3) = 2 < 2 \times 2$;
$ad(K_2 \boxtimes P_3) = 2 < 2 \times 2$; $ad(K_2 \bullet P_3) = 3 < 2 \times 2$;
$ad(K_2 \nabla P_3) = 2 < 2 \times 2$; $ad(K_2 \circledcirc P_3) = 2 < 2 \times 2$;
$ad(K_2 \boxtimes^c P_3) = 3 < 2 \times 2$; $ad(K_2 \,\Box\, {}^c P_3) = 3 < 2 \times 2$.

12. $ad(\overline{K}_2 \circledcirc \overline{K}_2) = 2 > 1 \times 1$; $ad(\overline{K}_2 \,\Box\, {}^c \overline{K}_2) = 2 > 1 \times 1$;
$ad(G_2 \times^c G_2) = 2 > 1 \times 1$.

Lemma 2.10. *For all graphs G and H, $\Gamma(G \bullet H) \leq \Gamma(G)\Gamma(H)$.*

Proof. Let D be an irredundant dominating set of maximum cardinality for $G \bullet H$. For $F \subset G \bullet H$, let $\prod_G(F) = S_F \cup C_F = X_F$ where S_F is the set of isolated vertices and C_F the union of the connected components in $\langle X_F \rangle$.

Suppose $\Gamma(H) = 1$ then $H \cong K_n$ for some n. Since D is irredundant, we have for each $a \in V(G)$, $|(\{a\} \times V(H)) \cap D| \leq 1$. Also $\prod_G(D)$ dominates G (Lemma 2.3(3)) and is irredundant (since D is). Therefore $\Gamma(G \bullet H) = |D| = |\prod_G(D)| \leq \Gamma(G) = \Gamma(G)\Gamma(H)$.

We may assume, therefore, that $\Gamma(H) > 1$. Choose D to be an irredundant dominating set of maximum cardinality for $G \bullet H$ which has the additional property that $|S_D|$ is maximum.

Suppose $a \in X_D$ and $ax, ay \in D$. Then, all their respective private neighbors must lie in $\{a\} \times H$ since they are adjacent to the same vertices of $(G - a) \bullet H$: In particular, this implies that $a \in S_D$. Also, if $a \in C_D$ then it has exactly one pre-image and we denote this vertex by ax_a. Moreover, if $a \in C_D$ then ax_a has no private neighbor in $\{a\} \times H$.

Now, C_D can be partitioned into two subsets, $C_D = C_D^1 \cup C_D^2$. Specifically, $a \in C_D^1$ if ax_a has a private neighbor of the form by, $b \notin X_D$; and $a \in C_D^2$ if all the private neighbors of ax_a are of the form by, $b \in X_D$.

We claim that $C_D^2 = \emptyset$. Suppose to the contrary that there exists $a \in C_D^2$. Let $Y = \{by | by \in I(D, ax_a)\}$, $Y_G = \prod_G(Y)$ and let Z be a maximum irredundant dominating set of H. Put $E = (D - (Y \cup \{ax_a\})) \cup (Y_G \times Z)$. Note that for any $b \in Y_G$, ax_a has a private neighbor in $\{b\} \times H$ and so b is isolated in X_E, that is, $S_E = S_D \cup Y_G$. Any vertex of $G \bullet H$ which is dominated by only vertices ax_a or vertices of Y and no other vertices of D, are in either $\{a\} \times H$ or $\{b\} \times H$, $b \in Y_G$. These vertices are dominated by $Y_G \times Z$ and the other vertices of $G \bullet H$ are dominated by $D - (Y \cup \{ax_a\})$, and so E is a dominating set of $G \bullet H$. The vertices of $D - (Y \cup \{ax_a\})$ have the same private neighbors with respect to E as with respect to D. If $by \in Y_G \times Z$ then y has a private neighbor, say z, in $V(H)$ and so $bz \in I(E, bx)$. Hence E is an irredundant set. Now $|E| = |D| - (|Y_G| + 1) + |Y_G|\Gamma(H)$ and since $\Gamma(H) > 1$, then $|E| \geq |D|$, which contradicts the choice of D. Therefore, $C_D^2 = \emptyset$.

Hence, $X_D = S_D \cup C_D^1$. By Lemma 2.3(3), X_D is a dominating set of G. By the definition of C_D^1, X_D is irredundant so that $|X_D| \leq \Gamma(G)$. Let $|S_D| = s$ and $|C_D^1| = c$. Then, $|D| \leq s\Gamma(H) + c \leq (s + c)\Gamma(H) \leq \Gamma(G)\Gamma(H)$. ∎

Corollary 2.11. *For all graphs G and H, $IR(G \bullet H) \leq IR(G)IR(H)$.*

Proof. Let D be a maximum-sized irredundant set for $G \bullet H$.

