STOCHASTIC DIFFERENTIAL INCLUSIONS OF LANGEVIN TYPE ON RIEMANNIAN MANIFOLDS*

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Abstract

We introduce and investigate a set-valued analogue of classical Langevin equation on a Riemannian manifold that may arise as a description of some physical processes (e.g., the motion of the physical Brownian particle) on non-linear configuration space under discontinuous forces or forces with control. Several existence theorems are proved.

Keywords: stochastic differential inclusions, Langevin equation, Riemannian manifolds.

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1. Introduction and physical motivations

In this paper, we introduce and investigate a certain type of second order stochastic differential inclusions on Riemannian manifolds. They may be interpreted as the laws of motion for mechanical systems whose configuration spaces are Riemannian manifolds, while the force fields are set-valued and contain both deterministic and random components such that neither of them can be neglected and the random force is a transformed white noise.

The set-valued force evidently arises in a system with control or may be obtained from a discontinuous force (for instance, the dry friction is

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considered or the motion takes place in a complicated medium, etc.). Recall that if the force is discontinuous there are well-known methods of transition to a set-valued force (for stochastic differential equations the pioneering paper was probably [4]). Examples of systems having discontinuous forces with random components of the above-mentioned sort are rather usual in physics, e.g., they describe the motion of the physical Brownian particle in a complicated medium. The use of Riemannian manifolds allows one to cover the mechanics on non-linear configuration spaces.

If the forces of above-mentioned type are single-valued, the motion of such a system is described by the Langevin equation. We introduce the term Langevin inclusion for the case of set-valued force.

Below we use the following notations and assumptions. We consider a finite-dimensional manifold \( M \) as the configuration space of our system. The tangent space at the point \( m \) to \( M \) is denoted by \( T_m M \). The Riemannian scalar product in \( T_m M \) is denoted by \( < \cdot , \cdot >_m \) or simply \( < \cdot , \cdot > \) if it does not yield confusion (recall that \( < \cdot , \cdot >_m \) is smooth in \( m \in M \) and the total family of those scalar products is called a Riemannian metric). The norm in the tangent space, generated by the above scalar product, will be denoted by the usual symbol \( \| \cdot \| \).

We assume the Riemannian manifold \( M \) to be complete. The mechanical meaning of this assumption is that a free particle on the configuration space \( M \) does not go to infinity in finite time. The Riemannian metric enables us to identify differential 1-forms and vector fields on \( M \) and henceforth we regard the force field as a vector field.

Let \( F(t, m, X) \) be the force vector field and \( A(t, m, X) \) a \((1,1)\)-tensor field on \( M \) (both maybe set-valued). In other words, for every \( t \in [0, l] \), \( m \in M \), and \( X \in T_m M \), we have a (set-valued) vector \( F(t, m, X) \in T_m M \) and a (set-valued) linear operator \( A(t, m, X) : T_m M \to T_m M \). Making use of the construction given in Chapter 15 of [10], fix a Wiener process \( w \) in \( \mathbb{R}^n \) and realize it in the tangent spaces to \( M \), i.e., send \( w(t) \) into all tangent spaces by a certain field of operators \( O_m : \mathbb{R}^n \to T_m M \) orthonormal with respect to standard scalar product in \( \mathbb{R}^n \) and Riemannian scalar product in \( T_m M \). Denote by \( \dot{w} \) the corresponding Itô white noise in tangent spaces.

Then the Langevin equation (inclusion) describes the evolution of a system with the force field:

\[
F(t, m, X) + A(t, m, X)\dot{w}.
\]
Since the white noise is a distribution-valued process, the direct investigation of systems with force (1) is a complicated problem.

In Section 2, we present the mathematically well-posed construction of the Langevin inclusion on Riemannian manifolds in integral form (based on Riemannian parallel translation) that allows us to avoid the use of distribution-valued processes.

In Section 3, we prove a general result on the existence of a weak solution of Langevin inclusion with \( A \) having continuous non-degenerate selection and \( F \) having an \( \varepsilon \)-approximation for any \( \varepsilon > 0 \) (a much more general case than, e.g., upper semicontinuous \( F \) with bounded closed convex images) under the assumption that \( F \) and \( A \) satisfy Itô condition with respect to velocities. It should be pointed out that we use some features of stochastic equations to construct here an a.s. selection of \( F \) as a limit of \( \varepsilon \)-approximations converging in some special probabilistic sense.

The case when \( A \) has no selection of the above sort requires additional assumptions. As an example we prove a theorem on the existence of strong solution for \( A \) with one-dimensional image when set-valued \( F \) and \( A \) are Borel measurable, uniformly bounded, dissipative and maximal. Here we reduce the problem to a certain theorem on first order stochastic differential inclusions in linear spaces from [15].

The main technical trick in both results is transition to the so-called velocity hodograph of the system that is described by a stochastic differential inclusion (equation) in a single tangent (i.e., linear) space to the manifold.

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\section{Mathematical description}

Our first goal is to give a rigorous mathematical meaning to the Langevin inclusion in analogy with that for the Langevin equation of [8] – [10], i.e., without using distributions. In doing so, we use an integral form (natural for stochastic equations) employing integrals with the Riemannian parallel translation (see detailed description in [11], [12]). Everywhere below we use the parallel translation with respect to the Levi-Civitá connection that is given along a piece-wise \( C^1 \)-curve on the Riemannian manifolds (see, e.g., [2]). Recall that this operation preserves the Riemannian scalar product.

Let \( m_0 \in M, I = [0, l] \) and let \( v: I \to T_{m_0} M \) be a continuous curve.
Theorem 1. There exists a unique $C^1$-curve $\gamma: I \to M$ such that $\gamma(0) = m_0$ and the tangent vector $\dot{\gamma}(t)$ is parallel to the vector $v(t) \in T_{m_0}M$ for every $t \in I$.

