TRANSPORTATION FLOW PROBLEMS WITH RADON MEASURE VARIABLES

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Abstract

For a multidimensional control problem (P)K involving controls \( u \in L_\infty \), we construct a dual problem (D)K in which the variables \( \nu \) to be paired with \( u \) are taken from the measure space \( rca(\Omega, B) \) instead of \( (L_\infty)^* \). For this purpose, we add to (P)K a Baire class restriction for the representatives of the controls \( u \). As main results, we prove a strong duality theorem and saddle-point conditions.

Keywords: multidimensional control problems, strong duality, saddle-point conditions.

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1 Introduction

a) The primal problem. We consider the following multidimensional control problem (P)K (1.1) – (1.4) ("classical deposit problem") introduced by Klötzler [8]:

(1.1) \[ J(x, u) = -\sum_{k=1}^{n} \int_{\Omega} x_k(t) \, d\alpha_k(t) \rightarrow \text{Min!} \]

subject to \( (x, u) \in W_1^{1,n}(\Omega) \times L_p^{nm}(\Omega) \), satisfying

(1.2) \[ x_{i;j}(t) = u_{ij}(t) \text{ a.e. on } \Omega, \quad i=1,\ldots,n; \quad j=1,\ldots,m; \]

(1.3) \[ u(t) \in U(t) = \left\{ z \in \mathbb{R}^{nm} \bigg| z^T v \leq r(t,v) \quad \forall v \in \mathbb{R}^{nm} \right\} \forall t \in \Omega \]

(1.4) \[ x(t) = \varphi(t) \quad \forall t \in \Gamma \text{ where } \Gamma \in \text{Comp}(\Omega), \quad \Gamma \neq \emptyset. \]
For $m = 2$ we may interpret $(P)_K$ as deposit problem [8, p. 394]: On a region $\Omega$ in the plane, $n$ infinitely divisible commodities have to be stored. $x_k(t)$ describes the deposit height of the $k^{th}$ commodity at the position $t$ (fixed in the case of $t \in \Gamma$), $(-\alpha_k)$ the related cost rate, $J(x, u)$ the total deposit cost which is to minimize. The control restrictions may be understood as generalized slope conditions for the resulting deposit hill. From [8] we take the following

Basic assumptions about the data of $(P)_K$:

(V1)$_K$: We have $m \geq 2$ and $p = \infty$. $\Omega \subset \mathbb{R}^m$ is a compact Lipschitz domain in strong sense, see [11, Definition 3.4.1, p. 72]. (In view of Lemma 2.1, we may assume $m < p < \infty$ instead of $p = \infty$.) Then functions $x \in W^{1,n}_p(\Omega)$, $m < p < \infty$, have continuous representatives, and functions $x \in W^{1,\infty}_p(\Omega)$ are Lipschitz representable [1, Theorem 5.5, p. 185].

(V2)$_K$: $r(\cdot, v)$ is summable on $\Omega$ for all $v \in \mathbb{R}^{nm}$; $r(t, \cdot)$ is positively homogeneous of degree one in $v$ (i.e. $r(t, \lambda v) = \lambda r(t, v)$ for all $\lambda > 0$) and convex; there exist constants $0 < \gamma_1 \leq \gamma_2$ with $\gamma_1|v| \leq r(t, v) \leq \gamma_2|v|$ for all $t \in \Omega$ and for all $v \in \mathbb{R}^{nm}$.

(V3)$_K$: $\alpha_k$ are signed regular measures on the $\sigma$-algebra $\mathcal{B}$ of the Lebesgue sets of $\Omega$ satisfying the balance condition $\alpha_k(\Omega) = 0$. (In the following, we only consider the uniquely determined restrictions of $\alpha_k$ on the $\sigma$-subalgebra $\mathcal{B} \subset \mathcal{B}$ of the Borel sets of $\Omega$.)

(V4)$_K$: There is $\Gamma = \{t_0\} \subset \partial \Omega$ and $x(t_0) = x_n$.

b) Outline and main results of the paper. In [8] and [9], a transportation flow problem $(T)_K$ in which the variables (“flows”) come from the space $(L_\infty)^*$ is opposited to $(P)_K$. Both problems are in strong duality. The aim of the present paper is the construction of a strong dual problem for $(P)_K$ with more regular variables, namely Radon measures, in place of $(L_\infty)^*$-functionals (which are representable only by finitely additive set functions, cf. [4, Theorem 16, p. 296]). For this purpose, we restrict the feasible domain of $(P)_K$ under conservation of the minimal value $\inf(P)_K$:

Definition 1.1. For $(P)_K$ and $k \in \mathbb{N}_0$, we consider the class-qualified problem $(P)_{K,B^k}$ (1.1) – (1.5) with

\begin{equation}
(1.5) \quad x_{i; t_j} \text{ admits (at least) one representative from } B_k(\Omega) \forall i, j.
\end{equation}

Here $B_k(\Omega)$ denotes the $k^{th}$ Baire function class on $\Omega$ (see below), thus we have to distinguish in $(P)_{K,B^k}$ feasible controls $u', u''$ taking different values
even on a $\lambda^m$-null set. The following theorem gives sufficient conditions under which the minimal value of $(P)_K$ is not influenced by addition of the class qualification (1.5) to (1.1) – (1.4).

**Theorem 1.2** (Sufficient conditions for $\inf (P)_K = \inf (P)_{K,B^k}$). Let $(P)_K$ satisfy assumptions $(V1)_K$ – $(V4)_K$. Assume further that the function $r(t,v)$ satisfies the condition $|r(t',v) - r(t'',v)| \leq L \cdot |t' - t''| \cdot \tilde{r}(v) \quad \forall \, \nu \in \mathbb{R}^{nm}$ $\forall \, t', t'' \in \Omega$ with $L > 0$ and $\tilde{r} \in C^0(\mathbb{R}^{nm})$.

Then $(P)_K$ admits a minimizing sequence $\{(x^N, u^N)\}$ with representatives of $0^\text{th}$ Baire class for $x^N_{i,j}$, and the minimal values of $(P)_K$ and $(P)_{K,B^k}$, $k = 0, 1, ..., \text{coincide. Furthermore, each} \ (x^N, u^N) \text{can be determined in such a way that the state equations (1.2) are satisfied everywhere on } \Omega$.

If the assumptions of Theorem 1.2 are satisfied then the problem $(D)_K (2.1) – (2.2)$

$$G(\nu) = \inf_{u \in B^{1,n}(\Omega)} \left[ - \sum_{i,j} \int_{\Omega} u_{ij}(t) \, d\nu_{ij}(t) \right] \to \text{Max!}$$

(2.1)

subject to $\nu \in (\text{rca} \, (\Omega, \mathcal{B}))^{nm}$, satisfying the continuity equation

$$\sum_{i,j} \int_{\Omega} \zeta_{i,j}(t) \, d\nu_{ij}(t) - \sum_k \int_{\Omega} \zeta_k(t) \, d\alpha_k(t) = 0$$

(2.2)

$\forall \zeta \in C^1(\Omega): \zeta(t_0) = 0$.

is strongly dual to $(P)_K$ (Theorem 3.4). In analogy to [8, p. 391 ff.], the feasible elements of $(D)_K$ may be understood as time-independent vectorial transportation flows: Assuming that we have to organize the shipment of infinitely divisible commodities within $\Omega$ where $\alpha_k(A)$ is the rate of supply resp. demand of the $k^{\text{th}}$ commodity in $A \in \mathcal{B}$, the average flow of the $k^{\text{th}}$ commodity in $A$ can be described by the vector $(\nu_{k,1}(A), ..., \nu_{k,m}(A))$.