Suppose $IR(H) = 1$ then $H \cong K_n$ for some n. Hence, for each $a \in V(G)$, $|(\{a\} \times V(H)) \cap D| \leq 1$, since D is irredundant. Also $\prod_G(D)$ is irredundant (since D is). Therefore $IR(G \bullet H) = |D| = |\prod_G(D)| \leq IR(G) = IR(G)IR(H)$.

In the proof of Lemma 2.10, note that if D is a maximum-sized irredundant set such that $|S_D|$ is maximized then it still follows (where Z is now a maximum-sized irredundant set of H) that $X_D = S_D \cup C_D^1$ and that X_D is an irredundant set for G. Hence, $|D| \leq |S_D|IR(H) + |C_D^1| \leq (|S_D| + |C_D^1|)IR(H) \leq IR(G)IR(H)$. ∎

Lemma 2.12. *For all graphs G and H, $\Gamma(G \boxtimes^c H) \leq \Gamma(G)\Gamma(H)$.*

Proof. Let D be an irredundant dominating set of $G \boxtimes^c H$ with $|D| = \Gamma(G \boxtimes^c H)$. From Lemma 2.3, we have that $\prod_G(D)$ is a dominating set of G or else $\prod_H(D)$ is a dominating set of H.

Suppose that $\prod_G(D)$ is a total-dominating set of G. Let E be a minimal total-dominating set of G contained in $\prod_G(D)$. For each $a \in E$ choose one pre-image $ax_a \in D$ and put $F = \{ax_a | a \in E\}$. Let $by \in V(G \boxtimes^c H)$, then there is some $a \in E$ such that $b \sim a$ and $by \sim ax_a$. Since $F \subset D$, F is irredundant and since it is dominating, $F = D$ and $\Gamma(G \boxtimes^c H) = |F| \leq \Gamma_t(G) \leq 2\Gamma(G)$. Similarly, if $\prod_G(H)$ is a total-dominating set of H then $\Gamma(G \boxtimes^c H) \leq 2\Gamma(H)$.

Suppose that $\Gamma(G) = \Gamma(H) = 1$ then both graphs are isomorphic to complete graphs and hence, so is $G \boxtimes^c H$. In this case $\Gamma(G \boxtimes^c H) = 1$.

Suppose that $\Gamma(G) = 1$ (i.e $G \cong K_n$) and $\Gamma(H) > 1$. Now, if $|\prod_G(D)| > 1$ then $\prod_G(D)$ is a total-dominating set and $\Gamma(G \boxtimes^c H) \leq 2 \leq \Gamma(G)\Gamma(H)$. If $\prod_G(D) = \{a\}$ then $D = \{a\} \times E$ where E is an irredundant dominating set of H and again we have $\Gamma(G \boxtimes^c H) \leq \Gamma(G)\Gamma(H)$. (By the commutativity of the product, we do not have to consider $\Gamma(H) = 1$ and $\Gamma(G) > 1$.)

We may suppose, therefore, that both $\Gamma(G)$ and $\Gamma(H)$ are at least two. Suppose that $\prod_G(D)$ is not a dominating set of G. Then there exists $a \in V(G) - N[\prod_G(D)]$. Since D dominates $\{a\} \times H$, for every vertex $ax \in \{a\} \times H$ there is a vertex $by \in D$ where $by \sim ax$. But, since $b \perp a$ it follows that

$y \sim x$: that is, $\prod_H(D)$ contains a total dominating set of H. Therefore, in this case, $\Gamma(G \boxtimes^c H) \leq 2\Gamma(H) \leq \Gamma(G)\Gamma(H)$.

Finally, therefore, we may assume that both $\prod_G(D)$ and $\prod_H(D)$ are dominating but not total-dominating sets in their respective graphs. Let $E(F)$ be an irredundant dominating set of $G(H)$ contained in $\prod_G(D)$ ($\prod_H(D)$). Let $E = C_G \cup S_G$ and $F = C_H \cup S_H$ where S_G is the set of isolated vertices and C_G is the union of the connected components in $\langle E \rangle$ and C_H, S_H are defined similarly for F. Let W be a set formed by taking one pre-image (not necessarily distinct) for each vertex of $C_G \cup C_H$. Now W dominates every vertex of $(N[C_G] \times H) \cup (G \times N[C_H])$ so that no other vertex of D can have a vertex of $(N[C_G] \times H) \cup (G \times N[C_H])$ as its private neighbor.