Indeed, the curve $\gamma$ is represented as $\gamma(t) = \delta^{-1}\left(\int_0^t v(\tau) \, d\tau\right)$ where $\delta$ is Cartan’s development and $\delta^{-1}$ is its inverse map developing $C^1$-curves from $T_{m_0}M$ into $M$ (see, e.g., [2] for details).

In what follows, we denote by $S_v(\cdot)$ the curve $\gamma$ constructed as above beginning with $v$.

Consider the Banach space $C^0(I, T_{m_0}M)$ of continuous maps from $I$ to $T_{m_0}M$ and the Banach manifold $C^1(I, M)$ of $C^1$-smooth maps from $I$ to $M$. As follows from Theorem 1, the operator $S: C^0(I, T_{m_0}M) \to C^1(I, M)$ is well-posed. If $M$ is the Euclidean space, $S_v$ is a primitive of $v$.

It is easy to see that $S$ is a homeomorphism between $C^0(I, T_{m_0}M)$ and its image $C^1_{m_0}(I, M)$ in $C^1(I, M)$, where the manifold $C^1_{m_0}(I, M)$ consists of all $C^1$-curves $\gamma$ with $\gamma(0) = m_0$.

Let $\gamma(t), t \in I, \gamma(0) = m_0$, be a $C^1$-curve in $M$ and $\alpha(t, m, X)$ be a single-valued deterministic force field (i.e., without stochastic component). Denote by $\Gamma_{\alpha}(t, m(t), \dot{m}(t))$ the curve in $T_{m_0}M$ such that the vector $\Gamma_{\alpha}(t, m(t), \dot{m}(t))$ is parallel to $\alpha(t, m(t), \dot{m}(t))$ along $m(\cdot)$ for every $t$ (i.e., $\Gamma_{\alpha}(t, m(t), \dot{m}(t))$ is obtained by parallel translation of vectors $\alpha(t, m(t), \dot{m}(t))$ along $\gamma(\cdot)$ at $T_{m_0}M$).

Specify a vector $C$ in $T_{m(0)}M$ and consider the integral equation

$$m(t) = S\left(\int_0^t \Gamma_{\alpha}(\tau, m(\tau), \dot{m}(\tau)) \, d\tau + C\right)$$

on $I = [0, l]$. It is shown in [10] (see also [11] and [12]) that (2) is the integral form of the second Newton’s law, i.e., its solution is the trajectory of mechanical system with force $\alpha$ having the initial conditions $\gamma(0) = m_0$ and $\dot{\gamma}(0) = C$.

Let $m(t), t \in I$, be a trajectory of the mechanical system, i.e., a solution to (2).

**Definition 2.** The velocity hodograph of the trajectory $m(t)$ is the curve $v: I \to T_{m(0)}M$ such that $v(t)$ is parallel to $\dot{m}(t)$ along $m(\cdot)$.

It is not hard to see that the velocity hodograph of a solution to (2) satisfies the equation
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\( v(t) = \int_0^t \Gamma \alpha \left( \tau, S v(\tau), \frac{d}{d\tau} S v(\tau) \right) d\tau + C. \)

It is obvious that if \( v \) is a solution to (3), then \( S v \) is a solution to (2), i.e., a trajectory of the mechanical system.

Substituting force (1) into (2) for single-valued \( F \) and \( A \) we obtain that the Langevin equation is the relation of the form

\[
\xi(t) = S \left( \int_0^t \Gamma F(\tau, \xi(\tau), \dot{\xi}(\tau)) d\tau + \int_0^t \Gamma A(\tau, \xi(\tau), \dot{\xi}(\tau)) d\omega(\tau) + C \right)
\]

and its velocity hodograph equation takes the form

\[
v(t) = \int_0^t \Gamma F(\tau, S v(\tau), \frac{d}{d\tau} S v(\tau)) d\tau + \int_0^t \Gamma A(\tau, S v(\tau), \frac{d}{d\tau} S v(\tau)) d\omega(\tau) + C
\]

(see [10] – [12] for details). Notice that by its construction the process \( \xi(t) \) a.s. has \( C^1 \)-smooth sample paths so that the derivative \( \dot{\xi}(t) \) is well-posed.

Since we deal with \( \xi(t) \) starting at the non-random point \( m_0 \) and \( v(t) \) lies in \( T_{m_0}M \), the realization of \( w(t) \) in tangent spaces to \( M \), mentioned above, is done as follows. We send \( w(t) \) from \( R^n \) into \( T_{m_0}M \) by an arbitrary operator that is orthonormal with respect to the standard scalar product in \( R^n \) and the scalar product in \( T_{m_0}M \) generated by the Riemannian metric tensor. Then we send \( w(t) \) into other points along \( \xi(\cdot) \) by Riemannian parallel translation. Everywhere below we use this realization of the Wiener process in tangent spaces to \( M \).

Now let \( F \) and \( A \) be set-valued. Introduce the Langevin inclusion as formal notation in the form

\[
\xi(t) \in S \left( \int_0^t \Gamma F(\tau, \xi(\tau), \dot{\xi}(\tau)) d\tau + \int_0^t \Gamma A(\tau, \xi(\tau), \dot{\xi}(\tau)) d\omega(\tau) + C \right).
\]

