ON CENTRALIZER OF SEMIPRIME INVERSE SEMIRING

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Abstract

Let $S$ be 2-torsion free semiprime inverse semiring satisfying $A_2$ condition of Bandlet and Petrich [1]. We investigate, when an additive mapping $T$ on $S$ becomes centralizer.

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1. Introduction and preliminaries

Throughout this paper, $S$ we will represent inverse semiring which satisfies $A_2$ condition of Bandlet and Petrich [1]. $S$ is prime if $aSb = (0)$ implies either $a = 0$ or $b = 0$ and $S$ is semiprime if $aSa = (0)$ implies $a = 0$. $S$ is n-torsion free if $nx = 0$, $x \in S$ implies $x = 0$. Following Zalar [12], we canonically define left(right) centralizer of $S$ as an additive mapping $T : S \to S$ such that

$T(xy) = T(x)y \ (xT(y)), \ \forall x, y \in S$ and $T$ is called centralizer if it is both right and left centralizer.

Bresar and Zalar [2] have proved that an additive mapping $T$ on 2-torsion free prime ring $R$ which satisfies weaker condition $T(x^2) = T(x)x$ is a left centralizer. Later, Zalar [12] generalized this result for semiprime rings. Motivated by the work of Zalar [12], Vukman [10] proved that an additive mapping on 2-torsion free semiprime ring satisfying $T(xyz) = xT(y)x$ is a centralizer. In this paper, our objective is to explore the result of Vukman [10] in the setting of inverse semirings as follows: Let $S$ be 2-torsion free semiprime inverse semiring and let
Let $T : S \to S$ be additive mapping such that $T(xy) + xT(y)x = 0$ holds $\forall x, y \in S$ then $T$ is a centralizer.

To prove this result we will first generalize Proposition 1.4 of [12] in the framework of inverse semirings.

By semiring we mean a nonempty set $S$ with two binary operations $'+'$ and '$\cdot'$ such that $(S, '+')$ and $(S, \cdot)$ are semigroups where $+$ is commutative with absorbing zero 0, i.e., $a + 0 = 0 + a = a$, $0.0 = 0a \forall a \in S$ and $a(b + c) = a.b + a.c$, $(b + c)a = b.a + c.a$ holds $\forall a, b, c \in S$. Introduced by Karvellas [6], a semiring $S$ is an inverse semiring if for every $a \in S$ there exist a unique element $\hat{a} \in S$ such that $a + \hat{a} + a = a$ and $\hat{a} + a + \hat{a} = \hat{a}$, where $\hat{a}$ is called pseudo inverse of $a$. Karvellas [6] proved that for all $a, b \in S$, $(a.b) = \hat{a}.b = a.\hat{b}$ and $\hat{a}b = ab$.

In this paper, inverse semirings satisfying the condition that for all $a \in S$, $a + \hat{a}$ is in center $Z(S)$ of $S$ are considered (see [4] for more details). Commutative inverse semirings and distributive lattices are natural examples of inverse semirings satisfying $A_2$. In a distributive lattice pseudo inverse of every element is itself. Also if $R$ is commutative ring and $I(R)$ is semiring of all two sided ideals of $R$ with respect to ordinary addition and product of ideals and $T$ is subsemiring of $I(R)$ then set $S_1 = \{(a, I) : a \in R, I \in T\}$. Define on $S_1$ addition $\oplus$ and multiplication $\odot$ by $(a, I) \oplus (b, J) = (a + b, I + J)$ and $(a, I) \odot (b, J) = (ab, IJ)$.

It is easy to see $S_1$ is an inverse semiring with $A_2$ condition where $(a, I) = (\hat{a}, I)$.

By [4], commutator $[\cdot, \cdot]$ in inverse semirings defines as $[x, y] = xy + yx$. We will make use of commutator identities $[x, y + z] = [x, y] + [x, z]$, $[x, y]z = [x, z]y + [x, y]z$ and $[x, y] + z = x[x, y]z + y[x, z]$ (see [4] for their proofs).