**Theorem 1.3** (Sufficient saddle-point conditions for the problems $(P)_{K,B^1} - (D)_K$). Let $(P)_K$ satisfy all assumptions of Theorem 1.2. Given some feasible element $(x^*, u^*)$ of $(P)_{K,B^1}$ (thus the weak derivatives $x^*_{i,j}$ admit representatives of first Baire class) and a measure $\nu^* \in (\text{rca} \, (\Omega, \mathcal{B}))^{nm}$.

If the following conditions $(M^*)_0^\ast$, $(K)_0^\ast$ and $(D)_0^\ast$ are satisfied then $(x^*, u^*)$ is a global minimizer of $(P)_{K,B^1}$ and $\nu^*$ is a global maximizer of $(D)_K$.
The paper is organized as follows: In the rest of this section, we compile some basic notations and definitions. In Section 2, we investigate the relations between the original deposit problem (1.1) – (1.4), its relaxed problem and the class-qualified problem (1.1) – (1.5) and prove Theorem 1.2. Then, in Section 3, we construct the announced dual problem (D)\textsubscript{c} and give the proof of Theorem 1.3. Finally, we prove that a partial converse of Theorem 1.3 is true (Theorem 3.5).

c) Notations. \( C^{k,n}(\Omega), L^p_\Omega(\Omega) \) and \( W^{k,n}_p(\Omega) \) \( (1 \leq p \leq \infty) \) denote the spaces of \( n \)-dimensional vector functions on \( \Omega \) whose components are \( k \)-times continuously differentiable, resp. belong to \( L^p(\Omega) \) or to the Sobolev space of \( L^p(\Omega) \)-functions having weak derivatives up to \( k^{th} \) order in \( L^p(\Omega) \). Instead of \( C^0_0(\Omega) \), we write shortly \( C^0(\Omega) \). For the classical as well as for the weak partial derivatives of \( x_i \) by \( t_j \) we use the notation \( x_{i;j} \). The Banach space of Radon measures (signed regular measures) acting on the \( \sigma \)-algebra \( B \) of the Borel sets of \( \Omega \) (equipped with the total variation norm) is denoted by \( rca(\Omega, B) \). Due to the compactness of \( \Omega \), there is an isometric isomorphism between the dual space \( (C^0(\Omega))^* \) and \( rca(\Omega, B) \) [4, Theorem 3, p. 265] so that each linear, continuous functional on \( C^0(\Omega) \) can be represented by an integral w. r. to a Radon measure \( \nu \in rca(\Omega, B) \). \( \delta_v \) denotes the Dirac measure concentrated in \( v \), \( \lambda^m \) the \( m \)-dimensional Lebesgue measure and \( \mathfrak{o} \) the zero element of the actual space (in particular, \( \mathfrak{o}_n \) is the \( n \)-dimensional zero vector).

d) Generalized controls. Let \( U = \bigcup_{t \in \Omega} U(t) \) (\( U \) is compact, see Lemma 2.1 below). A family \( \mu = \{ \mu_t | t \in \Omega \} \) of probability measures \( \mu_t \in rca(\Omega, \mathcal{B}_U) \) acting on the \( \sigma \)-algebra \( \mathcal{B}_U \) of the Borel sets of \( U \) is called a generalized control if 1) \( \text{supp} \mu_t \subseteq U(t) \) for all \( t \in \Omega \) and 2) for any continuous function \( f \in C^0(\Omega \times U) \) the function \( h_f : \Omega \times U \rightarrow \mathbb{R} \) with...
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\[ h(t) = \int_U f(t, v) \, d\mu_t(v) \] is measurable [5, p. 23]. Two families \( \mu', \mu'' \) can be identified if \( \mu'_t \equiv \mu''_t \) for a.e. \( t \in \Omega \). The set of all generalized controls is denoted by \( \mathcal{M}_U \). Let us equip \( \mathcal{M}_U \) with the following topology:

\[ \{\mu_N^T\} \to \mu^* \iff \lim_{N \to \infty} \int_{\Omega} \int_U f(t, v) \mu_N^T(v) \, dt = \int_{\Omega} \int_U f(t, v) \, d\mu^*(v) \, dt \]

for all \( f \in C^0(\Omega \times U) \). Due to the compactness of \( \Omega \) and \( U \), each family \( \{\mu_t\} \) is finite in the sense of [5, p. 21 f.], and each function \( h_f \) generated by some \( \mu \in \mathcal{M}_U \) is bounded and, consequently, integrable on \( \Omega \). The set \( \mathcal{M}_U \) is convex [5, p. 25] and, by [10, Theorem 20, p. 78], sequentially compact in the above introduced topology while the sets \( U(t) \) are nonempty, closed and uniformly bounded (Lemma 2.1) and the set-valued map \( U(t) : \Omega \to \mathcal{P}(\mathbb{R}^m) \) is upper semicontinuous (see Lemma 2.2).

e) Baire classification. We say that any continuous function \( \psi \) defined on the compact set \( \Omega \subset \mathbb{R}^m \) is of \( r \)th Baire class and write \( \psi \in B^r(\Omega) \). The limit functions of everywhere pointwise convergent sequences \( \{\psi^K\} \), \( \psi^K \in B^0(\Omega) \), form the first Baire class \( B^1(\Omega) \); the limit functions of everywhere pointwise convergent sequences \( \{\psi^K\} \), \( \psi^K \in B^1(\Omega) \), form the second Baire class \( B^2(\Omega) \) and so on. Obviously, we have \( B^0(\Omega) \subset B^1(\Omega) \subset B^2(\Omega) \subset ... \) If a finite function is contained in any Baire class then it is measurable [3, Theorem 4, p. 404]; conversely, any measurable, essentially bounded function on \( \Omega \) agrees a.e. with some function of second Baire class [3, Theorem 5, p. 406]. (Consequently, for \( k \geq 2 \) the minimal values of the problems (P)_K and (P)_K,\,B^k coincide.) Each Baire class is closed under (pointwise) addition and multiplication of finite functions [3, Theorems 6 and 7, p. 397]. For more details, see [3, p. 393 ff.].

f) Theorem 1.5 (Filippov’s lemma). Consider a measure space \((\Omega, \mathcal{A}, \lambda)\) with a \( \sigma \)-finite measure \( \lambda \) and a \( \sigma \)-algebra \( \mathcal{A} \) which is complete w. r. to subsets of \( \lambda \)-null sets. Further, let \( Y' \) and \( Y'' \) be separable, complete metric spaces, \( h(t, v) : \Omega \times Y' \to Y'' \) a Carathéodory function and \( S(t) : \Omega \to \mathcal{P}(Y') \) a measurable set-valued map [2, Definition 8.1.1, p. 307] with nonempty, closed images. Then for every measurable function \( z : \Omega \to Y'' \) satisfying \( z(t) \in \{h(t, v) \mid v \in S(t)\} \) for all \( t \in \Omega \) there exists a \( \mathcal{A} \cdot \mathcal{B}_{Y'} \)-measurable selection \( s : \Omega \to Y' \) with

\[ s(t) \in S(t) \quad \forall t \in \Omega \quad \text{and} \quad z(t) = h(t, s(t)) \quad \text{for all} \quad t \in \Omega. \]

[2, Theorem 8.2.10, p. 316, together with Theorem 8.2.9, p. 315].
2 Relations between \((P)_K\) and \((P)_{K,B^k}\)

a) Two auxiliary results.