(6) It can be understood in the sense that \( \xi(t) \) belongs to the set obtained by integration of all integrable selections in the right-hand side, but its precise meaning will be given in the definitions below.
Definition 3. We say that (6) has a weak solution on \([0, l] \subset \mathbb{R}\) with initial conditions \(\xi(0) = m_0, \dot{\xi}(0) = C\) if there exist: a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), an a.s. \(C^1\) stochastic process \(\xi(t)\) defined on \((\Omega, \mathcal{F}, \mathbb{P})\) and valued in \(M\) with initial condition \(\xi(0) = m_0\) and \(\dot{\xi}(0) = C\), a Wiener process \(w(t)\) in \(\mathbb{R}^n\), defined on \((\Omega, \mathcal{F}, \mathbb{P})\) and adapted to \(\xi(t)\), a single-valued vector field \(f(t, m, X)\) on \(M\) and a single-valued \((1,1)\)-tensor field \(a(t, m, X)\) such that

(i) for all \(t\) the random vector \(f(t, \xi(t), \dot{\xi}(t))\) belongs to \(F(t, \xi(t), \dot{\xi}(t))\) \(\mathbb{P}\)-almost surely (a.s.);

(ii) for all \(t\) the random tensor \(a(t, \xi(t), \dot{\xi}(t))\) belongs to \(A(t, \xi(t), \dot{\xi}(t))\) \(\mathbb{P}\)-a.s.;

(iii) the integrals \(\int_0^t \Gamma f(\tau, \xi(\tau), \dot{\xi}(\tau))d\tau\) and \(\int_0^t \Gamma a(\tau, \xi(\tau), \dot{\xi}(\tau))dw(\tau)\) are well-posed for \(\xi(t), w(t), f\) and \(a\) and the equality

\[
\xi(t) = \mathcal{S}\left(\int_0^t \Gamma f(\tau, \xi(\tau), \dot{\xi}(\tau))d\tau + \int_0^t \Gamma a(\tau, \xi(\tau), \dot{\xi}(\tau))dw(\tau) + C\right),
\]

holds for all \(t \in [0, l]\) \(\mathbb{P}\)-a.s.

Definition 4. We say that (6) has a strong solution on \([0, l] \subset \mathbb{R}\) with initial conditions \(\xi(0) = m_0, \dot{\xi}(0) = C\) if there exist: an a.s. \(C^1\) stochastic process \(\xi(t)\) defined on \((\Omega, \mathcal{F}, \mathbb{P})\), non-anticipating with respect to \(w(t)\) and valued in \(M\) with initial condition \(\xi(0) = m_0\) and \(\dot{\xi}(0) = C\), a single-valued vector field \(f(t, m, X)\) on \(M\) and a single-valued \((1,1)\)-tensor field \(a(t, m, X)\) such that

(i) for all \(t\) the random vector \(f(t, \xi(t), \dot{\xi}(t))\) belongs to \(F(t, \xi(t), \dot{\xi}(t))\) \(\mathbb{P}\)-a.s.;

(ii) for all \(t\) the random tensor \(a(t, \xi(t), \dot{\xi}(t))\) belongs to \(A(t, \xi(t), \dot{\xi}(t))\) \(\mathbb{P}\)-a.s.;

(iii) the integrals \(\int_0^t \Gamma f(\tau, \xi(\tau), \dot{\xi}(\tau))d\tau\) and \(\int_0^t \Gamma a(\tau, \xi(\tau), \dot{\xi}(\tau))dw(\tau)\) are well-posed for \(\xi(t), w(t), f\) and \(a\) and \(\mathbb{P}\)-a.s. (7) holds for all \(t \in [0, l]\).

Remark 5. The condition that \(f(t, \xi(t), \dot{\xi}(t))\) belongs to \(F(t, \xi(t), \dot{\xi}(t))\) \(\mathbb{P}\)-almost surely is less restrictive than the hypothesis that \(f\) is a selection of \(F\). In particular, if \(f\) is a selection of \(F\), then the former relation is evidently satisfied. The same is valid for \(a\) and \(A\).
One can easily prove that \( \xi(t) \) as above satisfies (7) if and only if its velocity hodograph \( v(t) \) (i.e., \( \xi(t) = Sv(t) \)) satisfies the velocity hodograph equation of the form

\[
v(t) = \int_0^t \Gamma f \left( \tau, Sv(\tau), \frac{d}{d\tau}Sv(\tau) \right) d\tau
\]

\[
+ \int_0^t \Gamma a \left( \tau, Sv(\tau), \frac{d}{d\tau}Sv(\tau) \right) dw(\tau) + C
\]

that is an equation of diffusion type in the tangent (i.e., linear) space at \( m_0 \) and so it is more convenient for investigation. Below, we shall find \( f \) and \( a \) as in Definitions 3 and 4 and corresponding \( v(t) \), being a solution to (8) in weak or strong sense, and then obtain \( \xi(t) = Sv(t) \) satisfying (7).

### 3. Existence of solutions

If both \( F \) and \( A \) have continuous selections satisfying Itô condition (see (9) below), the existence of weak solution trivially follows from that for Langevin equation obtained in \([8] - [10]\) (see also \([11]\) and \([12]\)). If it is not the case the existence problem for Langevin inclusions requires special constructions.

First, we need the definition of \( \varepsilon \)-approximation that in the case under consideration for \( t \in [0, l] \) looks as follows.

**Definition 6.** A continuous single-valued vector field \( f_\varepsilon(t, m, X) \) is called \( \varepsilon \)-approximation of the set-valued vector field \( F(t, m, X) \) on \( M \) if its graph \( (t, m, X, f(t, m, X)) \) lies in the \( \varepsilon \)-neighbourhood of \( (t, m, X, F(t, m, X)) \) (the graph of \( F \)) in \([0, l] \times TM \oplus TM \) where \( \oplus \) denotes the Whitney sum.