The following Lemmas are useful in establishing main result.

**Lemma 1.1.** For $a, b \in S$, $a + b = 0$ implies $a = \hat{b}$.

**Proof.** Let $a + b = 0$ which implies $a + \hat{a} + \hat{b} + \hat{b} = 0$ or $a + b + \hat{a} + \hat{b} + a = a$ or $a + b + \hat{b} = a$ and by hypothesis, we get $a = \hat{b}$.

However, converse of Lemma 1.1. is not true for instance, in distributive lattice $D$, for $a \in D$ we have $a = \hat{a}$ but $a + a = a$.

**Lemma 1.2.** If $x, y, z \in S$ then following identities are valid:

1. $[xy, x] = x[y, x]$, $[x, yx] = [x, y]x$, $[x, xy] = x[x, y]$, $[yx, x] = [y, x]x$
2. $y[x, z] = [x, yz] + [x, y]\hat{z}$, $[x, y]z = \hat{y}[x, z] + [x, yz]$
3. $x[y, z] = [xy, z] + [x, z]\hat{y}$, $[x, z]y = [xy, z] + \hat{x}[y, z]$

**Proof.**

1. $[xy, x] = xyx + \hat{x}xy = x(yx + \hat{y}x) = x[y, x]$.
2. $y[x, z] = (y + \hat{y} + y)(xz + \hat{z}x) = (y + \hat{y})xz + (y + \hat{y})\hat{z}x + yxz + y\hat{z}x = x(y + \hat{y})z + (y + \hat{y})\hat{z}x + yxz + y\hat{z}x = x[y, z] + [x, y]\hat{z}$.

Proof of the other identities can be obtained using similar techniques.
In the following, we extend Lemma 1.1 of Zalar [12] in a canonical fashion.

**Lemma 1.3.** Let $S$ be a semiprime inverse semiring such that for $a, b \in S$, $axb = ab = ba = 0$.

**Definition 1.4.** A mapping $f : S \times S \to S$ is biadditive if $f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$ and $f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$, for all $x, y, x_1, x_2, y_1, y_2 \in S$.

**Example.** Define mappings $f, g : S_1 \times S_1 \to S_1$ by $f((a, I), (b, J)) = (ab, IJ)$ and $g((a, I), (b, J)) = ([a, b], IJ)$. Then $f$ and $g$ are biadditive.

Also, if $(D, \land, \lor)$ is a distributive lattice then $h : D \times D \to D$ defined by $h(a, b) = a, \forall a, b \in D$ is a biadditive mapping.

**Lemma 1.5.** Let $S$ be semiprime inverse semiring and $f, g : S \times S \to S$ are biadditive mappings such that $f(x, y)wg(x, y) = 0, \forall x, y, w \in S$, then $f(x, y)wg(s, t) = 0, \forall x, y, s, t, w \in S$.

**Proof.** Replace $x$ with $x + s$ in $f(x, y)wg(x, y) = 0$, we get $f(s, y)wg(x, y) + f(x, y)wg(s, y) = 0$. By Lemma 1.1, we have $f(x, y)wg(s, y) = f(s, y)wg(x, y)$. This implies $(f(x, y)wg(s, y))z(f(x, y)wg(x, y)) = (f(s, y)wg(x, y))z(f(x, y)wg(s, y)) = 0$ and semiprimeness of $S$ implies that $f(x, y)wg(s, y) = 0$. Now replacing $y$ with $y + t$ in last equation and using similar approach we get the required result.

**Lemma 1.6.** Let $S$ be a semiprime inverse semiring and $a \in S$ some fixed element. If $a[x, y] = 0$ for all $x, y \in S$, then there exists an ideal $I$ of $S$ such that $a \in I \subset Z(S)$ holds.

**Proof.** By Lemma 1.2, we have $[z, a]x[z, a] = za[x, a] + zax[z, a] = za([z, xa] + [z, x]a) + [z, x]a \land [z, ax] = za[z, xa] + za[z, xa]a + a(z, xa)a = 0$.