**Lemma 2.1.** Under assumptions \((V1)_K\) and \((V2)_K\), the sets \(U(t)\) are nonempty, convex and compact, satisfying \(K(o_{nm}, \gamma_1) \subseteq U(t) \subseteq K(o_{nm}, \gamma_2)\). Consequently, the assumption “\(p = \infty\)” in \((V1)_K\) can be replaced by “\(m < p < \infty\)”.

**Proof.** By [8, Proof of Theorem 1, p. 394], all \(U(t)\) are nonempty, convex and compact. By \((V2)_K\), it holds for arbitrary \(z \in K(o_{nm}, \gamma_1)\) and \(v \in \mathbb{R}^{nm}: z^T v = |z| \cdot |v| \cdot \cos \angle(z, v) \leq \gamma_1 |v| \leq r(t, v)\), and we see that \(K(o_{nm}, \gamma_1) \subseteq U(t)\). Conversely, if \(z \in U(t)\) then, choosing \(v = z/|z|\), we compute \(z^T v = |z| \leq r(t, z/|z|) \leq \gamma_2 |z/|z|| = \gamma_2\) what proves the inclusion \(U(t) \subseteq K(o_{nm}, \gamma_2)\). If, consequently, a function \(u \in L^{nm}_p(\Omega)\) with \(m < p < \infty\) satisfies the control restrictions \((1.3)\) then \(u\) is automatically element of \(L^{\infty}_\Omega(\Omega)\), and \((V1)_K\) may be formulated with \(m < p < \infty\) instead of \(p = \infty\).

**Lemma 2.2.** If the function \(r(t,v)\) is continuous in \(t\) then the set-valued map \(U(t) : \Omega \to \mathbb{P}(\mathbb{R}^{nm})\) is upper semicontinuous in the sense of [2, Definition 1.4.1, p. 38].

**Proof.** We apply [2, Proposition 1.4.8, p. 42], taking the ball \(K(o_{nm}, \gamma_2)\) endowed with the Euclidean metric as compact image space. Obviously,

\[(5) \quad \text{Graph}(U) = \{(t, z) \in \mathbb{R}^m \times \mathbb{R}^{nm} | t \in \Omega, z^T v \leq r(t, v) \forall v \in \mathbb{R}^{nm}\}.
\]

Consider a sequence \(\{(t^N, z^N)\} \to (t^*, z^*)\) with \((t^N, z^N) \in \text{Graph}(U)\). Then \(t^* \in \Omega\) since \(\Omega\) is closed. From \((t^N, z^N) \in \text{Graph}(U)\) it follows \((z^N)^T v \leq r(t^N, v)\) for all \(v \in \mathbb{R}^{nm}\), and, by continuity of \(r(\cdot, v), (z^*)^T v = \lim_{N \to \infty} (z^N)^T v \leq \lim_{N \to \infty} r(t^N, v) = r(t^*, v)\). Thus \((t^*, z^*) \in \text{Graph}(U)\). Graph \((U)\) is a closed subset of \(\Omega \times \mathbb{R}^{nm}\), and the set-valued map \(U(t)\) is upper semicontinuous.

b) An approximation theorem. The following theorem generalizes a result of Hüseinov [6] about \(C^\infty\)-approximations of Lipschitz functions. For its proof, we refer on the author’s paper [15] to be published simultaneously.

**Theorem 2.3** (Generalized Hüseinov’s theorem). Consider a set-valued map \(S(t) : \Omega \to \mathbb{P}(\mathbb{R}^{nm})\) with convex, compact, uniformly bounded images containing the ball \(K(o, \omega)\) as subset. Assume that \(S(t)\) is Lipschitz
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[2, Definition 1.4.5, p. 41]. Given further a Lipschitz function \( x^* \in W^{1,n}_\infty(\Omega) \) with \((x^*_{i,t_j}(t))_{ij} \in S(t)\) for a.e. \( t \in \Omega \). Then \( x^* \) can be approximated by a sequence of functions \( x^N \in C^{\infty,n}(\Omega) \) with

1. \( \lim_{N \to \infty} \|x^N - x^*\|_{C^{0,n}(\Omega)} = 0 \), \( x^N(t_0) = x^*(t_0) \),
2. \( \lim_{N \to \infty} \|x^N_{i,t_j} - x^*_{i,t_j}\|_{L_1(\Omega)} = 0 \) \( \forall i, j \),
3. \((x^N_{i,t_j}(t))_{ij} \in S(t)\) for all \( t \in \Omega \). [15, Theorem 1.5, p. 2].

c) Relations between \((P)_K\) and its relaxed problem. The standard relaxation of \((P)_K\) by use of generalized controls (Young measures) leads to the problem \((\bar{P})_K\) (6.1) – (6.4)

\[
\bar{J}(x, \mu) = -\sum_{k=1}^{n} \int_{\Omega} x_k(t) \, d\alpha_k(t) \longrightarrow \text{Min!}
\]
subject to \((x, \mu) \in W^{1,\nu}_p(\Omega) \times \mathcal{M}_U\), satisfying

\[
x_{i,t_j}(t) = \int_{U} v_{ij} \, d\mu_t(v) \text{ a.e. on } \Omega, \ \forall i, j,
\]

\[
\text{supp } \mu_t \subseteq U(t) = \{ z \in \mathbb{R}^{nm} \mid z^T v \leq r(t, v) \ \forall v \in \mathbb{R}^{nm} \} \ \forall t \in \Omega
\]

\[
x(t) = \varphi(t) \ \forall t \in \Gamma \text{ where } \Gamma \in \text{Comp}(\Omega), \ \Gamma \neq \emptyset.
\]

Since \((P)_K\) itself has a linear-convex structure, the problems \((P)_K\) and \((\bar{P})_K\) are equivalent in a sense specified in the following Theorem 2.4. In particular, their minimal values coincide, and there is a one-to-one correspondence between their minimal solutions. Thus in the frame of the present investigation the relaxed problem is of merely technical interest: it allows to evaluate the conditions of the maximum principle from \([12]\) which is designed for relaxed problems. Moreover, the equivalence between \((P)_K\) and \((\bar{P})_K\) leads to a simple existence proof for global minimizers of \((P)_K\).

**Theorem 2.4** (Equivalence of the problems \((P)_K\) and \((\bar{P})_K\)). Let \((P)_K\) satisfy assumptions \((V1)_K - (V4)_K\), and let the function \(r(t,v)\) be continuous in \(t\) for all \(v \in \mathbb{R}^{nm}\). Then for each feasible element \((x, \mu)\) of \((\bar{P})_K\) there exists a generalized control of the form \( \{ \sum_{s=1}^{nm+1} \lambda_s(t) \delta_{u_s(t)} \} \) with the following properties:

1. \( u_s \in L^{nm}_\infty(\Omega), u_s(t) \in U(t) \) for all \( t \in \Omega \);
2. \( \lambda_s(t) \in L_\infty(\Omega), 0 \leq \lambda_s(t) \leq 1 \) and \( \sum_{s} \lambda_s(t) = 1 \) for all \( t \in \Omega \);
3. \( \int_{U} v_{ij} \, d\mu_t(v) = \sum_{s} \lambda_s(t) u_{s,ij}(t) \) for all \( t \in \Omega \ \forall i, j \).
4) \( \tilde{J}(x, \mu) = J(x, \sum_s \lambda_s u_s) \),
so that the element \((x, \sum_s \lambda_s u_s)\) is feasible in \((P)_K\). Consequently, the problems \((P)_K\) and \((\bar{P})_K\) have the same minimal value.