Among set-valued vector fields having an \( \varepsilon \)-approximation for any \( \varepsilon > 0 \) we can mention the following classes:
- upper semicontinuous vector fields with bounded closed convex images;
- upper semicontinuous vector fields with bounded closed images, aspherical in all dimensions from 1 to \( \dim M \) (see \([3]\)). This class of set-valued maps was introduced by A.D. Myshkis in 1954 \([16]\). In \([3]\) and \([7]\) topological characteristics of the Lefschetz number and index types were constructed for them. Later (in the 80-ies) this class of maps was rediscovered and called maps with \( uv^k \)-property for \( k = 1, \ldots, \dim M \) (see, e.g., \([14]\) for details).
Below all norms of vectors are generated by the Riemannian metric by usual formula $\|Y\| = \sqrt{\langle Y,Y \rangle}$. The norm of a set $F(t,m,X)$ is defined as $\|F(t,m,X)\| = \sup_{y \in F(t,m,X)} \|y\|$. The norm $\|A\|$ is the norm of linear operator $A$.

We say that $F$ and $A$ satisfy the Itô condition if they have linear growth in velocities, i.e., there exists a certain $\Theta > 0$ such that the following inequality

$$\|F(t,m,X)\| + \|A(t,m,X)\| < \Theta(1 + \|X\|) \tag{9}$$

holds.

Below we shall need some technical propositions. Denote by $\Omega$ the Banach space $C^0([0,l],\mathbb{R}^n)$ of continuous curves $x: [0,l] \rightarrow \mathbb{R}^n$ with usual uniform norm, and by $\mathcal{F}$ the $\sigma$-algebra generated by cylindrical sets on $\Omega$. The following statement is proved in Lemma III.2.1 of [6].

**Lemma 7.** For a solution of diffusion type stochastic differential equation

$$d\xi(t) = f(t,\xi(\cdot))dt + a(t,\xi(\cdot))dw(t)$$

in $\mathbb{R}^n$ with $t \in [0,l]$, such that $\|f(t,x(\cdot))\| + \|a(t,x(\cdot))\| < Q(1 + \|x(\cdot)\|)$ the inequality $E(\sup_{t \in [0,l]} \|\xi(t)\|^2) < c$ holds, where $c$ depends only on $Q$ and $l$.

Consider a sequence of equations $d\xi_k(t) = f_k(t,\xi(\cdot))dt + a_k(t,\xi(\cdot))dw(t)$ satisfying the hypothesis of Lemma 7 with the same $Q$ and $l$ for all $k$. Let there exist (weak) solutions $\xi_k$ to the above equations. Denote by $\mu_k$ the measures on $(\Omega,\mathcal{F})$ corresponding to $\xi_k$ and suppose that $\mu_k$ weakly converge to a certain measure $\mu$. Denote by $\xi$ the coordinate process $\xi(t,x(\cdot)) = x(t)$ on $(\Omega,\mathcal{F},\mu)$. Now introduce the measures $\nu_k$ by the relations $d\nu_k = (1 + \|x(\cdot)\|)d\mu_k$.

**Lemma 8.** The measures $\nu_k$ weakly converge to the measure $\nu$ defined by the relation $d\nu = (1 + \|x(\cdot)\|)d\mu$.

**Proof.** Specify an arbitrary bounded continuous function $f: \Omega \rightarrow \mathbb{R}$. From Lemma 7 it follows that the random variables $f(\xi_k)(1 + \|\xi_k\|)$ are uniformly integrable. Then, since $f(x(\cdot))(1 + \|x(\cdot)\|)$ is a continuous map from $\Omega$ to $\mathbb{R}$, the weak convergence of $\mu_k$ to $\mu$ yields $E(f(\xi_k)(1 + \|\xi_k\|)) \rightarrow E(f(\xi)(1 + \|\xi\|))$ as $k \rightarrow \infty$ (see [1]). Thus
\[
\lim_{k \to \infty} \int_{\Omega} f(x(\cdot))(1 + \|x(\cdot)\|)d\mu_k = \int_{\Omega} f(x(\cdot))(1 + \|x(\cdot)\|)d\mu
\]
and so \( \lim_{k \to \infty} \int_{\Omega} f(x(\cdot))d\nu_k = \int_{\Omega} f(x(\cdot))d\nu. \)

**Theorem 9.** Let the set-valued vector field \( F(t, m, X) \) be such that for every \( \varepsilon > 0 \) there exists an \( \varepsilon \)-approximation \( f_\varepsilon \) for \( F \). Let the linear operator \( A(t, m, X) : T_m M \to T_m M \) be single-valued, continuous jointly in \( (t, m, X) \) and non-degenerate for all \( t \in \mathbb{R}, (m, X) \in TM \). Let also \( F \) and \( A \) satisfy the Itô condition \((9)\) for a certain \( \Theta \).

Then for any \( m_0 \in M \) and \( C \in T_{m_0} M \) the Langevin inclusion has a weak solution with initial conditions \( \xi(0) = m_0, \dot{\xi}(0) = C \), well-posed for all \( t \in [0, \infty) \).

**Proof.** Since, \( A \) is single-valued and continuous we can define \( a \) from Definition 3 as \( a(t, m, X) = A(t, m, X) \).

Specify \( l > 0 \). Denote by \( \mathcal{B} \) the Borel \( \sigma \)-algebra on it and by \( \lambda \) an ordinary Lebesgue measure on \([0, l]\). Consider \( \Omega = C^0([0, l], T_{m_0} M) \), the Banach space of continuous curves \( x : [0, l] \to T_{m_0} M \) with usual uniform norm, and \( \mathcal{F} \), the \( \sigma \)-algebra generated by cylindrical sets on \( \Omega \). We shall use several measures on \((\Omega, \mathcal{F})\) and on the product space \([0, l] \times \Omega\) we shall introduce the corresponding product measures.

Take a sequence \( \varepsilon_i \to 0 \) and construct a sequence \( f_i(t, m, X) \) of \( \varepsilon_i \)-approximations of \( F(t, m, X) \).