Let X be a set formed by taking one pre-image (not necessarily distinct) for each vertex of $S_G \cup S_H$. Now X dominates every vertex of $(N(S_G) \times H) \cup (G \times N(S_H))$ so that no other vertex of D can have a vertex of $(N(S_G) \times H) \cup (G \times N(S_H))$ as its private neighbor. Therefore, only vertices of $S_G \times S_H$ are available as private neighbors. Consequently, since $|C_G| + |S_G| \leq \Gamma(G)$ and $|C_H| + |S_H| \leq \Gamma(H)$, we have that

$$\begin{aligned}
\Gamma(G \boxtimes^c H) &= |D| \leq |C_G| + |C_H| + \Gamma(G) - |C_G| + \Gamma(H) - |C_H| \\
&\quad +(\Gamma(G) - |C_G|)(\Gamma(H) - |C_H|) \\
&= \Gamma(G) + \Gamma(H) + (\Gamma(G) - |C_G|)(\Gamma(H) - |C_H|) \\
&\leq \Gamma(G)\Gamma(H).
\end{aligned}$$

The last inequality follows since all of $|C_G|$, $|C_H|$, $\Gamma(G)$ and $\Gamma(H)$ are at least 2. ■

2.4. Conjectures

In addition to Vizing's conjecture, we believe the following statements to be true but we were not able to find proofs. For all graphs G and H

1. $ir(G \,\square\, H) \geq ir(G)ir(H)$.
2. $i(G \times H) \geq i(G)i(H)$.
3. $\gamma(G \,\square\, H) \geq \gamma(G)\gamma(H)$ – Vizing's conjecture.
4. $\Gamma(G \times H) \geq \Gamma(G)\Gamma(H)$; $\quad \Gamma(G \,\square\, H) \geq \Gamma(G)\Gamma(H)$.
5. $IR(G \boxtimes^c H) \leq IR(G)IR(H)$; $\quad IR(G \times^c H) \leq IR(G)IR(H)$.
6. $\chi(G \,\square^c H) \geq \chi(G)\chi(H)$; $\quad \chi(G \times^c H) \geq \chi(G)\chi(H)$.
7. $\psi(G \circledcong H) \geq \psi(G)\psi(H)$; $\quad \psi(G \times^c H) \geq \psi(G)\psi(H)$.

The other missing entries in our tables we believe to have counterexamples, but we have not been able to find any.

3. OTHER MULTIPLICATIVE RESULTS

Some of the inequalities presented in the previous sections can be improved by using combinations of different parameters. These are included in this section and are previewed in the next table. The entries in the table are

Table 3

$\otimes =$	$\times$	$\square$	$\boxtimes$	$\bullet$	∇
ir	$\leq \gamma_t\gamma$ (1.1)			$\leq \gamma_t$ (1.1)	$\leq \gamma_t$ (i^1)
γ	$\leq \gamma_t\gamma_t$ (x)	$\leq \gamma\|V\|$ (x)	$\leq \gamma^2$ (2.2)	$\leq \gamma_t$ (e^2)	$\leq \gamma_t$ (e^1)
	$\geq P_2\gamma_t$ (x^2)	$\geq P_2\gamma$ [19]	$\geq P_2\gamma$ (3.4)		
i	$\leq i\|V\|$ (x^2)		$\leq i^2$ (2.2)	$= i^2$ (2.8)	$\leq i^2$ $(\mathrm{x})^3$
β	$\geq \beta\|V\|$ (x)	$\geq \beta^2$ (2.2)	$\geq \rho^2$ [27]	$= \beta^2$ [7]	$\geq \beta^2$ (x)
Γ	$\geq \beta\|V\|$ (x)	$\geq \beta^2$ (1.1)	$\geq \Gamma^2$ (2.2)	$\geq \beta\Gamma$ (x)	$\geq \beta^2$ (x)
				$\leq \Gamma^2$ (2.10)	
IR	$\geq IR^2$ (2.2)	$\geq IR^2$ (2.2)	$\geq IR^2$ (2.2)	$\geq \beta IR$ (x)	$\geq \beta^2$ (x)
				$\leq IR^2$ (2.11)	
γ_t	$\leq \gamma_t^2$ (x)		$\leq \gamma_t\gamma$ (x)	$\leq \gamma_t$ (2.5)	$\leq \gamma_t$ (e^1)
Γ_t				$\geq \beta\Gamma_t$ (x)	
χ	$\leq \min\{\chi,\chi\}$	$= \max\{\chi,\chi\}$[28]	$\leq \chi^2$ (2.2)	$\leq \chi^2$ [7]	$\leq \chi^2$ [3]
				$\geq \chi(G) + 2\chi(H)$ -2[7]	
ψ	$\geq \max\{\psi,\psi\}$([13])		$\geq \psi^2$ (2.2)	$\geq \psi^2$ (2.2)	
d			$\geq d^2$ (2.2)	$\geq d^2$ (2.2)	