**Proof.** At first, let us define for fixed \(t \in \Omega\) the set-valued maps \(M_U(t) : \Omega \to \mathcal{P}(\text{rca}(U, \mathcal{B}U))\) and \(Z(t) : \Omega \to \mathcal{P}(\mathbb{R}^{nm})\) by

\[
M_U(t) = \{ \mu_t \in \text{rca}(U, \mathcal{B}U) | \mu_t \geq 0, \text{supp } \mu_t \subseteq U(t), \mu_t(U(t)) = 1 \};
\]

\[
Z(t) = \{ z \in \mathbb{R}^{nm} | z_{ij} = \int_U v_{ij} d\mu_t(v), \mu_t \in M_U(t) \}.
\]

Choosing \(z', z'' \in Z(t)\) and \(\lambda \in [0, 1]\), it follows

\[
\lambda z_{ij} + (1 - \lambda) z''_{ij} = \int_U v_{ij} [\lambda d\mu^t(v) + (1 - \lambda) d\mu''_t(v)]
\]

with \(\text{supp } [\lambda \mu^t + (1 - \lambda) \mu''_t] \subseteq \text{supp } \mu^t \cup \text{supp } \mu''_t \subseteq U(t)\) and, consequently, \(\lambda \mu^t + (1 - \lambda) \mu''_t \in M_U(t)\). This proves the convexity of \(Z(t)\). Given a sequence \(\{z^N\} \to z^*\) with \(z^N \in Z(t)\) then there are representations

\[ z_{ij}^N = \int_U v_{ij} d\mu^N_t(v) \]

with \(\mu^N_t \in M_U(t)\), and the norm-bounded sequence \(\{\mu^N_t\}\) admits some subsequence \(\{\mu'^N_t\}\) converging to \(\mu^*_t\) in the sense of (3). It holds

\[
z_{ij}^* = \lim_{N' \to \infty} z_{ij}^{N'} = \lim_{N' \to \infty} \int_U v_{ij} d\mu'^N_t(v) = \int_U v_{ij} d\mu^*_t(v),
\]

and from [14, Proposition 1.5.1. (iii), p. 47 f.] it follows that \(\mu^*_t\) is also a probability measure. Thus \(Z(t)\) is closed, and from the continuity of the integrand and the uniform boundedness of the sets \(U(t)\) (Lemma 2.1) it follows also compactness. Since the cost functional does not depend on the control variables, the proof can be completed now as in [5, Assertion 8.3, p. 157 ff.], using the version of Filippov’s lemma given in Theorem 1.5 above.

**Theorem 2.5.** \((P)_K\) admits a global minimizer with \(\inf (P)_K = \inf (\bar{P})_K\).

Let \((P)_K\) satisfy assumptions \((V1)_K - (V4)_K\), and let \(r(t, v)\) be continuous in \(t\) for all \(v \in \mathbb{R}^{nm}\). Then there exists a global minimizer \((x^*, u^*)\) for \((P)_K\), and the problems \((P)_K\) and \((\bar{P})_K\) have the same minimal value. Furthermore, \((x^*, u^*)\) can be determined in such a way that the state equations (1.2) are satisfied everywhere on \(\Omega\).
Proof. In view to Theorem 2.4, it suffices to prove that the relaxed problem $(P)_K$ admits a global minimizer. Then by [12, Remark after Theorem 2.2, p. 224 f.] we have to check that 1) the basic assumptions (V1) – (V4) from [12] are satisfied (together with the feasibility of the zero solution $(o_n, o_{nm})$), this follows from our assumptions (V1)$_K$ – (V4)$_K$ and Lemma 2.1) and 2) $U(t): \Omega \to \Psi(\mathbb{R}^{nm})$ is upper semicontinuous in the sense of [2, p. 38, Definition 1.4.1] with nonempty, closed and uniformly bounded images (this is true by Lemmata 2.1 and 2.2). Finally, the assertion about the state equation (1.2) is proved by Theorem 2.4, 3).

**d) Comparison of the minimal values of $(P)_K$ and $(P)_{K,B_k}$.** In Theorem 1.2, sufficient conditions for the coincidence of the minimal values of $(P)_K$ and $(P)_{K,B_k}$ were formulated. We continue with its proof.

**Proof of Theorem 1.2.**

**Step 1.** We prove first that the set-valued map $U(t)$ is Lipschitz [2, Definition 1.4.5, p. 41]. Choosing $t', t'' \in \Omega$ and $z \in U(t')$, we have for arbitrary $v \in \mathbb{R}^{nm}$:

\begin{equation}
(11) \quad z^T v \leq r(t', v) = r(t'', v) + \left( r(t', v) - r(t'', v) \right).
\end{equation}

If $v = o_{nm}$ then from (V2)$_K$ it follows $r(t, o_{nm}) = 0$ for all $t \in \Omega$, and (11) gives $z^T o_{nm} \leq r(t'', o_{nm})$. Let $v \neq o_{nm}$, then it holds in consequence of the homogeneity of $r(t, \cdot)$ and of the assumption of the theorem:

\begin{equation}
(12) \quad r(t', v) = r(t'', v) + |v| \left( |r(t', v/|v|)| - |t'', v/|v|) \right) \\
\leq r(t'', v) + |v| \cdot L \cdot |t' - t''| \cdot \tilde{r}(v/|v|).
\end{equation}

Since $z \in K(o_{nm}, o)$ $\iff$ $z^T v \leq \omega \cdot v$ for all $v \in \mathbb{R}^{nm}$, it follows

\begin{equation}
(13) \quad z \in U(t'') + K(o_{nm}, o) \iff z^T v \leq r(t'', v) + \omega \cdot v \quad \forall v \in \mathbb{R}^{nm}.
\end{equation}

The nonnegative continuous function $\tilde{r}(v)$ takes on its maximum $c$ on the unit sphere of $\mathbb{R}^{nm}$, thus, by (12), $z$ is element of $U(t'') + K(o, cL|t' - t''|)$ what proves the Lipschitz continuity of $U(t)$.

**Step 2. Application of the generalized Hüseinov’s theorem.** In consequence of the assumptions, $r(t, v)$ is continuous in $t$, and we know then from Theorem 2.5 that $(P)_K$ possesses a global minimizer $(x^*, u^*)$. By Lemma 2.1
and Step 1, we can apply Theorem 2.3 to \(x^*_t\) and the set-valued map \(U(t)\). So there exists a sequence of functions \(x^N \in C^{\infty,n}(\Omega)\) with the following properties: They converge to \(x^*\) uniformly on \(\Omega\) and share the boundary value with \(x^*\) (so that the boundary condition (1.4) is satisfied), their weak derivatives come from the space \(C^{\infty,nm}(\Omega)\) and satisfy the inclusions \((x^N_i; t^N_j(t))_{ij} \in U(t)\) for all \(t \in \Omega\). Thus all pairs \((x^N, u^N)\) with \(u^N_{ij}(t) = x^N_{ij}(t)\) are feasible in \((P)_K\), and these elements satisfy the state equations (1.2) everywhere on \(\Omega\). From the uniform convergence of \(\{x^N\}\) it follows that 

\[ J(x^N, u^N) \to J(x^*, u^*) \]

and we find some subsequence of \(\{(x^N, u^N)\}\) being a minimizing sequence for \((P)_K\). Since all functions \(x^N_i; t^N_j\) are contained in \(C^{\infty,nm}(\Omega) \subset B^{0,nm}(\Omega) \subset B^{1,nm}(\Omega) \subset \ldots\), the proof is complete.