Taking into account Definition 6 and inequality \((9)\) we are able to construct \( f_i(t, m, X) \) so that \( \|f_i(t, m, X)\| + \|a(t, m, X)\| < Q(1 + \|X\|) \) for a certain \( Q > \Theta \) and for all \( i \).

Pass from the sequence \( f_i(t, m, X) \) to the sequence \( \tilde{f}_i : [0, l] \times \Omega \to TM \), where \( \tilde{f}_i(t, x(\cdot)) = f_i(t, Sx(t), \tfrac{d}{dt} Sx(t)) \). Introduce also \( \tilde{a}(t, x(\cdot)) = a(t, Sx(t), \tfrac{d}{dt} Sx(t)) \).

Consider the maps \( \Gamma \tilde{f}_i(t, x(\cdot)) \) from \([0, l] \times \Omega \) into \( T_{m_0} M \) and \( \Gamma \tilde{a}(t, x(\cdot)) \) from \([0, l] \times \Omega \) into linear endomorphisms on \( T_{m_0} M \).

Since \( \tfrac{d}{dt} Sx(t) \) is by the construction parallel to \( x(t) \) along \( Sx(\cdot) \) and the parallel translation preserves the norms, we get

\[
\|\Gamma \tilde{f}_i(t, x(\cdot))\| + \|\Gamma \tilde{a}(t, x(\cdot))\| < Q(1 + \|x(\cdot)\|). \]
By the construction, $\tilde{\Gamma}(t,x(\cdot))$ is continuous on $[0,l] \times \Omega$ (this follows from the continuity of $\Gamma$, see [10] – [12]) and measurable with respect to the $\sigma$-subalgebra $P_t$ in $\mathcal{F}$ generated by cylindrical sets with bases over $[0,t]$. Since it also satisfies (10), there exists a weak solution $v_0(t)$ to the equation

$$v_0(t) = \int_0^t \tilde{\Gamma}(\tau,v_0(\cdot))dw(\tau) + C$$

(see Theorem III. 2.4 of [6]). Denote by $\mu_0$ the measure on $(\Omega,\mathcal{F})$ corresponding to $v_0$. Recall that $v_0(t)$ is represented as the coordinate process $v_0(t,x(\cdot)) = x(t)$ on the probability space $(\Omega,\mathcal{F},\mu_0)$. Thus from Lemma 7 it follows that

$$\int_{[0,l] \times \Omega} \|x(\cdot)\|^2 d\lambda \times d\mu_0 < lc.$$

Taking into account the above inequality together with (10) we get that the maps $\tilde{\Gamma_i}(t,x(\cdot))$ from $[0,l] \times \Omega$ into $T_{\mu_0}M$ belong to $L^2(([0,l] \times \Omega,\mathcal{B} \times \mathcal{F},\lambda \times \mu_0),T_{\mu_0}M)$ and are uniformly bounded there. Thus they form a weakly compact set and so there exists a subsequence weakly converging in $L^2(([0,l] \times \Omega,\mathcal{B} \times \mathcal{F},\lambda \times \mu_0),T_{\mu_0}M)$ to a certain map $\tilde{f}(t,x(\cdot))$. To avoid changing notations, let $\tilde{\Gamma_i}(t,x(\cdot))$ denote the last subsequence. It follows from Mazur’s lemma (see, e.g., [19]) that there exists a sequence of finite convex combinations $f^*_k(t,x(\cdot))$ of $\tilde{\Gamma_i}(t,x(\cdot))$ that converges to $\tilde{f}(t,x(\cdot))$ strongly in $L^2(([0,l] \times \Omega,\mathcal{B} \times \mathcal{F},\lambda \times \mu_0),T_{\mu_0}M)$. Convex combinations $f^*_k(t,x(\cdot))$ take the form

$$f^*_k(t,x(\cdot)) = \sum_{i=j(k)}^{n(k)} \alpha_i \Gamma_i(t,x(\cdot)),$$

$$\alpha_i \geq 0, i = j(k), \ldots, n(k), \quad \sum_{i=j(k)}^{n(k)} \alpha_i = 1,$$

where $j(k) \to \infty$ as $k \to \infty$. Thus this sequence converges in probability and there exists a subsequence that converges almost surely (see, e.g., [18]). As usual we do not change the notation and say that $f^*_k(t,x(\cdot))$ converges to $\tilde{f}(t,x(\cdot))$ almost surely (a.s.) with respect to $\lambda \times \mu_0$ on $[0,l] \times \Omega$.

By the construction all $f^*_k(t,x(\cdot))$ are continuous on $[0,l] \times \Omega$ (as well as above this follows from the continuity of $\Gamma$) and measurable with respect
functions (1 + \parallel \cdot \parallel) converge to $\nu$ for all $v$ subsequence. Denote by $\mu$ a certain probability measure is weakly compact and so there exists a subsequence converging weakly to $\nu$ that for any $v$ satisfy (10) with the same $v$ sequence and so $f$ all measures $K$ limit of continuous functions on each $\delta > \lambda$.

By routine method (see, e.g., [6]), since all $f_k^*(t, x(\cdot))$ and $\Gamma \tilde{u}(t, x(\cdot))$ satisfy (10) with the same $Q$, one can show that the set of measures $\{\mu_k\}$ is weakly compact and so there exists a subsequence converging weakly to a certain probability measure $\mu$ on $(\Omega, \mathcal{F})$. For the sake of convenience we do not change the notations and say that $\mu_k$ itself is that converging subsequence. Denote by $v(t)$ the coordinate process on $(\Omega, \mathcal{F}, \mu)$. Recall that for any $\delta > 0$ there exists a compact $K_\delta \subset \Omega$ such that $\mu_k(K_\delta) > 1 - \delta$ for all $k$ from 0 to $\infty$ where $\mu_\infty = \mu$.

