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ON LOWER LIPSCHITZ CONTINUITY OF MINIMAL POINTS

Ewa M. Bednarczuk

Systems Research Institute, PAS
01–447 Warsaw, Newelska 6, Poland

e-mail: bednarcz@ibspan.waw.pl

Abstract

In this paper we investigate the lower Lipschitz continuity of minimal points of an arbitrary set A depending upon a parameter u. Our results are formulated with the help of the modulus of minimality. The crucial requirement which allows us to derive sufficient conditions for lower Lipschitz continuity of minimal points is that the modulus of minimality is at least linear. The obtained results can be directly applied to stability analysis of vector optimization problems.

Keywords: minimal points, Lipschitz continuity, vector optimization.

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1 Introduction

Let $(Y, \|\cdot\|)$ be a normed space and let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y. Let $A \subset Y$ be a subset of Y. We say that $y \in A$ is a minimal point of A with respect to $\mathcal{K}$ if $(y - \mathcal{K}) \cap A = \{y\}$ (see [12]). By $Min(A|\mathcal{K})$ we denote the set of all minimal points of A with respect to $\mathcal{K}$. We say that the domination property (DP) holds for A if $A \subset Min(A|\mathcal{K}) + \mathcal{K}$ (see [12, 15]).

Let $U = (U, \|\cdot\|)$ be a normed space and let $\Gamma : U \to Y$ be a set-valued mapping. Define a set-valued mapping $M : U \to Y$ as follows

$$M(u) = Min(\Gamma(u)|\mathcal{K}).$$

The set-valued mapping M is called the minimal point multifunction.

In the present paper we give sufficient conditions which ensure that M is lower Lipschitz continuous and/or locally Lipschitz at a given $u_0 \in U$.

Lipschitz behaviour of solutions to optimization problems is one of central topics of stability analysis in optimization. For scalar optimization it was investigated by many authors, see e.g. [2, 20, 16, 21, 11, 13, 14, 23, 18, 19, 24, 1] and many others. In vector optimization the results on Lipschitz continuity of solutions are not so numerous, and concern some classes of problems, for linear problems see e.g. [7, 8, 9], for convex problems see e.g. [6, 10].

We say that a multivalued mapping $F : U \to Y$ is locally Lipschitz at u_0, [2], if there exist a neighbourhood $U_0 \subset \mathrm{dom}F$ of u_0 and a positive constant ℓ such that

$$F(u_1) \subset F(u_2) + \ell \cdot \|u_1 - u_2\| \quad \text{for} \quad u_1, u_2 \in U\,.$$

We say that $F : U \to Y$ is lower Lipschitz continuous at $u_0 \in U$ if there exist a constant L and a neighbourhood U_0 of u_0 such that $F(u_0) \subset F(u) + L\|u - u_0\|$ for $u \in U_0$. $F : U \to Y$ is upper Lipschitz continuous at $u_0 \in U$ if there exist a constant L and a neighbourhood U_0 of u_0 such that $F(u) \subset F(u_0) + L\|u - u_0\|$ for $u \in U_0$.

2 Modulus of minimality

Let $(Y, \|\cdot\|)$ be a normed space and let $\mathcal{K}$ be a closed convex and pointed cone in Y. By $B(a, r)$ we denote the open ball of centre a and radius r, $B(0, 1) = B$.

It was shown in [4], and [5] that for the lower continuity of minimal point multifunction M at $u_0 \in U$ the crucial requirement is that strictly minimal points are dense in $Min(\Gamma(u_0)|\mathcal{K})$. Some conditions assuring this kind of density are given in [5].

Let $A \subset Y$ be a subset of Y.

Definition 21 ([4, 5])**.** (Strict minimality) We say that $x \in Min(A|\mathcal{K})$ is a **strictly minimal point**, $x \in SM(A|\mathcal{K})$, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$[A \setminus B(x, \varepsilon)] \cap [(x + \delta B) - \mathcal{K}] = \emptyset\,.$$

Clearly, each strictly minimal point is minimal. Other properties of strictly minimal points were investigated in [4, 5]. For minimality notions of similar type, see e.g. [17, 22].
To derive our continuity results we introduce the modulus of minimality of a set A.