$\otimes =$	$\textcircled{$\cong$}$	$\boxtimes^c$	$\square^c$	$\times^c$
ir	$\leq ir + ir$ (2e)	$\leq \min\{ir, ir\}$(e)	≤ 3(1.1)	$\leq \min\{\gamma,\gamma\}$(1.1)
γ	$\leq \gamma + \gamma - 1$(3.5)	$\leq \min\{\gamma,\gamma\}$(2.5)	≤ 3 (3.7)	$= \min\{\gamma,\gamma\}$(3.11)
i		$= i^2$ (2.9)	$= \min\{i, i\}$ (3.6)	$= \min\{i, i\}$(3.10)
β		$= \beta^2$ [4]	$= \max\{\beta,\beta\}$ (3.6)	$= \min\{\beta,\beta\}$(3.10)
Γ		$\leq \Gamma^2$ (2.12)	$\leq \max\{\Gamma,\Gamma,4\}$ (3.7)	$= \max\{\Gamma,\Gamma\}$(3.11)
			$\geq \max\{\Gamma,\Gamma\}$ (2.2)	
IR		$\geq \beta^2$ (x)	$\leq \max\{4, IR, IR\}$(3.6)	
			$\geq \max\{IR, IR\}$(3.6)	
γ_t	$\leq \gamma_t^2$ (2.2)	$\leq \gamma_t$ (o)	≤ 6 (1.1)	$\leq \gamma_t$ (o)
χ		$\leq \chi^2$ [3]		
ψ		$\geq \psi^2$ (2.2)	$\geq \|V(G)\|(\psi(H) - \|S\|)$ $+\psi(G)\|S\|$ (3.8)	$\geq \psi\|V\|$ (x^4)
d	$\geq d^2$(2.2)	$\geq d\|V\|$ (x)	$\geq \|V\|^2/4$ (3.9)	$\geq d\|V\|$ (x)

parameters that have been stripped of references to the factors. For the commutative products the order is unimportant and there is an implied optimization operator. For the lexicographic product, the order is important and the parameters refer to the factors in that order, or if only one parameter is given, this refers to the first factor.

In the table, (x) means the cartesian product of two of the indicated sets results in a set of the required type; (e) means that an appropriate set in one graph multiplied by a single vertex from the other graph is a set of the required type, (2e) means take the union of two such sets where an appropriate set is taken from both factors; (o) means that the inequality is true because of the inclusion order of the products; square brackets are references; all other numbers refer to the Lemma or Corollary where the proof can be found. The superscript 1 means the result is only true for graphs with a minimum-sized total-dominating set D where for all vertices of G there is a vertex of D to which it is not adjacent. The superscript 2 means that the result is true if G has no isolated vertices. The superscript 3 means the result is only true for graphs with a minimum-sized independent dominating set I such that for all vertices of G there is a vertex of I to which it is not adjacent. The superscript 4 means the result is only true for graphs with an independence partition of maximum size where no subset is of cardinality 1. The set S refers to a set of singletons in an independence partition of G.

3.1. Categorical Product

Theorem 3.1. *Let G be a graph with no isolated vertices. Then, for every graph H,*

$$\gamma(G \times H) \geq P_2(H)\gamma_t(G).$$

Proof. Let P be a maximum 2-packing of H. Let D be any dominating set of $G \times H$. For each $x \in V(H)$, set $G_x = V(G) \times \{x\}$ and $E_x = D \cap V(G \times N[x])$.

Note that, if, for some $x \in V(H)$, $E_x \cap G_x = \emptyset$ then every $ax \in G_x$ must be adjacent to some $by \in E_x$ with $b \in N(a)$ and $y \in N(x)$. In this case, let $A = \{c|cy \in E_x\}$ then A is a total-dominating set of G and $|E_x| \geq |A| \geq \gamma_t(G)$. On the other hand, if $x \in V(H)$, $E_x \cap G_x \neq \emptyset$, then replace each vertex $ax \in E_x$ with a vertex by where $b \sim a$ and $y \sim x$ to form the set F. F still dominates G_x, $F \cap G_x = \emptyset$ and so by the previous argument, $|E_x| \geq |F| \geq \gamma_t(G)$.
Therefore, if $P = \{y_1, y_2, \ldots, y_r\}$ is a maximum 2-packing of H then $|D \cap V(G \times N[y_i])| \geq \gamma_t(G)$, and for $i \neq j$, $V(G \times N[y_i]) \cap V(G \times N[y_j]) = \emptyset$.