We introduce measures $\nu_k$ on $(\Omega, \mathcal{F})$ by the relations $d\nu_k = (1 + \parallel x(\cdot) \parallel) d\mu_k$. By Lemma 8 they weakly converge to $\nu$ defined by the relation $d\nu = (1 + \parallel x(\cdot) \parallel) d\mu$.

As $f_k^*(t, x(\cdot))$ converge to $\tilde{f}(t, x(\cdot))$ a.s. with respect to all $\lambda \times \mu_i$, the functions $f_k^*(t, x(\cdot))$ converge to $\tilde{f}(t, x(\cdot))$ a.s. with respect to all $\lambda \times \nu_i$. Specify $\delta > 0$. By Egorov’s theorem (see, e.g., [13]) for any $i$ there exists a subset $\tilde{K}_i \subset [0, l] \times \Omega$ such that $(\lambda \times \nu_i)(\tilde{K}_i) > (\lambda \times \nu_i)([0, l] \times \Omega) - \delta$, and the sequence $f_k^*(t, x(\cdot))$ converges to $\tilde{f}(t, x(\cdot))$ uniformly on $\tilde{K}_i$. Introduce $(\tilde{K}_\delta = \bigcup_{i=0}^{\infty} \tilde{K}_i)$. The sequence $f_k^*(t, x(\cdot))$ converges to $\tilde{f}(t, x(\cdot))$ uniformly on $\tilde{K}_\delta$ and $(\lambda \times \nu_i)(\tilde{K}_\delta) > (\lambda \times \nu_i)([0, l] \times \Omega) - \delta$ for all $i = 0, \ldots, \infty$.

Notice that $\tilde{f}(t, x(\cdot))$ is continuous on a set of full measure $\lambda \times \nu$ on $[0, l] \times \Omega$. Indeed, consider a sequence $\delta_i \to 0$ and the corresponding sequence $\tilde{K}_\delta$ from Egorov’s theorem. By the above construction $\tilde{f}(t, x(\cdot))$ is a uniform limit of continuous functions on each $\tilde{K}_\delta$. Thus it is continuous on each $\tilde{K}_\delta$. }
and so on every finite union $\bigcup_{i=1}^{n} \tilde{K}_{\delta_{i}}$. Evidently, $\lim_{n \to \infty}(\lambda \times \nu)(\bigcup_{i=1}^{n} \tilde{K}_{\delta_{i}}) = (\lambda \times \nu)([0, l] \times \Omega)$.

Hence $\tilde{f}(t, x(\cdot))$ is continuous on a set of full measure $\lambda \times \nu$ on $[0, l] \times \Omega$.

Let $g_{t}(x(\cdot))$ be a bounded (say, $|g_{t}(x(\cdot))| < \Xi$ for all $x(\cdot) \in \Omega$) and continuous $\mathcal{P}_{t}$-measurable function on $\Omega$. Notice that

$$\int_{\Omega} (x(t + h) - x(t) - \int_{t}^{t+h} \tilde{f}_{k}(\tau, x(\cdot))d\tau)g_{t}(x(\cdot))d\mu_{k} = 0$$

since

$$\int_{\Omega} (x(t + h) - x(t))g_{t}(x(\cdot))d\mu_{k} = E[(v_{k}(t + h) - v_{k}(t))g_{t}(v_{k}(t))],$$

$$\int_{\Omega} \left( \int_{t}^{t+h} f^{*}_{k}(\tau, x(\cdot))d\tau \right)g_{t}(x(\cdot))d\mu_{k} = E \left[ \left( \int_{t}^{t+h} f^{*}_{k}(\tau, v_{k}(\tau))d\tau \right)g_{t}(v_{k}(t)) \right]$$

and $v_{k}(t)$ is a solution to (13).

Because of the above uniform convergence on $\tilde{K}_{\delta}$ for all $k$ and boundedness of $g_{t}$ we get that for $k$ large enough

$$\| \int_{K_{\delta} \cap \tilde{K}_{\delta}} \left( \int_{t}^{t+h} (f^{*}_{k}(\tau, x(\cdot)) - \tilde{f}(\tau, x(\cdot)))d\tau \right)g_{t}(x(\cdot))d\mu_{k} \|$$

$$= \| \int_{K_{\delta} \cap \tilde{K}_{\delta}} \left( \int_{t}^{t+h} \frac{f^{*}_{k}(\tau, x(\cdot)) - \tilde{f}(\tau, x(\cdot))}{1 + \|x(\cdot)\|}d\tau \right)g_{t}(x(\cdot))d\nu_{k} \| < \delta.$$  

Since $(\lambda \times \mu_{k})(K_{\delta} \cap \tilde{K}_{\delta}) > 1 - 2\delta$ for all $k$, $\| \frac{f^{*}_{k}(t, x(\cdot))}{1 + \|x(\cdot)\|} \| < Q$ for all $k = 0, 1, \ldots, \infty$ (i.e., $\tilde{f} = f^{*}_{\infty}$ is included) and $|g_{t}(x(\cdot))| < \Xi$ (see above), we get

$$\| \int_{\Omega \setminus K_{\delta} \cap \tilde{K}_{\delta}} \left( \int_{t}^{t+h} (f^{*}_{k}(\tau, x(\cdot)) - \tilde{f}(\tau, x(\cdot)))d\tau \right)g_{t}(x(\cdot))d\mu_{k} \|$$

$$= \| \int_{\Omega \setminus K_{\delta} \cap \tilde{K}_{\delta}} \left( \int_{t}^{t+h} \frac{f^{*}_{k}(\tau, x(\cdot)) - \tilde{f}(\tau, x(\cdot))}{1 + \|x(\cdot)\|}d\tau \right)g_{t}(x(\cdot))d\nu_{k} \| < 4Q\Xi\delta.$$
From the fact that $\delta$ is an arbitrary positive number it follows that
\[ \lim_{k \to \infty} \int_{\Omega} \left( \int_{t}^{t+h} f_k^*(\tau, x(\cdot))d\tau \right) g_\ell(\cdot) d\mu_k = 0. \]