Definition 22. (*Modulus of minimality*) *The modulus of minimality of a set $A \subset Y$ is the function $m : R_+ \to R$, defined as*

$$m(\varepsilon) = \inf_{x \in SM(A|\mathcal{K})} \nu(\varepsilon, x) \tag{1}$$

where $\nu : R_+ \times A \to R$, is the modulus of minimality of $x \in A$ defined as

$$\nu(\varepsilon, x) = \sup\{\delta : (A \setminus B(x, \varepsilon)) \cap [x + \delta B - \mathcal{K}] = \emptyset\}. \tag{2}$$

For each $x \in A$, $\nu(\varepsilon, x) \leq \varepsilon$, and for $x \in SM(A|\mathcal{K})$, $0 < \nu(\varepsilon, x) \leq \varepsilon$. Clearly, $[A \setminus B(x, \varepsilon)] \cap [(x + \nu(\varepsilon, x) \cdot B) - \mathcal{K}] = \emptyset$ for $x \in SM(A \setminus \mathcal{K})$.

3 Lower Lipschitz continuity

We start with sufficient conditions for lower Hausdorff continuity of minimal point multifunction M. By $SM(u)$ we denote the set of strictly minimal points of the set $\Gamma(u)$, $\mathrm{cl}(\cdot)$ stands for the closure.

Theorem 31. *Let Y be a normed space and let $\mathcal{K} \subset Y$ be a closed convex pointed cone. Assume that $\Gamma : U \to Y$ is a set-valued mapping defined on a normed space U, $u_0 \in U$. If*

(i) $M(u_0) \subset \mathrm{cl}(SM(u_0))$,

(ii) (DP) *holds for all $\Gamma(u)$ in some neighbourhood U_1 of u_0,*

(iii) Γ *is Hausdorff continuous at u_0, i.e., for each $\varepsilon > 0$ there exists a neighbourhood U_2 of u_0 such that*

$$\Gamma(u) \subset \Gamma(u_0) + \varepsilon \cdot B,$$

and

$$\Gamma(u_0) \subset \Gamma(u) + \varepsilon \cdot B,$$

for $u \in U_2$,

then M is lower Hausdorff semicontinuous at u_0, i.e. for each $\varepsilon > 0$

$$M(u_0) \subset M(u) + \varepsilon \cdot B$$

for $u \in U_1 \cap U_2$.

Proof. If $M(u_0) = \emptyset$, then, by the assumptions, $\Gamma(u) = \emptyset$, and, consequently, $M(u) = \emptyset$, for $u \in U_0 \cap U_1$. Hence, we can suppose that $M(u_0) \neq \emptyset$.

Take any $\varepsilon > 0$, and $y \in M(u_0)$. By (i) there exists $y_1 \in SM(u_0)$ such that $y_1 \in y + \frac{1}{4}\varepsilon \cdot B$, and

$$\Big[\Big(\Gamma(u_0) \setminus \Big(y_1 + \frac{1}{2}\varepsilon \cdot B\Big)\Big) + \nu\Big(\frac{1}{2}\varepsilon, y_1\Big) \cdot B\Big] \cap (y_1 - \mathcal{K}) = \emptyset.$$

Hence,

$$\Big[\Big(\Gamma(u_0)\setminus\Big(y_1+\frac{1}{2}\varepsilon\cdot B\Big)\Big)+\frac{1}{2}\nu\Big(\frac{1}{2}\varepsilon, y_1\Big)\cdot B\Big]\cap\Big(y_1+\frac{1}{2}\nu\Big(\frac{1}{2}\varepsilon, y_1\Big)\cdot B-\mathcal{K}\Big) = \emptyset. \tag{3}$$

I. Consider first the case where $\nu(\varepsilon, y_1) \leq \frac{1}{2}\varepsilon$. By the upper Hausdorff semicontinuity of Γ

$$\begin{aligned} \Gamma(u) &\subset \Gamma(u_0) + \frac{1}{2}\nu\Big(\frac{1}{2}\varepsilon, y_1\Big) \cdot B \\ &\subset \Big[\Big(\Gamma(u_0) \setminus \Big(y_1 + \frac{1}{2}\varepsilon \cdot B\Big)\Big) + \frac{1}{2}\nu\Big(\frac{1}{2}\varepsilon, y_1\Big) \cdot B\Big] \\ &\quad \cup \Big[y_1 + \Big(\frac{1}{2}\nu\Big(\frac{1}{2}\varepsilon, y_1\Big) + \frac{1}{2}\varepsilon\Big) \cdot B\Big], \end{aligned} \tag{4}$$