Using semiprimeness of $S$ and then Lemma 1.1, we get $a \in Z(S)$. By Lemma 1.2, we have $zaw[x, y] = za([x, w]y) = 0, \forall x, y, z, w \in S$. By similar argument, we can show that $zaw \in Z(S)$ and hence $SaS \subset Z(S)$. Now it is easy to see that ideal generated by $a$ is central.

**Lemma 1.7.** Let $S$ be semiprime inverse semiring and $a, b, c \in S$ such that

\[(1)\quad axb + bxc = 0\]

holds for all $x \in S$ then $(a + c)xb = 0$ for all $x \in S$.

**Proof.** Replace $x$ with $xyb$ in (1), we get

\[(2)\quad axbyb + bxbyc = 0, \quad x, y \in S.\]
Post multiplying (1) by \(yb\) gives

\[
(3) \quad axbyb + bxcyb = 0, \quad x, y \in S.
\]

Applying Lemma 1.1 on (2) and using it in (3), we have

\[
(4) \quad bx(byc + cyb) = 0, \quad x, y \in S.
\]

Replace \(x\) with \(ycx\) in (4), we get

\[
(5) \quad bycx(byc + cyb) = 0, \quad x, y \in S.
\]

Pre multiplying (4) by \(cy\) gives

\[
(6) \quad cybx(byc + cyb) = 0, \quad x, y \in S.
\]

Adding pseudo inverse of (5) and (6) we get

\[
(byc + cyb) x(byc + cyb) = 0, \quad x, y \in S.
\]

Using semiprimeness of \(S\) and Lemma 1.1, we get \(byc = cyb, y \in S\). By using last relation in (1) we get the required result.

2. Main results

**Theorem 2.1.** Let \(S\) be a 2-torsion free semiprime inverse semiring and \(T : S \rightarrow S\) be an additive mapping which satisfies \(T(x^2) + T(x)\dot{x} = 0, \forall x \in S\). Then \(T\) is a left centralizer.

**Proof.** Take,

\[
(7) \quad T(x^2) + T(x)\dot{x} = 0, \quad x \in S.
\]

Linearization of (7) gives

\[
(8) \quad T(xy + yx) + T(x)\dot{y} + T(y)\dot{x} = 0, \quad x, y \in S.
\]

Replace \(y\) with \(xy + yx\) in (8), we get

\[
(9) \quad T(x^2y + yx^2) + 2T(xy) + T(xy)\dot{x} + T(yx)\dot{x} + T(x)yx + T(x)xy = 0.
\]

Using Lemma 1.1 in (8) and using it in (9), we have

\[
(10) \quad T(x^2y + yx^2) + 2T(xy) + T(x)y\dot{x} + T(y)\dot{x}^2 + T(x)y\dot{x} + T(x)xy = 0.
\]
Using Lemma 1.1 in (7) and using it in (10) we get
\[(11) \quad T(x^2y + yx^2) + 2T(xy^2) + T(x)y^2 + T(y)x^2 + T(x^2)y = 0.\]
Replace \(x\) with \(x^2\) in (8) we get
\[(12) \quad T(x^2y + yx^2) + T(x^2)y + T(y)x^2 = 0.\]
Using (12) in (11), we get
\[
2T(xy^2) + 2T(x)y^2 = 0.
\]
As \(S\) is 2-torsion free, so we have
\[(13) \quad T(xyx) + T(x)y^2 = 0.\]
Linearization (by \(x = x + z\)) of (13) gives
\[(14) \quad T(xyz + yzx) + T(x)y^2 + T(z)y^2 = 0.\]
Replace \(x\) with \(xy, z \) with \(yx\) and \(y\) with \(z\) in (14), we get
\[(15) \quad T(xyzx + yzx) + T(xy)zy^2 + T(yzx) = 0.\]
Replace \(y\) with \(yzy\) in (13), we get
\[(16) \quad T(xyzx) + T(x)yzy^2 = 0.\]
Replace \(x\) with \(y\) and \(y\) with \(xzx\) in (13), we get
\[(17) \quad T(yxx) + T(y)zx^2 = 0.\]
By adding (16) and (17), we get
\[(18) \quad T(xyzx + yzx) + T(xy)zy^2 + T(y)xzx^2 = 0.\]
Using Lemma 1.1 in (15) and using the result in (18), we get
\[(19) \quad T(xyzx) + T(xy)zy^2 + T(y)zxx^2 = 0.\]
Now if we define biadditive function \(f : S \times S \to S\) by \(f(x, y) = T(xy) + T(x)y^2\),
then (19) can be written as
\[(20) \quad f(x, y)zyx + f(y, x)zx = 0.\]
From (8) and Lemma 1.1, we have
\[
(f(x, y))' = f(y, x).
\]
Thus (20) can be rewritten as