**Remark.** For more general boundary conditions with \(\Gamma \subseteq \partial \Omega\) and \(\varphi|\Gamma = c \in \mathbb{R}^n\), Theorems 1.2 and 2.5 remain true if there exists a feasible solution at all.

e) The maximum principle for \((P)_K\). By use of Theorem 2.4, the statements [12, Theorem 3.1, p. 225, and Theorem 3.4, p. 231] can be carried over to the unrelaxed deposit problem \((P)_K\).

**Theorem 2.6** (\(\varepsilon\)-maximum principle for \((P)_K\)). Let \((x^*, u^*)\) be a global minimizer of the problem \((P)_K\) under all assumptions of Theorem 2.4. Then for arbitrary \(\varepsilon > 0\) there exist multipliers \(y^\varepsilon \in L_q^{nm}(\Omega)\) \((p^{-1} + q^{-1} = 1)\) satisfying the \(\varepsilon\)-maximum condition (in integrated form), \((M)_\varepsilon\), and the canonical equation \((K)_\varepsilon\):

\[ (M)_\varepsilon: \quad \varepsilon + \sum_{i,j} \int_{\Omega} (u_{ij}(t) - u_{ij}(t)) y_{ij}^\varepsilon(t) dt \geq 0 \]

\[ \forall u \in L_{\infty}^{nm}(\Omega): u(t) \in U(t) \forall t \in \Omega \]

\[ (K)_\varepsilon: \quad \sum_{i,j} \int_{\Omega} y_{ij}^\varepsilon(t) \zeta_{ij}(t) dt - \sum_k \int_{\Omega} \zeta_k(t) d\alpha_k(t) = 0 \]

\[ \zeta \in W_p^{1,n}(\Omega): \zeta(t_0) = \varphi_n. \]

**Proof.** As mentioned above, the relaxed problem \((\overline{P})_K\) satisfies assumptions \((V1) - (V4)\) from [12], and thus we can apply [12, Theorem 3.1, p. 225]. Its proof in [12] is not influenced by the use of the generalized control restrictions supp \(\mu \subseteq U(t)\) in the definition of \(\mathfrak{S}_U\). If \((x^*, u^*)\) is a global minimizer of \((P)_K\) then \((x^*, \mu^*)\) with \(\mu^*_t = \delta_{u^*_t(t)}\) forms a global minimizer of \((\overline{P})_K\) since both problems have the same minimal value (Theorem 2.4) and 

\[ J(x^*, u^*) = \overline{J}(x^*, \mu^*). \]

By the above cited theorem, we find for arbitrary
$\varepsilon > 0$ multipliers $y_{ij}^\varepsilon \in L^q_{nm}(\Omega)$ which fulfill its $\varepsilon$-maximum condition and the canonical equation together with $(x^*, \mu^*)$. In the $\varepsilon$-maximum condition from [12],

\begin{equation}
\varepsilon + \sum_{i,j} \int_{\Omega} \int_U v_{ij} \left[ d\delta_{u^*(t)}(v) - d\mu_t(v) \right] y_{ij}^\varepsilon(t) \, dt \geq 0 \quad \forall \mu \in \mathcal{M}_U,
\end{equation}

we can substitute each generalized control $\mu \in \mathcal{M}_U$ by ordinary controls in the sense of Theorem 2.4 and vice versa, so that we arrive at $(\mathcal{M})_\varepsilon$ while $(\mathcal{K})_\varepsilon$ carries over formally unchanged. □

**Remark.** Theorem 2.6 differs from [8, Theorem 2, p. 395] in the choice of the spaces of the multipliers $y_{ij}^\varepsilon$ as well as of the test functions in the canonical equation.

**Theorem 2.7** (Maximum principle for $(P)_{K,B_1}$ with $\varepsilon = 0$). Let $(x^*, u^*)$ be a global minimizer of the problem $(P)_{K,B_1}$ (the weak derivatives $x^*_i; t_j$ have representatives from first Baire class) under all assumptions of Theorem 2.4. Then there exist multipliers $\nu \in (rca(\Omega, \mathcal{B}))^{nm}$ satisfying the maximum condition with $\varepsilon = 0$ (in integrated form), $(\mathcal{M})_0$, and the canonical equation $(\mathcal{K})_0$:

\[
(\mathcal{M})_0 : \quad \sum_{i,j} \int_{\Omega} (u^*_{ij}(t) - u_{ij}(t)) \, d\nu_{ij}(t) \geq 0
\]

\[
(\mathcal{K})_0 : \quad \sum_{i,j} \int_{\Omega} \zeta_{i; t_j}(t) \, d\nu_{ij}(t) - \sum_k \int_{\Omega} \zeta_k(t) \, d\alpha_k(t) = 0
\]

\[
\zeta \in C^{1,n}(\Omega); \zeta(t_0) = 0,
\]

**Proof.** By Lemma 2.1, the relaxed problem $(\overline{P})_K$ satisfies all assumptions of [12, Theorem 3.4, p. 231]. Its proof in [12] is also not influenced by the formal difference in the definition of $\mathcal{M}_U$. If $(x^*, u^*)$ is a global minimizer of $(P)_K$ having weak derivatives $x^*_{i; t_j}$ with representatives from the first Baire class then, as in the proof of Theorem 2.6, $(x^*, \mu^*)$ with $\mu^*_t = \delta_{u^*(t)}$ is a global minimizer of $(\overline{P})_K$. After correcting the error in the choose of the test function space in $(\mathcal{K})'_0$ ($\zeta \in C^{1,n}(\Omega)$ instead of $\zeta \in W^{1,n}_\infty(\Omega)$ with $\zeta_{i; t_j} \in \mathcal{B}^1(\Omega)$, see [13, Erratum]) and replacing in $(\mathcal{M})'_0$ the generalized controls $\mu \in \mathcal{M}_U$ by ordinary controls in the sense of Theorem 2.4. (even generating functions $x_{i; t_j}$ from the first Baire class on the whole
domain $\Omega$), one has derived from the conditions $(\mathcal{K})'_0$ and $(\mathcal{M})'_0$ of the above cited theorem the demanded conditions $(\mathcal{K})_0$ and $(\mathcal{M})_0$.

\section{Duality theorems}

\textbf{a) Construction of the dual problem.} Two optimization problems, a minimizing problem (P) and a maximizing problem (D), are said to be \textit{weakly dual} in the case that $\inf (P) \geq \sup (D)$, and \textit{strongly dual} if equality holds: $\inf (P) = \sup (D)$ (cf. Klötzler [7]). Under the assumptions of Theorem 1.2, the minimal values of the problems $(\mathcal{P})_{K}$, $(\mathcal{P})_{K,0}$ and $(\mathcal{P})_{K,B}$ coincide, and the dual problem can be formulated in relation to $(\mathcal{P})_{K,B}$. Thus it is possible to use Radon measures as dual variables.