for $u \in U_2$, and by the lower Hausdorff semicontinuity of Γ, for $u \in U_1$ there exists $y_2 \in \Gamma(u)$ such that

$$y_2 \in y_1 + \frac{1}{2}\nu\Big(\frac{1}{2}\varepsilon, y_1\Big) \cdot B$$

and

$$y_2 - \mathcal{K} \subset y_1 + \frac{1}{2}\nu\Big(\frac{1}{2}\varepsilon, y_1\Big) \cdot B - \mathcal{K}\,.$$

By (3)

$$(y_2 - \mathcal{K}) \cap \Big[\Big(\Gamma(u) \setminus \Big(y_1 + \frac{1}{2}\varepsilon \cdot B\Big)\Big) + \frac{1}{2}\nu\Big(\frac{1}{2}\varepsilon, y_1\Big) \cdot B\Big] = \emptyset\,.$$

Now, by (4) for $u \in U_1$

$$(y_2 - \mathcal{K}) \cap \Gamma(u) \subset y_1 + \Big(\frac{1}{2}\nu\Big(\frac{1}{2}\varepsilon, y_1\Big) + \frac{1}{2}\varepsilon\Big) \cdot B\,.$$

Since (DP) holds for $\Gamma(u)$, for $u \in U_1 \cap U_2$ there exists $\eta_2 \in M(u)$ such that

$$\eta_2 \subset (y_2 - \mathcal{K}) \cap \Gamma(u) \subset y_1 + \Big(\frac{1}{2}\nu\Big(\frac{1}{2}\varepsilon, y_1\Big) + \frac{1}{2}\varepsilon\Big) \cdot B\,,$$

and since $\nu(\varepsilon, y_1) \le \frac{1}{2}\varepsilon$,

$$\eta_2 \in y_1 + \frac{3}{4}\varepsilon \cdot B \subset y + \varepsilon \cdot B\,.$$

This means that for $u \in U_1 \cap U_2$

$$M(u_0) \subset M(u) + \varepsilon \cdot B$$

which completes the proof in the case I.

II. Consider now the case where $\nu(\varepsilon, y_1) > \frac{1}{2}\varepsilon$. By the upper Hausdorff semicontinuity of Γ we have for $u \in U_2$

$$\begin{aligned}\Gamma(u) &\subset \Gamma(u_0) + \frac{1}{8}\varepsilon \cdot B \\ &\subset \Big[\Big(\Gamma(u_0) \setminus \Big(y + \frac{1}{2}\varepsilon \cdot B\Big)\Big) + \frac{1}{8}\varepsilon \cdot B\Big] \cup \Big[y_1 + \Big(\frac{1}{8}\varepsilon + \frac{1}{2}\varepsilon\Big) \cdot B\Big]\,,\end{aligned} \tag{5}$$

and by the lower Hausdorff semicontinuity of Γ there exists $y_2 \in \Gamma(u)$, $u \in U_2$ such that

$$y_2 \in y_1 + \frac{1}{2}\nu\Big(\frac{1}{2}\varepsilon, y_1\Big) \cdot B\,.$$

In consequence,

$$y_2 - \mathcal{K} \subset y_1 + \frac{1}{2}\nu\Big(\frac{1}{2}\varepsilon, y_1\Big) \cdot B - \mathcal{K}\,,$$

and by (3),

$$(y_2 - \mathcal{K}) \cap \Big[\Big(\Gamma(u_0) \setminus \Big(y_1 + \frac{1}{2}\varepsilon \cdot B\Big)\Big) + \frac{1}{2}\nu\Big(\frac{1}{2}\varepsilon, y_1\Big) \cdot B\Big] = \emptyset\,.$$

Since $\frac{1}{2}\nu(\frac{1}{2}\varepsilon, y_1) > \frac{1}{8}\varepsilon$ the latter implies that

$$(y_2 - \mathcal{K}) \cap \Big[\Big(\Gamma(u_0) \setminus \Big(y_1 + \frac{1}{2}\varepsilon \cdot B\Big)\Big) + \frac{1}{8}\varepsilon \cdot B\Big] = \emptyset\,.$$