\[ f(x, y)zyx + f(x, y)zxy = 0, \quad \text{or} \]
\[ f(x, y)z[x, y] = 0, \quad x, y, z \in S. \]

Using Lemma 1.5 and then Lemma 1.3, we have \( f(x, y)[s, t] = 0, \) \( x, y, s, t \in S. \)

Now fix \( x, y \) then by Lemma 1.6, there exist ideal \( I \subset Z(S) \) such that \( f = f(x, y) \in I \subset Z(S) \). This implies that \( bf, fb \in Z(S), \forall b \in S \), thus we have

(21) \[ xfy = xyf = fxy = yfx \quad \text{and} \]
(22) \[ xf^2y = f^2xy = yf^2x = f^2yx. \]

Replace \( y \) with \( f^2y \) in (8), we get

\[ 2T(xf^2y + f^2yx) + 2T(x)f^2\dot{y} + 2T(f^2y)\dot{x} = 0. \]

Using (22), we get

(23) \[ 2T(yf^2x + f^2xy) + 2T(x)f^2\dot{y} + 2T(f^2y)\dot{x} = 0. \]

By Lemma 1.1, (8), (7) and (23), we have

\[ 2T(y)f^2x + 2T(f^2x)y + 2T(x)f^2\dot{y} + 2T(f^2y)\dot{x} = 0, \quad \text{or} \]
\[ 2T(y)f^2x + T(f^2x + f^2x)y + 2T(x)f^2\dot{y} + T(f^2y + f^2y)\dot{x} = 0, \quad \text{or} \]
\[ 2T(y)f^2x + T(f^2x + xf^2)y + 2T(x)f^2\dot{y} + T(f^2y + yf^2)\dot{x} = 0, \quad \text{or} \]
\[ 2T(y)f^2x + T(f^2)x y + T(x)f^2\dot{y} + T(f^2)y \dot{x} + T(y)f^2x = 0, \quad \text{or} \]
\[ 2T(y)f^2x + T(y)f^2\dot{x} + T(f^2)xy + T(x)f^2y + 2T(x)f^2\dot{y} + T(f^2)y \dot{x} = 0, \quad \text{or} \]
\[ T(y)f^2x + T(f)fxy + T(x)f^2\dot{y} + T(f)fy\dot{x} = 0, \quad \text{or} \]
(24) \[ T(f)x f^2 + T(x)\dot{f} ^2y + T(f)y (\dot{x} + x) = 0. \]

Now replace \( x \) with \( xy \) and \( y \) with \( f^2 \) in (8) and then using (21) and (22), we get

\[ 2T(fxfy + fgyf) + 2T(xy)f^2 + 2T(f^2)\dot{xy} = 0. \]