\(\tilde{f}(t,x(\cdot))\) is $\lambda \times \nu$-a.s. continuous (see above) and bounded on $[0, l] \times \Omega$. Hence by Theorem VI.4.4 of [5] from the weak convergence of $\nu_k$ to $\nu$ it follows that
\[ \lim_{k \to \infty} \int_{\Omega} \left( \int_{t}^{t+h} \tilde{f}(\tau, x(\cdot))d\tau \right) g_\ell(\cdot) d\mu_k = \int_{\Omega} \left( \int_{t}^{t+h} \tilde{f}(\tau, x(\cdot))d\tau \right) g_\ell(\cdot) d\nu. \]

Thus
\[ \lim_{k \to \infty} \int_{\Omega} \left( x(t + h) - x(t) - \int_{t}^{t+h} \tilde{f}_k(\tau, x(\cdot))d\tau \right) g_\ell(\cdot) d\mu_k = \int_{\Omega} \left( x(t + h) - x(t) - \int_{t}^{t+h} \tilde{f}(\tau, x(\cdot))d\tau \right) g_\ell(\cdot) d\mu = 0. \]

Since the last equality holds for arbitrary bounded continuous $g_\ell(\cdot)$ that is $\mathcal{P}_t$-measurable, it means that $z(t) = x(t + h) - x(t) - \int_{t}^{t+h} \tilde{f}(\tau, x(\cdot))d\tau$ is a martingale with respect to $\mathcal{P}_t$ on the probability space $(\Omega, \mathcal{F}, \mu)$. By standard Girsanov technique one shows that there exists a Wiener process $w(t)$, adapted to $\mathcal{P}_t$ and the $\mathcal{P}_t$-nonanticipative coordinate process $v(t)$ on $(\Omega, \mathcal{F}, \mu)$, satisfying
\[ v(t) = \int_{0}^{t} \tilde{f}(\tau, v(\cdot)) d\tau + \int_{0}^{t} \Gamma \tilde{a}(\tau, v(\cdot))dw(\tau) + C. \]
of the map \((Sv(t), \frac{d}{dt}Sv(t))\) from \([0, l] \times \Omega\) to \(TM\). Consider the conditional expectation \(E(\Gamma_t \hat{f} | N_t)\). It is a routine fact of probability theory (see, e.g., [17]) that there exists a Borel vector field \(f(t, m, X)\) such that \(E(\Gamma_t \hat{f} | N_t) = f(t, Sv(t), \frac{d}{dt}Sv(t))\).

All maps \(f_i(t, Sv(t), \frac{d}{dt}Sv(t))\) are measurable with respect to \(N_t\) and so \(E(f_i(t, Sv(t), \frac{d}{dt}Sv(t)))|N_t) = f_i(t, Sv(t), \frac{d}{dt}Sv(t))\). Introduce \(f_k^*(t, m, X) = \sum_{i=j(k)}^{n(k)} \alpha_if_i(t, m, X)\), where \(\alpha_i\) are the same as in (12). Evidently \(f_k^*(t, m, X)\) are \(\tilde{\varepsilon}_k\)-approximations for some sequence \(\tilde{\varepsilon}_k \rightarrow 0\) and also

\[
E(f_k^*(t, Sv(t), \frac{d}{dt}Sv(t))|N_t) = f_k^*(t, Sv(t), \frac{d}{dt}Sv(t)).
\]

From the property of convergence under conditional expectation it follows that for above sequences of indices \(f_k^*(t, Sv(t), \frac{d}{dt}Sv(t)) \rightarrow f(t, Sv(t), \frac{d}{dt}Sv(t))\) in \(L^2\) and a.s., respectively. Hence \(f(t, Sv(t), \frac{d}{dt}Sv(t)) \in F(t, Sv(t), \frac{d}{dt}Sv(t))\) \(\mu\)-a.s. Otherwise it would contradict the fact that \(f(t, Sv(t), \frac{d}{dt}Sv(t))\) is an \(\tilde{\varepsilon}_k\)-approximation for any \(k\).

Introduce \(\Gamma f(t, x(\cdot)) = \Gamma f(t, Sv(t), \frac{d}{dt}Sv(t))\). Immediately from the construction it follows that \(\mu\)-a.s. \(\Gamma f(t, x(\cdot)) = \hat{f}(t, x(\cdot))\). Then (14) is transformed into

\[
v(t) = \int_0^t \Gamma f(\tau, Sv(t), \frac{d}{dt}Sv(t)) d\tau + \int_0^t \Gamma a(\tau, Sv(t), \frac{d}{dt}Sv(t)) dw(\tau) + C.
\]

Thus \(\xi(t) = Sv(t)\) satisfies (7) with the above \(f\), \(a\) and \(w(t)\). Since \(l\) is an arbitrary positive number, the Theorem follows.

**Corollary 10.** Let \(F\) be as in Theorem 9. Let \(A\) be set-valued and such that there exists a selection \(a(t, m, X)\) of \(A(t, m, X)\), continuous jointly in \((t, m, X)\) and non-degenerate for all \(t \in R\), \((m, X) \in TM\), and let (9) be fulfilled.

Then for any \(m_0 \in M\) and \(C \in T_{m_0}M\) the Langevin inclusion has a weak solution with initial conditions \(\xi(0) = m_0\), \(\xi(0) = C\), well-posed for all \(t \in [0, \infty)\).