Thus $|D| \geq P_2(H)\gamma_t(G)$. ■

Corollary 3.2. *Let G and H be graphs. Then*

1. $\gamma(G \times H) \geq P_2(G)\gamma(H)$.
2. *If G is a graph with $P_2(G) = \gamma(G)$ then $\gamma(G \times H) \geq \gamma(G)\gamma(H)$.*

Proof. 1. Let $G = E \cup F$ where E is the set of isolated vertices of G. Then $\gamma(G \times H) \geq P_2(H)(\gamma_t(F) + |E|) \geq P_2(H)\gamma(G)$.
2. Immediate. ■

These lower bounds can be achieved. For example, $P_2(P_4) = \gamma(P_4) = 2$, $\gamma_t(C_6) = 4$ and it is straightforward to verify that $\gamma(C_6 \times P_4) = 8$.

3.2. Strong Product

Theorem 3.3. *If G is a graph with $P_2(G) = \gamma(G)$ then $\gamma(G \boxtimes H) = \gamma(G)\gamma(H)$.*

Proof. Let D be a dominating set of $G \boxtimes H$ and $a \in V(G)$. Let $E = D \cap (N[a] \boxtimes V(H))$. E dominates $\{a\} \times H$ and so every vertex of H is dominated by $\prod_H(E)$ and therefore $|D \cap (N[a] \boxtimes V(H))| \geq \gamma(H)$.

Consider $g = P_2(G) = \gamma(G)$ disjoint closed neighborhoods $N[a_1]$, $N[a_2], \ldots, N[a_g]$ in G. Then $N[a_1] \times V(H)$, $N[a_2] \times V(H), \ldots, N[a_g] \times V(H)$ are pairwise disjoint. If D is any dominating set of $G \boxtimes H$, then by the preceding paragraph, for each i, $|D \cap (N[a_i] \boxtimes V(H))| \geq \gamma(H)$. It follows that

$$|D| \geq |D \cap (\cup_{i=1}^{g} N[a_i] \boxtimes V(H))|$$

$$= | \cup_{i=1}^{g} D \cap (N[a_i] \boxtimes V(H))| \geq \gamma(G)\gamma(H).$$

This, together with Corollary 2.2(2), gives the result. ■

Corollary 3.4. $\gamma(G \boxtimes H) \geq P_2(G)\gamma(H)$.

As an application of this result, note that if T is a tree then $\gamma(T \boxtimes H) = \gamma(T)\gamma(H)$.

3.3. Equivalence Product

Although this product produces many edges, note that if $G \cong K_n$ then $G \boxtimes H \cong G \circledcong H$. Thus to construct an equivalence product with $\gamma(G \circledcong H) = r$ let $G = K_3$, and $H = P_{3r}$.

Theorem 3.5. *Let G and H be graphs.*

1. *If $diam(G) \geq 5$, then $\gamma(G \circledcong H) \leq \gamma(G)$.*
2. *Suppose that $P_2(G) \geq 3$ and $P_2(H) \geq 3$, then $\gamma(G \circledcong H) \leq 3$.*
3. *Suppose that $P_2(G) \geq 2$ and $P_2(H) \geq 2$, then $\gamma(G \circledcong H) \leq 4$.*
4. *$\gamma(G \circledcong H) \leq \gamma(G) + \gamma(H) - 1$.*

Proof. 1. Let D be a minimum dominating set of G and $x \in V(H)$. Consider the set $D \times \{x\}$. Let $bz \in V(G \circledcong H)$. Suppose that $z \simeq x$ then for some $g \in D$, $b \simeq g$ and so $bz \simeq gx$. If $z \perp x$ then there is some $g \in D$ such that $g \perp b$ and so again, $bz \simeq gx$.

2. Let $\{a, b, c\}$ and $\{x, y, z\}$ be 2-packings of G and H, respectively, then consider $D = \{ax, by, cz\}$. Let $dv \in V(G \circledcong H)$. If $d \notin N[c]$ and $v \notin N[z]$ then cz dominates dv. If $d \in N[c]$ and $v \in N[z]$ then again cz dominates dv. If $d \in N[c]$ and $v \notin N[z]$ then $d \notin N[a] \cup N[b]$ and either $v \notin N[x]$ or $v \notin N[y]$. Therefore either ax or by dominates dv. The case $d \notin N[c]$ and $v \in N[z]$ is similar to the last case. Therefore, D dominates $G \circledcong H$.