By Lemma 1.1, (8) and (7), we have

\[ 2T(f)xfy + 2T(fy)f + 2T(xy)f^2 + 2T(f^2)\dot{xy} = 0, \quad \text{or} \]
\[ T(f + fx)fy + T(fy + fy)fx + 2T(xy)f^2 + 2T(f^2)\dot{xy} = 0. \]
\[ T(fx + xf)fy + T(fy + yf)fx + 2T(xy)f^2 + 2T(f^2)\dot{x}y = 0 \]
\[ T(f)xy + T(x)fy + T(y)fxf + 2T(xy)f^2 + 2T(f^2)\dot{x}y = 0 \]
\[ T(f)xfy + T(x)f^2y + T(y)fxy + 2T(xy)f^2 + 2T(f)\dot{x}y = 0 \]
\[ T(f)fxy + 2T(f)\dot{x}y + T(f)fxy + T(x)f^2y + yT(y)f^2x + 2T(xy)f^2 = 0 \]
\[ T(f)fxy(\dot{x} + x) + T(x)f^2y + T(y)f^2x + 2T(xy)f^2 = 0. \]

Using Lemma 1.1 in (24) and using the result in last equation, we get
\[ 2T(x)f^2y + 2T(xy)f^2 = 0, \text{ or} \]
\[ T(x)f^2y + T(xy)f^2 = 0, \text{ or} \]
\[ (T(x)\dot{y} + T(xy))f^2 = 0 \text{ or } f^3 = 0 \text{ which implies} \]
\[ f^3Sf^2 = f^4 = (0) \Rightarrow f^2 = 0. \]

Thus \( fSf = f^2S = (0) \Rightarrow f = 0. \) Therefore \( T(xy) + T(x)\dot{y} = 0 \) and then Lemma 1.1 implies that \( T \) is a left centralizer.

**Theorem 2.2.** Let \( S \) be a 2-torsion free semiprime inverse semiring and let \( T : S \to S \) be an additive mapping such that
\[ (26) \quad T(xy) + xT(y)\dot{x} = 0, \forall x, y \in S. \]

Then \( T \) is a centralizer.

**Proof.** First we show that
\[ [[T(x), x], x] = 0. \]

Linearization of (26) gives
\[ (27) \quad T(xy + zy) + xT(y)\dot{z} + zT(y)\dot{x} = 0, \forall x, y, z \in S. \]

Replace \( y \) with \( x \) and \( z \) with \( y \) in last equation, we get
\[ (28) \quad T(x^2y + yx^2) + xT(x)\dot{y} + yT(x)\dot{x} = 0. \]

Replace \( z \) with \( x^3 \) in (27), we get
\[ (29) \quad T(xy^3 + x^3yx) + xT(y)\dot{x}^3 + x^3T(y)\dot{x} = 0. \]
Replace $y$ with $xyx$ in (28), we get

\[(30) \quad T(x^3y + x^3yx) + xyT(x)x + xT(x)xyx = 0.\]

Replace $y$ with $x^2y + yx^2$ in (26), we have

\[(31) \quad T(x^3y + x^3yx) + xT(x^2y + yx^2)x = 0.\]

Using Lemma 1.1 in (30) and using the result in (31), we get

\[(32) \quad xyT(x)x + xT(x)xyx + xT(x^2y + yx^2)x = 0, \quad \text{or} \quad x[T(x), x]yx + xyT(x), x]x = 0.\]

Using Lemma 1.7 in (32), we have

\[(33) \quad ([T(x), x]yx = 0, \quad \text{or} \quad x[T(x), x] + [T(x), x]x = 0.\]

Replace $y$ with $y[T(x), x]$ in (33), we have

\[(34) \quad [[T(x), x], x]y[T(x), x]x = 0.\]

Post multiplication (33) with $[T(x), x]$ gives

\[(35) \quad [[T(x), x], x]yx[T(x), x] = 0.\]

Adding pseudo inverse of (35) and (34), we have $[[T(x), x], y[[T(x), x], x] = 0$ and then semiprimeness of $S$ implies that

\[(36) \quad [[T(x), x], x] = 0, \quad \forall x \in S \quad \text{or} \quad [T(x), x]x + x[T(x), x] = 0 \quad \text{or} \quad [T(x), x]x + (x + \dot{x})[T(x), x] = x[T(x), x], \quad \text{or} \quad [T(x), x]x + [T(x), x][x + \dot{x}] = x[T(x), x], \quad \text{or} \quad [T(x), x]x = x[T(x), x], \quad \forall x \in S.\]
Linearization of (36) gives

\[
[T(x), x], y + [T(y), x] + [T(y), x] + [T(y), x] = 0.
\]