\textbf{Definition 3.1.} We define the sets $X_0$, $X_1$ and $Y_0$ and a functional $\Phi: X_0 \times Y_0 \to \mathbb{R}$ by

\begin{equation}
X_0 = \{ (x, u) \in W^{1,n}_\infty (\Omega) \times L^{nm}_\infty (\Omega) | x_{i; t_j} \in \mathcal{B}^0(\Omega), u_{ij} \in \mathcal{B}^1(\Omega), u(t) \in U(t) \ \forall \ t \in \Omega, x(t_0) = o_n \};
\end{equation}

\begin{equation}
X_1 = \{ (x, u) \in W^{1,n}_\infty (\Omega) \times L^{nm}_\infty (\Omega) | x_{i; t_j}(t) = u_{ij}(t) \ \text{a.e. on} \ \Omega \};
\end{equation}

\begin{equation}
Y_0 = (rca (\Omega, \mathfrak{B}))^{nm};
\end{equation}

\begin{equation}
\Phi(x, u, \nu) = J(x, u) + \sum_{i,j} \int_{\Omega} [x_{i; t_j}(t) - u_{ij}(t)] d\nu_{ij}(t).
\end{equation}

\textbf{Lemma 3.2.} Let $(\mathcal{P})_{K}$ satisfy all assumptions of Theorem 1.2. Then the functional $\Phi(x, \mu, \nu)$ satisfies the equivalence condition

$$\inf_{(x,u)\in X_0 \cap X_1} J(x,u) = \inf_{(x,u)\in X_0} \sup_{\nu \in Y_0} \Phi(x, \mu, \nu).$$

\textbf{Proof.} Given a pair $(x, u) \in X_0$ where $x_{i_0; t_{j_0}}(t') - u_{i_0,j_0}(t') > 0$ (without loss of generality) for certain indices $i_0, j_0$ at a point $t' \in \Omega$. Then we have along the sequence of the measures $\nu^N \in (rca (\Omega, \mathfrak{B}))^{nm}$ with $\nu^N_{i_0,j_0} = N \cdot \delta_{t'}$ and $\nu^N_{ij} = o$ for $i \neq i_0$ or $j \neq j_0$

$$\lim_{N \to \infty} \Phi(x, u, \nu^N) = J(x, u) + \lim_{N \to \infty} N \cdot \left[ x_{i_0; t_{j_0}}(t') - u_{i_0,j_0}(t') \right] = +\infty.$$
It follows that \( \sup_{\nu \in Y_0} \Phi(x, u, \nu) = J(x, u) \) if \( (x, u) \in X_0 \) satisfies (1.2) for all \( t \in \Omega \) (consequently, \( (x, u) \in X_1 \)), and \( \sup_{\nu \in Y_0} \Phi(x, u, \nu) = +\infty \) else. By Theorem 1.2, \( (P)_{K,B_0} \) admits a minimizing sequence \( \{(x^N, u^N)\} \) of feasible processes which fulfill the state equations (1.2) everywhere on \( \Omega \). Along this sequence, we have \( \inf_{(x,u)\in X_0 \cap X_1} J(x, u) = \lim_{N \to \infty} J(x^N, u^N) = \lim_{N \to \infty} \sup_{\nu \in Y_0} \Phi(x^N, u^N, \nu) = \inf_{(x,u)\in X_0} \sup_{\nu \in Y_0} \Phi(x, u, \nu) \), and the proof is complete. \( \blacksquare \)

**Theorem 3.3** (Weak duality theorem for \( (P)_K \)). Let \( (P)_{K} \) satisfy all assumptions of Theorem 1.2. Then there is weak duality between each of the problems \( (P)_{K} \), \( (P)_{K,B_0} \) and \( (P)_{K,B_1} \); and the following problem \( (D)'_K \) (17.1) – (17.2):

\[
\text{(17.1) } G'(\nu) = \inf_{(x,u)\in X_0} \Phi(x, u, \nu) \to \text{Max!}
\]

\[
\text{(17.2) } \nu \in Y_0 = (\text{rca} (\Omega, \mathcal{B}))^{nm}.
\]

**Proof.** It holds \( \inf (P)_K = \inf (P)_{K,B_0} = \inf (P)_{K,B_1} \) (by Theorem 1.2); \( \inf (P)_{K,B_0} = \inf_{(x,u)\in X_0 \cap X_1} J(x, u) \) (by construction); \( \inf_{(x,u)\in X_0 \cap X_1} J(x, u) = \inf_{(x,u)\in X_0} \sup_{\nu \in Y_0} \Phi(x, u, \nu) \) (by Lemma 3.2) and, finally,

\[
\inf_{(x,u)\in X_0} \sup_{\nu \in Y_0} \Phi(x, u, \nu) \geq \sup_{\nu \in Y_0} \inf_{(x,u)\in X_0} \Phi(x, u, \nu) = \sup (D)'_K. \quad \blacksquare
\]

**b) Strong duality.** Note that \( G'(\nu) \) can be expressed as follows:

\[
G'(\nu) = \inf_{x \in C_1(\Omega), x(t_0) = x_0, u(t) \in B_1^{1,nm}(\Omega), \forall t \in \Omega} \left[ -\sum_k \int_{\Omega} x_k(t) d\alpha_k(t) + \sum_{i,j} \int_{\Omega} x_{i}(t) u_{ij}(t) dt - \sum_{i,j} \int_{\Omega} u_{ij}(t) dt \right].
\]

Then, by restriction of the feasible domain, we receive from \( (D)'_K \) the problem \( (D)_K \) (2.1) – (2.2) mentioned in the introduction. Obviously, it holds

\[
\sup (D)'_K \geq \sup (D)_K ; \quad G'(\nu) = G(\nu) \quad \text{for all } \nu \text{ feasible in } (D)_K.
\]

The feasible set of \( (D)_K \) is weak*-closed and convex, the cost functional \( G(\cdot) \) is concave in \( \nu \), and thus the set of the global maximizers of \( (D)_K \) is convex.

**Theorem 3.4** (Strong duality theorem for \( (P)_{K} \)). Let \( (P)_{K} \) satisfy all assumptions of Theorem 1.2. Then the problems \( (D)'_K \) and \( (D)_K \) have the same maximal value, and each of the problems \( (D)'_K \) and \( (D)_K \) is strongly dual to each of the problems \( (P)_{K}, (P)_{K,B_0}, (P)_{K,B_1} \) and \( (P)_K \).
Proof. By Theorem 2.5, \((P)_K\) admits a global minimizer \((x^*, u^*)\) (which is eventually not feasible in \((P)_{K,B^0}\) or \((P)_{K,B^1}\)). Then, by Theorem 2.6, for each \(\varepsilon_N = 1/N, N \in \mathbb{N}_1\), there exists a multiplier \(y^\varepsilon_N = y^N\) satisfying the conditions \((M)_{\varepsilon}^N\) and \((K)_{\varepsilon}^N\) together with \((x^*, u^*)\), and \(y^N\) can be interpreted as the density of a \(\lambda^m\)–absolutely continuous measure \(\nu^N\). By \((K)_{\varepsilon}\), each of the measures \(\nu^N\) is feasible in \((D)'_K\) as well as in \((D)_K\) (since \(C^{1,n}(\Omega) \subset W^{1,n}_p(\Omega)\)). Then it follows from (18):

\[
G'(\nu^N) = \inf_{u \in B^{1,\infty}(\Omega), u(x_0) = \phi, u(t) \in U(t) \forall t \in \Omega} \left[ J(x^*, u^*) - \sum_k \int_{\Omega} (x_k(t) - x_k^*(t)) d\alpha_k(t) + \sum_{i,j} \int_{\Omega} (x_i^*(t) - x_i^*(t)) d\nu^N_{ij}(t) dt \right]
\]