3. Let $\{a, b\}$ and $\{x, y\}$ be 2-packings of G and H respectively. Then $D = \{ax, ay, bx, by\}$ is a dominating set for $G \circledcong H$.

4. Let $D = \{g_1, g_2, \ldots, g_k\}$ and $E = \{h_1, h_2, \ldots, h_s\}$ be minimum dominating sets of G and H. Consider the set $F = (D \times \{h_1\}) \cup (\{g_1\} \times E)$. Let $bz \in V(G \circledcong H)$. If $b \simeq g_1$ then there is some j such that $z \simeq h_j$ and therefore $bz \simeq g_1 h_j$; if $z \simeq h_1$ then there is some j such that $b \simeq g_j$ and therefore $bz \simeq g_j h_1$; if $b \perp g_1$ and $z \perp h_1$ then $bz \simeq g_1 h_1$. Therefore, F is a dominating set of $G \circledcong H$. ■

3.4. Cartesian complement

Theorem 3.6. *Let G and H be graphs. Then*

1. *if $\min\{i(G), i(H)\} = 1$, then $i(G \,\square\, {}^c H) = \max\{i(G), i(H)\}$, otherwise $i(G \,\square\, {}^c H) = \min\{i(G), i(H)\}$.*
2. *$\beta(G \,\square\, {}^c H) = \max\{\beta(G), \beta(H)\}$.*
3. *$\max\{\Gamma(G), \Gamma(H)\} \leq \Gamma(G \,\square\, {}^c H) \leq \max\{4, \Gamma(G), \Gamma(H)\}$.*
4. *$\max\{IR(G), IR(H)\} \leq IR(G \,\square\, {}^c H) \leq \max\{4, IR(G), IR(H)\}$.*

Proof. Let S and T be dominating sets of G and H, respectively. If $|S| > 1$ then for each $x \in V(H)$ the set $S \times \{x\}$ is a dominating set for $G \,\square\, {}^c H$. If $|S| = 1$ then for each $a \in V(G)$ the set $\{a\} \times T$ is a dominating set for $G \,\square\, {}^c H$. In addition, suppose that S is an independent or irredundant set of G.

Then for each $x \in V(H)$ the set $S \times \{x\}$ is an, respectively, independent or irredundant set for $G \,\square\, {}^c H$. Therefore, by the symmetry of the product, we have $i(G \,\square\, {}^c H) \leq \min\{i(G), i(H)\}$, if $\min\{i(G), i(H)\} > 1$ else $i(G \,\square\, {}^c H) \leq \max\{i(G), i(H)\}$; $\beta(G \,\square\, {}^c H) \geq \max\{\beta(G), \beta(H)\}$, $\max\{\Gamma(G), \Gamma(H)\} \leq \Gamma(G \,\square\, {}^c H)$ and $\max\{IR(G), IR(H)\} \leq IR(G \,\square\, {}^c H)$.

Let $a \in V(G)$ and $x \in V(H)$. Then the only vertices of $G \,\square\, {}^c H$ not adjacent to ax are in the subsets $\{a\} \times \overline{N[x]}$ and $\overline{N[a]} \times \{x\}$. Moreover, any vertex in either subset dominates all the vertices of the other. Therefore, the only independent sets containing more than one vertex are of the form $\{a\} \times X$ or $A \times \{x\}$ where A and X are independent sets. Thus $i(G \,\square\, {}^c H) = \min\{i(G), i(H)\}$ and $\beta(G \,\square\, {}^c H) = \max\{\beta(G), \beta(H)\}$.

Let ax and by be two vertices with $a \neq b$ and $x \neq y$. The only vertices of $G \,\square\, {}^c H$ not dominated by $\{ax, by\}$ are bx and ay. Therefore, if D is a maximal irredundant or minimal dominating set such that both $|\prod_G(D)|$ and $|\prod_H(D)|$ are at least two then $|D| \leq 4$. Thus, if $|D| \geq 5$ then either $D \subseteq \{a\} \times H$ for some vertex a or $D \subseteq G \times \{x\}$ for some vertex x.