(38)

Replace \( x \) with \( \dot{x} \) in (38) and using again (38) and the fact that \( (T(x)) = T(\dot{x}) \) we have

\[
2[[T(x), x], y] + 2[[T(x), x], x] + [[T(y), x], y + \dot{y}]
\]

(39)

Adding (38) in (39) and then using (38) again, we get

\[
[[T(x), x], y] + [[T(x), y], x] + [[T(y), x], x] = 0, \forall x, y \in S.
\]

(40)

Replacing \( y \) with \( xyx \) in (40), we have

\[
[[T(x), x], xyx] + [[T(x), x], x] + [[T(y), x], x] = 0, \forall x, y \in S.
\]

Using Lemma 1.1 in (26) and using it in last equation, we get

\[
[[T(x), x], y] + [[T(x), y], x] + [[xT(y)x], x] = 0.
\]

Using Lemma 1.2, we have

\[
[[T(x), x], y] + x[[T(y), y], x] + [xT(y), x], x]
\]

Using (36) and Lemma 1.2, we get

\[
x[[T(x), x], y] + x[[T(y), x], x] + [[T(x), x], x] + x[yT(x)], x] = 0.
\]

Again using Lemma 1.2, and (36) we have

\[
x[[T(x), x], y] + x[[T(y), x], x] + [T(x), x]y]x + [T(x), x][y, x]x + xT(x), x] = 0.
\]

Using (40) in last equation, we get

\[
[T(x), x][y, x] + x[y, x][T(x), x] = 0
\]

\[
[T(x), x](y + \dot{y}x + x(y, x)]T(x), x] = 0
\]
\[ [T(x), x]yx^2 + [T(x), x]\dot{y}x + xy[T(x), x] + x^2y[T(x), x] = 0. \]

Using (37), we get
\[ [T(x), x]yx^2 + x^2\dot{y}[T(x), x] + \dot{x}[T(x), x]yx + xy[T(x), x]x = 0. \]

Using (32), we have
\[ (41) \quad [T(x), x]yx^2 + x^2\dot{y}[T(x), x] = 0. \]

Pre multiply (41) by \( x \) gives
\[ (42) \quad x[T(x), x]yx^2 + x^3\dot{y}[T(x), x] = 0. \]

Using Lemma 1.1 in (32) and using it in (42), we get
\[ (43) \quad xy[T(x), x]x^2 + x^3\dot{y}[T(x), x] = 0. \]

Pre multiply last equation by \( T(x) \), we get
\[ (44) \quad T(x)xy[T(x), x]x^2 + T(x)x^3\dot{y}[T(x), x] = 0. \]

Replace \( y \) with \( T(x)y \) in (43), we get
\[ (45) \quad xT(x)y[T(x), x]x^2 + x^3T(x)\dot{y}[T(x), x] = 0. \]

Adding pseudo inverse of (45) and (44), we get
\[ (46) \quad [T(x), x]y[T(x), x]x^2 + [T(x), x^3]y[T(x), x] = 0. \]

By applying Lemma 1.7 in (46), we get
\[
\begin{align*}
([T(x), x]\dot{x}^2 + [T(x), x^3])y[T(x), x] &= 0 \\
([T(x), x]\dot{x}^2 + [T(x), x]x^2 + x[T(x), x^2])y[T(x), x] &= 0 \\
([T(x), x]\dot{x}^2 + [T(x), x]x^2 + x[T(x), x]x + x^2[T(x), x])y[T(x), x] &= 0.
\end{align*}
\]
Using (37) and the fact that $S$ is inverse semiring, we have
$$x[T(x), x]xy[T(x), x] = 0.$$ And then semiprimeness of $S$ implies that

(47) \hspace{1cm} x[T(x), x]x = 0, \forall x \in S.