Now, by (5)

$$(y_2 - \mathcal{K}) \cap \Gamma(u) \subset y_1 + \frac{5}{8}\varepsilon \cdot B\,.$$

Since (DP) holds for $\Gamma(u)$, $u \in U_1$, there exists $\eta_2 \in M(u)$, $u \in U_1 \cap U_2$ such that

$$\eta_2 \in (y_2 - \mathcal{K}) \cap \Gamma(u) \subset y_1 + \frac{5}{8}\varepsilon \cdot B$$

and

$$\eta_2 \in y + \frac{7}{8}\varepsilon \cdot B \subset y + \varepsilon \cdot B\,.$$

This means that for $u \in U_1 \cap U_2$

$$M(u_0) \subset M(u) + \varepsilon \cdot B$$

which completes the proof.

Now, by strengthening the assumption (i) of Theorem 31 we prove sufficient conditions for lower Lipschitz continuity of M at u_0.

Theorem 32. *Let Y be a normed space and let $\mathcal{K} \subset Y$ be a closed convex pointed cone. Assume that $\Gamma : U \to Y$ is a set-valued mapping defined on a normed space U, $u_0 \in U$. If*

(i) $M(u_0) \subset \mathrm{cl}(SM(u_0))$, *and the modulus of minimality $m(\varepsilon)$ of $\Gamma(u_0)$, satisfies the condition $m(\varepsilon) \geq c\varepsilon$, where $c \in R$, $c > 0$,*

(ii) (DP) *holds for all $\Gamma(u)$ in some neighbourhood U_0 of u_0,*

(iii) Γ *is upper and lower Lipschitz at u_0, i.e.*

$$\Gamma(u) \subset \Gamma(u_0) + L\|u - u_0\| \cdot B\,,$$

$$\Gamma(u_0) \subset \Gamma(u) + \frac{1}{2}L\|u - u_0\| \cdot B$$

for u in a neighbourhood U_1 of u_0,

then M is lower Lipschitz at u_0, i.e. for $u \in U_0 \cap U_1$

$$M(u_0) \subset M(u) + \Big(1 + \frac{2}{c}\Big)L\|u - u_0\|\,.$$

Proof. As previously, we can assume that $M(u_0) \neq \emptyset$.

Let $u \in U_0 \cap U_1$ and $y \in M(u_0)$. By (i) there exists $y_1 \in SM(u_0)$ such that $y_1 \in y + \frac{1}{c}L\|u - u_0\| \cdot B$. Since $y_1 \in SM(u_0)$

$$\Big[\Big(\Gamma(u_0) \setminus \Big(y_1 + \frac{1}{c}L\|u - u_0\| \cdot B\Big)\Big) + m\Big(\frac{1}{c}L\|u - u_0\|\Big) \cdot B\Big] \cap (y_1 - \mathcal{K}) = \emptyset\,,$$

and hence

$$\begin{aligned}\Big[\Gamma(u_0)\setminus\Big(y_1+\frac{1}{c}L\|u_1-u_2\|\cdot B\Big)+\frac{1}{2}m\Big(\frac{1}{c}L\|u_1-u_2\|\Big)\cdot B\Big]\\ \cap\Big(y_1+\frac{1}{2}m\Big(\frac{1}{c}L\|u_1-u_2\|\Big)\cdot B-\mathcal{K}\Big)=\emptyset\,.\end{aligned}\tag{6}$$

By the upper Lipschitz continuity of Γ

$$\begin{aligned}\Gamma(u)&\subset\Gamma(u_0)+L\|u-u_0\|\cdot B\\ &\subset\Big[\Big(\Gamma(u_0)\setminus\Big(y_1+\frac{1}{c}L\|u-u_0\|\cdot B\Big)\Big)+L\|u-u_0\|\cdot B\Big]\\ &\quad\cup\Big[y_1+\Big(\frac{1}{c}+1\Big)L\|u-u_0\|\cdot B\Big]\,,\end{aligned}\tag{7}$$

and since $y_1\in\Gamma(u_0)$, by the lower Lipschitz continuity there exists $y_2\in\Gamma(u)$ such that

$$y_2\in y_1+\frac{1}{2}L\|u-u_0\|\cdot B\,,$$

and, since $\frac{1}{2}L\|u-u_0\|\le\frac{1}{2}m(\frac{1}{c}L\|u-u_0\|)$

$$y_2-\mathcal{K}\subset y_1+\frac{1}{2}L\|u-u_0\|\cdot B-\mathcal{K}\subset y_1+\frac{1}{2}m\Big(\frac{1}{c}L\|u-u_0\|\Big)\cdot B-\mathcal{K}\,.$$