Together with the conditions \((M)_{\varepsilon}^N\), \((K)_{\varepsilon}^N\) and the feasibility of \((x^*, u^*)\) for \((P)_K\) we conclude that \(G'(\nu^N) \geq J(x^*, u^*) - 1/N\). Using Theorem 3.3 and (19), we arrive at the inequalities \(J(x^*, u^*) = \inf (P)_K \geq \sup (D)'_K \geq G(\nu^N) = G'(\nu^N) \geq J(x^*, u^*) - 1/N\) for all \(N \in \mathbb{N}_1\), and the relation \(\inf (P)_K = \sup (D)_K\) is proved.

c) Saddle-point conditions. Proof of Theorem 1.3. Assume that \(\nu^* \in (\text{rca } (\Omega, \mathcal{B}))/\lambda^m\) and a feasible pair \((x^*, u^*)\) of \((P)_{K,B^1}\) satisfy the conditions \((M)^*_0\), \((K)^*_0\) and \((D)^*_0\). By \((K)^*_0\), \(\nu^*\) is a feasible element of \((D)_K\). From \((M)^*_0\) we deduce

\[
G(\nu^*) = \inf_{u \in B^{1,\infty}(\Omega), u(t) \in U(t) \forall t \in \Omega} \left[ - \sum_{i,j} \int_{\Omega} u_{ij}(t) d\nu^*_{ij}(t) \right] = - \sum_{i,j} \int_{\Omega} u_{ij}(t) d\nu^*_{ij}(t),
\]

from which, together with \((D)^*_0\), it follows that \(J(x^*, u^*) = G(\nu^*)\), and \((x^*, u^*)\) and \(\nu^*\) form a saddle point for the problems \((P)_{K,B^1} - (D)_K\).

Remark. Since the value of the cost functional does not depend on \(u\) one can construct from a given global minimizer \((x^*, u^*)\) of \((P)_{K,B^1}\) non-denumerably many different global minimizers \((x^*, u^{**})\) of \((P)_{K,B^1}\) by the setting \(u^{**}_{ij}(t) = \chi_{(0 \setminus N_{ij})}(t) \cdot u^*_ij(t) + \chi_{N_{ij}}(t) \cdot u_{ij}(t)\). Here \(N_{ij}\) are \(\lambda^m\)-null sets with characteristic functions from the first Baire class while \(u \in B^{1,\infty}(\Omega)\) with \(u(t) \in U(t)\) for all \(t \in \Omega\) can be chosen arbitrarily. On this fact, it can be founded a partial converse of Theorem 1.3.
Theorem 3.5 (Partial converse of Theorem 1.3). Let \((P)_K\) satisfy all assumptions of Theorem 1.2. Assume that \((x^*, u^*)\) and \(\nu^*\) are feasible elements of \((P)_{K,B^1}\) resp. \((D)_K\) with \(J(x^*, u^*) = G(\nu^*)\). Let \(\nu_{ij}'\) and \(\nu_{ij}''\) denote the absolutely continuous resp. singular parts of the components of \(\nu^*\) in the Lebesgue decomposition with respect to \(\lambda^m\). Further assume that 
\[ \lambda^m(\text{supp} \nu_{ij}'') = 0, \lambda^m(\text{supp} \nu_{ij}'') = 0 \quad \text{and} \quad \text{supp} \nu_{ij}' + \text{supp} \nu_{ij}'' = \emptyset \forall i, j. \]

Then there exists a function \(u^{**} \in B^{1,nm}(\Omega)\) with the following properties:

1) \(u^*(t) = u^{**}(t)\) for a.e. \(t \in \Omega\) (\(u^*\) and \(u^{**}\) belong to the same \(L^{nm}_\infty\)-equivalence class).

2) \(u^{**}(t) \in U(t) \forall t \in \Omega\) (the pair \((x^*, u^{**})\) is feasible in \((P)_{K,B^1}\)).

3) \(J(x^*, u^*) = J(x^*, u^{**})\) (the triple \((x^*, u^{**}, \nu^*\)) forms also a saddle point for the problems \((P)_{K,B^1} - (D)_K\).

4) The triple \((x^*, u^{**}, \nu^*)\) satisfies the saddle-point conditions \((\mathcal{M})^*_0\), \((\mathcal{K})^*_0\) and \((D)_0^*\) of Theorem 1.3.

Proof. From the feasibility of \(\nu^*\) in \((D)_K\) it follows that \((\mathcal{K})^*_0\) is valid. Now we distinguish two cases:

Case 1. \(u^*\) and \(\nu^*\) satisfy \((\mathcal{M})^*_0\), i.e.

\[ -\sum_{i,j} \int_{\Omega} u^*_{ij}(t) \, d\nu_{ij}^* = \inf_{u \in B^{1,nm}(\Omega)} \left[ -\sum_{i,j} \int_{\Omega} u_{ij}(t) \, d\nu_{ij}^*(t) \right] = G(\nu^*). \]

Then from \(J(x^*, u^*) = G(\nu^*)\) it results \((D)_0^*\), and the theorem is valid with \(u^*(t) = u^{**}(t)\) for all \(t \in \Omega\).

Case 2. \(u^*\) and \(\nu^*\) violate \((\mathcal{M})^*_0\) what means

\[ -\sum_{i,j} \int_{\Omega} u^*_{ij}(t) \, d\nu_{ij}^* > \inf_{u \in B^{1,nm}(\Omega)} \left[ -\sum_{i,j} \int_{\Omega} u_{ij}(t) \, d\nu_{ij}^*(t) \right] = G(\nu^*). \]

Here and below, the infimum is taken over the same function set as in (21). Using the members \((x^N, u^N) \in C^{\infty,n}(\Omega) \times C^{\infty,nm}(\Omega)\) of the minimizing sequence \(\{(x^N, u^N)\}\) from Theorem 1.2 as test functions in \((\mathcal{K})^*_0\), it follows:

\[ J(x^N, u^N) = -\sum_k \int_{\Omega} x^N_k(t) \, d\lambda(t) = -\sum_{i,j} \int_{\Omega} u^N_{ij}(t) \, d\nu^N_{ij}(t) \implies \]
\[ J(x^*, u^*) = \lim_{N \to \infty} J(x^N, u^N) = \lim_{N \to \infty} \left[ -\sum_{i,j} \int_{\Omega} u_{ij}^N(t) \, d\nu_{ij}^*(t) \right] = G(\nu^*). \]

We subject each of the measures \( \nu_{ij}^* \) to the Lebesgue decomposition w. r. to the measure \( \lambda^m \) into the absolutely continuous part \( \nu_{ij}' \) and the singular part \( \nu_{ij}'' \) [4, Theorem 14, p. 132]. The densities of the absolutely continuous parts are denoted by \( y_{ij}' \in L_1(\Omega) \). Since the functions \( u_{ij}^N \) are bounded on \( \Omega \), from \( u_{ij}^N \to L_1(\Omega) \) \( u_{ij}^* \) it follows the convergence \( u_{ij}^N \to y_{ij}' \) in \( L_1(\Omega) \) and we have

\[
\inf_{u \in \ldots} \left[ -\sum_{i,j} \int_{\Omega} u_{ij}(t) \, d\nu_{ij}^*(t) \right] = \lim_{N \to \infty} \left[ -\sum_{i,j} \int_{\Omega} u_{ij}^N(t) \, d\nu_{ij}^*(t) \right]