Replace $y$ with $yx$ in (32) and using (47) we have

(48) \hspace{1cm} x[T(x), x]yx^2 = 0.

Replace $y$ with $yT(x)$ in (48), we get

(49) \hspace{1cm} x[T(x), x]yT(x)x^2 = 0.

Post multiplying (48) by $T(x)$, we get

(50) \hspace{1cm} x[T(x), x]yx^2T(x) = 0.

Adding pseudo inverse of (50) in (49), we get

$$x[T(x), x]y + \left[ T(x), y \right]_x + \left[ T(y), x \right]_x = 0.$$ Using (37) and the fact that $S$ is 2-torsion free, we have

(51) \hspace{1cm} x[T(x), x] = 0 = [T(x), x]_x, \ x \in S.

As (40) obtained from (36), we can get following from (51)

(52) \hspace{1cm} [T(x), x]y + [T(x), y]_x + [T(y), x]_x = 0.

Post multiplying (52) by $[T(x), x]$ and using (51), we get $[T(x), x]y[T(x), x] = 0$, $\forall y \in S$ which implies that

(53) \hspace{1cm} [T(x), x] = 0.

Replace $y$ with $xy + yx$ in (26), we have

(54) \hspace{1cm} T(x^2yx + xyx^2) + xT(xy + yx)T = 0.

Replace $z$ with $x^2$ in (27), we get

(55) \hspace{1cm} T(xy + x^2yx) + xT(y)x^2 + x^2T(y)T = 0.
Using Lemma 1.1 in (54) and using the result in (55) we get

\[ x(T(xy + yx) + xT(y) + T(y)x) = 0. \]

Now if we define biadditive function \( g : S \times S \to S \) by \( g(x, y) = T(xy + yx) + T(y)x + xT(y) \) then last equation can be written as

\[ (56) \quad xg(x, y)x = 0. \]

As (40) obtained from (36), we can obtain following from (56)

\[ (57) \quad xg(x, y)z + xg(z, y)x + zg(x, y)x = 0, \quad \forall x, y, z \in S. \]

Post multiplication (57) by \( g(x, y)x \) and using (56) we get

\[ (58) \quad xg(x, y)zg(x, y)x = 0. \]

Linearization of (53) gives

\[ (59) \quad [T(x), y] + [T(y), x] = 0. \]

Replace \( y \) with \( xy + yx \) in above equation and using (53) we get

\[ [T(xy + yx), x] + x[T(x), y] + [T(x), y]x = 0. \]

Using Lemma 1.1 in (59) and using the result in last equation, we get

\[ x[T(y), x] + [T(y), x]x + [T(xy + yx), x] = 0. \]

Using Lemma 1.2 in last equation, we get

\[ [xT(y), x] + [T(y)x, x] + [T(xy + yx), x] = 0, \quad \text{or} \]

\[ [xT(y) + T(y)x + T(xy + yx), x] = 0, \quad \text{or} \]

\[ (60) \quad [g(x, y), x] = 0. \]

which gives

\[ (61) \quad g(x, y)x = xg(x, y), \quad x, y \in S. \]

By (58) and (61), \( g(x, y)xzg(x, y)x = 0 \) this and (61) implies

\[ (62) \quad xg(x, y) = 0 = g(x, y)x. \]

Linearization of (62) gives \( g(x, y)z + g(z, y)x = 0 \).
Post multiplying last equation by \( g(x, y) \) and using (62), we get \( g(x, y)zg(x, y) = 0 \) and this implies \( g(x, y) = 0, x, y \in S \). Put \( x = y \), we get

\[
2T(x^2) + \dot{x}T(x) + T(x)\dot{x} = 0.
\]

From (53) we can get \( T(x)x = xT(x) \), using this and the fact that \( S \) is 2-torsion free, in (63), we get

\[
T(x^2) + \dot{x}T(x) = 0 \quad \text{and} \quad T(x^2) + T(x)\dot{x} = 0.
\]

And therefore by Theorem 2.1, it follows that \( T \) is right and left centralizer. This completes the proof.

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References


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