By (6)

$$(y_2-\mathcal{K})\cap\Big[\Gamma(u_0)\setminus\Big(y_1+\frac{1}{c}L\|u-u_0\|\cdot B\Big)+\frac{1}{2}m\Big(\frac{1}{c}L\|u-u_0\|\Big)\cdot B\Big]=\emptyset\,,$$

and since $L\|u-u_0\|\le m(\frac{1}{c}L\|u-u_0\|)$

$$(y_2-\mathcal{K})\cap\Big[\Gamma(u_0)\setminus\Big(y_1+\frac{1}{c}L\|u-u_0\|\cdot B\Big)+L\|u-u_0\|\cdot B\Big]=\emptyset\,.$$

Now, by (7)

$$(y_2-\mathcal{K})\cap\Gamma(u)\subset y_1+\Big(1+\frac{1}{c}\Big)L\|u-u_0\|\cdot B\,.$$

Since (DP) holds for $\Gamma(u)$ there exists $\eta_2\in M(u)$ such that

$$\eta_2\subset(y_2-\mathcal{K})\cap\Gamma(u)\subset y_1+\Big(1+\frac{1}{c}\Big)L\|u-u_0\|\cdot B\subset y+\Big(1+\frac{2}{c}\Big)L\|u-u_0\|.$$

This means that for $u\in U_0\cap U_1$

$$M(u_0)\subset M(u)+\frac{c+2}{c}L|u-u_0\|\cdot B$$

which completes the proof.

Theorem 33. *Let Y be a normed space and let $\mathcal{K} \subset Y$ be a closed convex pointed cone. Assume that $\Gamma : U \to Y$ is a set-valued mapping defined on a normed space U, $u_0 \in U$. If*

(i) $M(u) \subset \mathrm{cl}(SM(u))$, *in some neighbourhood U_2 of u_0 and for any $\varepsilon > 0$,*

$$\tilde{m}(\varepsilon) = \inf_{u \in U_2} m_u(\varepsilon) \geq 2c\varepsilon > 0\,,$$

where $m_u(\cdot)$ is the modulus of minimality of $\Gamma(u)$, $c \in R$, $c > 0$,

(ii) *(DP) holds for all $\Gamma(u)$ in some neighbourhood U_0 of u_0,*

(iii) *Γ is locally Lipschitz at u_0, i.e.*

$$\Gamma(u_1) \subset \Gamma(u_2) + L\|u_1 - u_2\| \cdot B$$

for $u_1,\ u_2$ in a neighbourhood U_1 of u_0,

then M is locally Lipschitz at u_0, i.e. for each $u_1,\ u_2 \in U_0 \cap U_1 \cap U_2$

$$M(u_1) \subset M(u_2) + \Big(1 + \frac{2}{c}\Big) L \cdot B\,.$$

Proof. By (i), for any $\varepsilon > 0$, $u \in U_2$, and any $z \in SM(u)$,

$$[(\Gamma(u) \setminus (z + \varepsilon B)) + \tilde{m}(\varepsilon)B] \cap (z - \mathcal{K}] = \emptyset\,.$$

Let $u_1,\ u_2 \in U_0 \cap U_1 \cap U_2$, and $y \in M(u_1)$. By (i) there exists $y_1 \in SM(u_1)$ such that $y_1 \in y + \frac{1}{c} L\|u_1 - u_2\| \cdot B$. Since $y_1 \in SM(u_1)$

$$\Big[\Big(\Gamma(u_1) \setminus \Big(y_1 + \frac{1}{c} L\|u_1 - u_2\| \cdot B\Big)\Big) + \tilde{m}\Big(\frac{1}{c} L\|u_1 - u_2\|\Big) \cdot B\Big] \cap (y_1 - \mathcal{K}) = \emptyset\,,$$

and hence,

$$\begin{aligned} &\Big[\Gamma(u_1) \setminus \Big(y_1 + \frac{1}{c} L\|u_1 - u_2\| \cdot B\Big) + \frac{1}{2}\tilde{m}\Big(\frac{1}{c} L\|u_1 - u_2\|\Big) \cdot B\Big] \\ &\qquad \cap \Big(y_1 + \frac{1}{2}\tilde{m}\Big(\frac{1}{c} L\|u_1 - u_2\|\Big) \cdot B - \mathcal{K}\Big) = \emptyset\,. \end{aligned} \tag{8}$$