The above existence results do not cover the case of the Langevin inclusion whose diffusion term \(A\) has no continuous non-degenerate selections. In this case the existence of solutions can be proved under some additional
conditions. Moreover, sometimes these new conditions yield the existence of strong solutions. We illustrate this by proving a result based on [15], where the existence of strong solution for a certain class of first order stochastic differential inclusions with degenerate diffusion term in linear spaces has been obtained. In some sense, this case is opposite to the one mentioned above: here the (set-valued) linear operator of diffusion term admits a one-dimensional image and so it cannot have non-degenerate selections.

In what follows we use $[0, l], B, \Omega, F$ and $P_t$ introduced in the proof of Theorem 9. By $B_t$ we denote the Borel $\sigma$-algebra on $[0, t]$ for $t \in [0, l]$.

Introduce the notation $\text{comp} \mathcal{Z}$ for the space of compact subsets in the metric space $\mathcal{Z}$. Thus, we say that the set-valued vector field $B(t, m, X)$ sends $[0, l] \times TM$ into $\text{comp} TM$ if for any $(t, m, X) \in [0, l] \times TM$ the image $B(t, m, X) \subset T_m M$ is compact.

Recall several definitions.

**Definition 11.** A single-valued map $\beta : [0, l] \times \Omega \rightarrow \mathbb{R}^n$ is called $\{P_t\}$-progressively measurable if for every $t$ it is measurable with respect to $B_t \times P_t$.

**Definition 12.** A set-valued map $B : [0, l] \times \Omega \rightarrow \text{comp} \mathbb{R}^n$ is called $\{P_t\}$-progressively measurable if $\{(t, \omega) \in [0, l] \times \Omega | B(t, \omega) \cap C \neq \emptyset \} \in B_t \times P_t$ for every closed set $C \subset \mathbb{R}^n$.

**Definition 13.** We say that a set-valued vector field $B : [0, l] \times TM \rightarrow \text{comp} TM$

(i) is dissipative if for all $t \in [0, l], \ m \in M, X, Y \in T_m M$ and $U \in B(t, m, X), V \in B(t, m, Y)$ the inequality $< X - Y, U - V > \leq 0$ holds.

(ii) is maximal if for $t, m, X, Y$ and $V$ as in (i) the inequality $< X - Y, U - V > \leq 0$ is equivalent to the assumption that $U \in B(t, m, X)$.

Denote by $w(t)$ a certain one-dimensional Wiener process. Let $F(t, m, X)$ and $G(t, m, X)$ be set-valued vector fields on $M$ as above. Then we can consider the stochastic differential inclusion of the Langevin type

$$\xi(t) \in \mathcal{S} \left( \int_0^t \Gamma F(\tau, \xi(\tau), \dot{\xi}(\tau)) d\tau + \int_0^t \Gamma G(\tau, \xi(\tau), \dot{\xi}(\tau)) dw(\tau) + C \right).$$

Inclusion (15) is a particular case of (6) since $\Gamma G(\tau, \xi(\tau), \dot{\xi}(\tau)) dw(\tau)$ can be represented as $\Gamma G(\tau, \xi(\tau), \dot{\xi}(\tau))(P dW(\tau))$ where $P$ is the orthogonal projection onto the linear span of vector $\Gamma G(\tau, \xi(\tau), \dot{\xi}(\tau))$. 
**Theorem 14.** Let the set-valued vector fields \( F(t, m, X) \) and \( G(t, m, X) \), \( F, G : [0, l] \times TM \to \text{comp} TM \), be Borel measurable, uniformly bounded, dissipative and maximal. Then there exists a strong solution to (15), well posed for \( t \in [0, l] \), with initial conditions \( \xi(0) = m_0 \) and \( \dot{\xi}(0) = C \) for any \( m_0 \in M \) and \( C \in T_{m_0}M \).

**Proof.** Let \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}) \) be a probability space admitting a one-dimensional Wiener process \( w(t) \). Denote by \( \mathcal{P}_t^w \) the \( \sigma \)-subalgebra of \( \tilde{\mathcal{F}} \) generated by all \( w(s) \) for \( 0 \leq s \leq t \) and completed by all sets of zero probability. Let \( Y : \tilde{\Omega} \to \Omega \) be a measurable map. From the properties of parallel translation and the assumed hypothesis one can easily derive that the coefficients

\[
\Gamma F(t, \omega, Y) = \Gamma F(t, SY(\omega)(t), \frac{d}{dt}SY(\omega)(t))
\]

and

\[
\Gamma G(t, \omega, Y) = \Gamma G(t, SY(\omega)(t), \frac{d}{dt}SY(\omega)(t))
\]

for \( \omega \in \tilde{\Omega} \) satisfy all conditions of Theorem 1 [15] and so on \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}) \) there exists a continuous \( \mathcal{P}_t^w \)-progressively measurable process \( v(t) \) \((v(0) = 0)\) in \( T_{m_0}M \) and \( L^2 \)-selections \( f(t, \omega) \) of \( \Gamma F(t, \omega, v) \) and \( g(t, \omega) \) of \( \Gamma G(t, \omega, v) \) such that a.s.

\[
(16) \quad v(t) = \int_0^t f(\tau, \omega)d\tau + \int_0^t g(\tau, \omega)d\omega(\tau) + C.
\]

Consider the \( M \)-valued process \( \xi(t) = Sv(t) \) with \( v(t) \) satisfying (16). In the same manner as in the proof of Theorem 9 we can construct Borel measurable selections \( f(t, m, X) \) of \( F(t, m, X) \) and \( g(t, m, X) \) of \( G(t, m, X) \) such that a.s.

\[
\xi(t) = S \left( \int_0^t \Gamma f(\tau, \xi(\tau), \dot{\xi}(\tau))d\tau + \int_0^t \Gamma g(\tau, \xi(\tau), \dot{\xi}(\tau))d\omega(\tau) + C \right).
\]

**References**


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