Suppose D is an irredundant set of $G \,\square\, {}^c H$, $|D| \geq 5$ and $D \subseteq \{a\} \times H$ for some vertex a. Let $X \subseteq V(H)$, $|X| \geq 3$. If for some $x \in X$, $N[x] \subseteq N[X - x]$ then $\{a\} \times X$ is not an irredundant set of $G \,\square\, {}^c H$ since $I(\{a\} \times X, ax) = \emptyset$. Therefore, D is an irredundant set of $\{a\} \times H$. Thus, by the symmetry of the product, we have $IR(G \,\square\, {}^c H) \leq \max\{4, IR(G), IR(H)\}$.

Suppose D is an irredundant dominating set of $G \,\square\, {}^c H$, $|D| \geq 5$ and $D \subseteq \{a\} \times H$ for some vertex a. Trivially, D must be a dominating set of $\{a\} \times H$ and, by the previous paragraph, D must also be an irredundant set of $\{a\} \times H$. Thus $\Gamma(G \,\square\, {}^c H) \leq \max\{4, \Gamma(G), \Gamma(H)\}$. ∎

Since $IR(2K_2 \,\square\, {}^c 2K_2) = \Gamma(2K_2 \,\square\, {}^c 2K_2) = 4$, it is not possible to remove the '4' from the previous result.

Theorem 3.7. *Let G and H be graphs. If one of G and H contains an edge then $\gamma(G \,\square\, {}^c H) \leq 2$; if both are trivial graphs then $\gamma(G \,\square\, {}^c H) \leq 3$.*

Proof. Let a, b be distinct vertices of G and x, y distinct vertices of H. If one or both of (a, b) and (x, y) are edges then $\{ax, by\}$ is a dominating set of $G \,\square\, {}^c H$. If both graphs are trivial then $\{ax, ay, bx\}$ dominates. ∎

For a graph G let $s(G)$ be the least number of singleton sets in an independence partition of G where the size of the partition is $\psi(G)$.

Theorem 3.8. *Suppose that G and H are graphs, then $\psi(G \,\square\, {}^c H) \geq$*

$$\max\{|V(G)|(\psi(H) - s(G)) + s(G)\psi(G),\ |V(H)|(\psi(G) - s(H)) + s(H)\psi(H)\}.$$

Proof. Let A_1, A_2, …, A_k, S_1, S_2, …, S_s be the sets of an independence partition of G where $|S_1| = |S_2| = \ldots = |S_s| = 1$. Also let Y_1, Y_2, …, Y_f be an independence partition of H. Consider the sets $A_i \times \{x\}$, $1 \leq i \leq k$, $x \in V(H)$ and $\{a\} \times Y_j$, $1 \leq j \leq f$, $a \in \cup_{i=1}^{s} S_i$.

All the sets in this partition of $V(G \times H)$ are independent. Consider two sets of the form $A_i \times \{x\}$ and $A_j \times \{y\}$. If $i = j$, $x \neq y$ then for $a, b \in A_i$, $a \neq b$, it follows that $ax \sim by$. If $i \neq j$ then there exists $a \in A_i$ and $b \in A_j$ such that $a \sim b$ so again $ax \sim by$. Thus the union of two sets such sets is not an independent set.

Similar arguments show that the same is true for any two sets of the form $\{a\} \times Y_i$ and $\{b\} \times Y_j$.

Consider then a set $A_i \times \{x\}$ and $\{a\} \times Y_j$. If either $a \in N(A_i)$ or $x \in N(Y_j)$ then the union of these two sets is not independent. If $a \notin N(A_i)$ and $x \notin N(Y_j)$ then for any $b \in A_i$ and $y \in Y_j$ we have $bx \sim ay$.

Thus, the sets $A_i \times \{x\}$, $1 \leq i \leq k$, $x \in V(H)$ and $\{a\} \times Y_j$, $a \in V(G)$, $1 \leq j \leq f$ form an independence partition of $G \,\square^{c} H$. ■

Theorem 3.9. *Suppose that G and H are graphs that have maximum-matchings with g and h edges respectively. Then*

$$d(G \,\square^{c} H) \geq \Big(\Big\lfloor \frac{|V(G)|}{2} \Big\rfloor + g\Big)\Big(\Big\lfloor \frac{|V(H)|}{2} \Big\rfloor + h\Big) - 2gh.$$

In particular, $d(G \,\square^{c} H) \geq |V(G)||V(H)|/4$.

Proof. Let F and W be maximum matchings of G and H respectively. Then $G - F$ and $H - W$ are independent sets. In each of these independent sets partition the vertices into pairs. If either set is of odd cardinality then form a single group of size three in that set. The following are all dominating sets:

1. If both $(a, b) \in F$ and $(x, y) \in W$ then take the sets $\{ax, by\}$ and $\{ay, bx\}$.

2. If $(a, b) \in F$ and x, y are paired in $H - W$ then take the sets $\{ax, by\}$ and $\{ay, bx\}$; if x, y, z is the group of size three then take $\{ax, by, az\}$ and $\{ay, bx, bz\}$.