By local Lipschitz continuity of Γ

$$\begin{aligned} \Gamma(u_2) &\subset \Gamma(u_1) + L\|u_1 - u_2\| \cdot B \\ &\subset \Big[\Big(\Gamma(u_1) \setminus \Big(y_1 + \frac{1}{c} L\|u_1 - u_2\| \cdot B\Big)\Big) + L\|u_1 - u_2\| \cdot B\Big] \\ &\quad \cup \Big[y_1 + \Big(1 + \frac{1}{c}\Big) L\|u_1 - u_2\| \cdot B\Big]\,, \end{aligned} \tag{9}$$

and, since $y_1 \in \Gamma(u_1)$ there exists $y_2 \in \Gamma(u_2)$ such that

$$y_2 \in y_1 + L\|u_1 - u_2\| \cdot B$$

and, since $L\|u_1 - u_2\| \leq \frac{1}{2}\tilde{m}(\frac{1}{c}L\|u_1 - u_2\|)$,

$$y_2 - \mathcal{K} \subset y_1 + L\|u_1 - u_2\| \cdot B - \mathcal{K} \subset y_1 + \frac{1}{2}\tilde{m}\Big(\frac{1}{c}L\|u_1 - u_2\|\Big) \cdot B - \mathcal{K}\,.$$

By (8)

$$(y_2 - \mathcal{K}) \cap \Big[\Gamma(u_1) \setminus \Big(y_1 + \frac{1}{c}L\|u_1 - u_2\| \cdot B\Big) + \frac{1}{2}\tilde{m}\Big(\frac{1}{c}L\|u_1 - u_2\|\Big) \cdot B\Big] = \emptyset$$

and since $L\|u_1 - u_2\| \leq \frac{1}{2}\tilde{m}(\frac{1}{c}L\|u_1 - u_2\|)$

$$(y_2 - \mathcal{K}) \cap \Big[\Gamma(u_1) \setminus \Big(y_1 + \frac{1}{c}L\|u_1 - u_2\| \cdot B\Big) + L\|u_1 - u_2\| \cdot B\Big] = \emptyset\,.$$

Now, by (9)

$$(y_2 - \mathcal{K}) \cap \Gamma(u_2) \subset y_1 + \Big(1 + \frac{1}{c}\Big)L\|u_1 - u_2\| \cdot B\,.$$

Since (DP) holds for $\Gamma(u_2)$ there exists $\eta_2 \in M(u_2)$ such that

$$\eta_2 \subset (y_2 - \mathcal{K}) \cap \Gamma(u_2) \subset y_1 + \Big(1 + \frac{1}{c}\Big)L\|u_1 - u_2\| \cdot B \subset y + \Big(1 + \frac{2}{c}\Big)L\|u_1 - u_2\| \cdot B\,.$$

This means that for $u_1, u_2 \in U_0 \cap U_1 \cap U_2$

$$M(u_1) \subset M(u_2) + \Big(1 + \frac{2}{c}\Big)L\|u_1 - u_2\| \cdot B$$

which completes the proof.

4 Final remarks

Parametric vector optimization problem

$$\mathcal{K} - \min\{f(x) \mid x \in A(u)\}$$

consists of finding all $x \in A(u)$ such that $f(x) \in Min(f(A(u))|\mathcal{K})$, where $f : X \to Y$ is a mapping defined on a space X to be minimized and $A : U \to X$ is a feasible set multifunction. By taking $\Gamma(u) = f(A(u))$ Theorems 31, 32, 33 can be directly applied to derive sufficient conditions for lower Lipschitz continuity of the set-valued mapping $\mathcal{M}(u) = Min(A(u)|\mathcal{K})$. Conditions ensuring that $M(u_0) \subset \mathrm{cl}SM(u_0)$ can be found in [5].

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