= -\sum_{i,j} \int_{\Omega} u_{ij}^*(t) y_{ij}'(t) \, dt - \lim_{N \to \infty} \sum_{i,j} \int_{\Omega} u_{ij}^N(t) \, d\nu_{ij}''(t).
\]

Further, the singular parts are subjected to the Jordan decomposition \( \nu_{ij}'' = \nu_{ij}''^+ - \nu_{ij}''^- \) [4, p. 98, Theorem 8]; both parts are still Radon measures [4, Lemma 12, p. 137] whose supports, by assumption, are compact \( \lambda^m \)-null sets. We abbreviate: supp \( \nu_{ij}''^+ = \text{N}_{ij}^+ \), supp \( \nu_{ij}''^- = \text{N}_{ij}^- \), \( \text{N}_{ij} = \text{N}_{ij}^+ \cup \text{N}_{ij}^- \) and define the functions

\[
u_{ij}^*(t) = \chi_{(\Omega \cap \text{N}_{ij})}(t) u_{ij}^*(t) + \chi_{\text{N}_{ij}^+}(t) \inf_N u_{ij}^N(t) + \chi_{\text{N}_{ij}^-}(t) \sup_N u_{ij}^N(t).
\]

All \( u_{ij}^* \) are contained in the first Baire class since the characteristic functions \( \chi_{(\Omega \cap \text{N}_{ij})} \), \( \chi_{\text{N}_{ij}^+} \) and \( \chi_{\text{N}_{ij}^-} \) (cf. [12, Lemma 1.4, p. 220]) as well as the pointwise infimum resp. supremum of the sequence \( \{u_{ij}^N\} \) of continuous functions have the same property [3, Theorem 10, p. 398]. The values of the functions \( u_{ij}^* \) and \( u_{ij}^* \) differ at most on the null sets \( \text{N}_{ij} \). By Theorem 1.2, we have \( u_{ij}^N(t) \in U(t) \) for all \( N \in \mathbb{N}_1 \) and for all \( t \in \Omega \); then it follows from the closedness of the sets \( U(t) \) (Lemma 2.1) that \( \inf_N u_{ij}^N(t) \in U(t) \) as well as \( \sup_N u_{ij}^N(t) \in U(t) \) for all \( t \in \Omega \). Together with \( \text{N}_{ij}^+ \cap \text{N}_{ij}^- = \emptyset \) (by assumption) it results that \( u^{**}(t) \in U(t) \) for all \( t \in \Omega \). Thus \( u^{**} \) fulfills the assertions 1) – 3) of our theorem.

We have still to prove that \( (x^*, u^{**}, \nu^*) \) satisfies the saddle-point conditions. For this purpose, let us introduce the following abbreviations:

\[
L = \lim_{N \to \infty} \sum_{i,j} \int_{\Omega} u_{ij}^N(t) \, d\nu_{ij}^*(t); \quad L' = \lim_{N \to \infty} \sum_{i,j} \int_{\Omega} u_{ij}^N(t) \, (y_{ij}')(t) \, dt;
\]

\[
L'' = \lim_{N \to \infty} \sum_{i,j} \int_{\text{N}_{ij}} u_{ij}^N(t) \, d\nu_{ij}''(t);
\]
As immediate consequence of (28), the single equations (28)

\[ J = \inf_{u \in \mathcal{U}} \left[ - \sum_{i,j} \int_{\Omega} u_{ij}(t) \, dv_{ij}^*(t) \right]; \quad J' = \inf_{u \in \mathcal{U}} \left[ - \sum_{i,j} \int_{\Omega} u_{ij}(t) \, y_{ij}(t) \, dt \right]; \]

\[ J'' = \inf_{u \in \mathcal{U}} \left[ - \sum_{i,j} \int_{\Omega} u_{ij}(t) \, dv_{ij}''(t) \right]. \]

In these notations, it holds obviously

(26) \[ L' + L'' = L = J \geq J' + J''. \]

Here \( J > J' + J'' \) leads to a contradiction since one could choose then functions \( u', u'' \in \mathcal{B}_{1,nm}^1(\Omega) \) with \( u'(t) \in \mathcal{U}(t) \) and \( u''(t) \in \mathcal{U}(t) \) for all \( t \in \Omega \) in such a way that

(27) \[ J > - \sum_{i,j} \int_{\Omega} u_{ij}'(t) y_{ij}'(t) \, dt - \sum_{i,j} \int_{N_{ij}} u_{ij}''(t) \, dv_{ij}''(t) \]

but \( u_{ij}(t) = \chi_{(\Omega, N_{ij})}(t) \cdot u'(t) + \chi_{N_{ij}}(t) \cdot u''(t) \) would be feasible for the construction of \( J \) as a function of first Baire class. So we have

(28) \[ L' + L'' = J' + J''. \]

As immediate consequence of (28), the single equations \( L' = J' \) and \( L'' = J'' \) result since \( L' < J' \) as well as \( L'' < J'' \) are impossible. \( L' = J' \) means

(29) \[ - \sum_{i,j} \int_{\Omega} u_{ij}'(t) y_{ij}'(t) \, dt = - \sum_{i,j} \int_{\Omega} u_{ij}''(t) y_{ij}'(t) \, dt \]

\[ = \inf_{u \in \mathcal{U}} \left[ - \sum_{i,j} \int_{\Omega} u_{ij}(t) \, y_{ij}'(t) \, dt \right]. \]

Further, it holds for all \( N \in \mathbb{N}_1 \):

(30) \[ - \sum_{i,j} \int_{\Omega} u_{ij}^N(t) \, dv_{ij}''(t) = - \sum_{i,j} \int_{N_{ij}}^+ u_{ij}^N(t) \, dv_{ij}''^+(t) + \sum_{i,j} \int_{N_{ij}}^- u_{ij}^N(t) \, dv_{ij}''^-(t) \]

\[ \geq - \sum_{i,j} \int_{N_{ij}}^N \inf(u_{ij}^N(t)) \, dv_{ij}''^+(t) + \sum_{i,j} \int_{N_{ij}}^- \sup(u_{ij}^N(t)) \, dv_{ij}''^- (t) \]

\[ = - \sum_{i,j} \int_{\Omega} u^*_{ij}(t) \, dv_{ij}''(t) \geq J''. \]

The last inequality results from the fact that \( u^* \) is feasible for the construction of \( J'' \). After the limit passage \( N \to \infty \) in (30) we arrive at

(31) \[ J'' = L'' \geq - \sum_{i,j} \int_{\Omega} u^*_{ij}(t) \, dv_{ij}''(t) \geq J''. \]
Together with (28), it results from equations (29) and (31):

\[
(32) \quad -\sum_{i,j} \int_{\Omega} u_{ij}^{**}(t) \, d\nu_{ij}^*(t) = J' + J'' = J = \inf_{u \in \mathcal{M}} \left[ -\sum_{i,j} \int_{\Omega} u_{ij}(t) \, d\nu_{ij}^*(t) \right],
\]

so that \( u^{**} \) and \( \nu^* \) satisfy condition \((\mathcal{M})_0^*\). As in the former case, \((\mathcal{D})_0^*\) is then satisfied also, and the proof is complete. 

References


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