3. If a, b are paired in $G - F$ and $xy \in W$ then take the sets $\{ax, by\}$ and $\{ay, bx\}$; if a, b, c is the group of size three then take $\{ax, by, cx\}$ and $\{ay, bx, cy\}$.

4. For a pair or triple, A, grouped in $G - F$ and a pair or triple, B grouped in $H - W$ take the set $A \times B$.

These sets partition $V(G \times H)$ so that if $g = |F|$ and $h = |W|$ then we have

$$d(G \,\Box^c H) \geq 2gh + 2g\Big\lfloor \frac{|V(H)| - 2h}{2} \Big\rfloor + 2h\Big\lfloor \frac{|V(G)| - 2g}{2} \Big\rfloor$$

$$+\Big\lfloor \frac{|V(H)| - 2h}{2} \Big\rfloor \Big\lfloor \frac{|V(G)| - 2g}{2} \Big\rfloor = (\Big\lfloor \frac{|V(G)|}{2} \Big\rfloor + g)(\Big\lfloor \frac{|V(H)|}{2} \Big\rfloor + h) - 2gh.$$

This expression is always at least $|V(G)||V(H)|/4$. ■

3.5. Categorical complement

Theorem 3.10. *For all graphs G and H, $i(G \times^c H) = \min\{i(G), i(H)\}$ and $\beta(G \times^c H) = \min\{\beta(G), \beta(H)\}$.*

Proof. Let I be a maximal independent set of $G \times^c H$. From Lemma 2.4 both $\prod_G(I)$ and $\prod_H(I)$ are independent and at least one of them is maximal independent. It follows that

$$\min\{\beta(G), \beta(H)\} \geq |I| \geq \min\{i(G), i(H)\}.$$

Let $X = \{g_1, g_2, \ldots, g_s\}$ and $Y = \{h_1, h_2, \ldots, h_r\}$ be maximal independent sets of G and H respectively, with $k = \min\{s, r\}$. Let $M = \{g_1h_1, \ldots, g_kh_k\}$. M is an independent set of $G \times^c H$. Also, it is easily seen that if $ax \in V(G \times^c H)$, then $ax \in N[M]$ and so M is a dominating set. Therefore, if $|X| = i(G)$ and $|Y| = i(H)$ then $i(G \times^c H) \leq |M| = \min\{i(G), i(H)\}$, that is, $i(G \times^c H) = \min\{i(G), i(H)\}$. Similarly, if $|X| = \beta(G)$ and $|Y| = \beta(H)$ then $\beta(G \times^c H) \geq |M| = \min\{\beta(G), \beta(H)\}$, and so $\beta(G \times^c H) = \min\{\beta(G), \beta(H)\}$. ■

Theorem 3.11. *For all graphs G and H,*

1. $\gamma(G \times^c H) = \min\{\gamma(G), \gamma(H)\}$.
2. $\Gamma(G \times^c H) = \max\{\Gamma(G), \Gamma(H)\}$.

Proof. Let D be a minimal dominating set of $G \times^c H$.

1. By Lemma 2.3(1) it follows that one of $\prod_G(D)$ or $\prod_H(D)$ is a dominating set hence $\gamma(G \times^c H) \geq \min\{\gamma(G), \gamma(H)\}$.

If E is a dominating set of G and $x \in H$ then $E \times \{x\}$ is a dominating set of $G \times^c H$. Consequently, $\gamma(G \times^c H) = \min\{\gamma(G), \gamma(H)\}$.

2. By Lemma 2.3(1), we may suppose that $\prod_G(D)$ is a dominating set. It is minimal since if any vertex is redundant in $\prod_G(D)$ its pre-image is also redundant in D. In addition, no two vertices of D project to the same vertex in G, since if they did both would have the same closed neighborhood

so that at least one would be redundant in D. Hence, $\Gamma(G \times^c H) \leq \max\{\Gamma(G), \Gamma(H)\}$.

Let E be a dominating set of G. For each $a \in E$ let $p_a \in I(E, a)$. Choose $x, y \in V(H)$, $x \perp y$. Then $E \times \{x\}$ is a dominating set of $G \times^c H$ and $p_a y$ is a private neighbor for ax. Consequently, $\Gamma(G \times^c H) = \max\{\Gamma(G), \Gamma(H)\}$.

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