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QUASIORDER LATTICES ARE FIVE-GENERATED

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Abstract

A quasiorder (relation), also known as a preorder, is a reflexive and transitive relation. The quasiorders on a set A form a complete lattice with respect to set inclusion. Assume that A is a set such that there is no inaccessible cardinal less than or equal to $|A|$; note that in Kuratowski's model of ZFC, all sets A satisfy this assumption. Generalizing the 1996 result of Ivan Chajda and Gábor Czédli, also Tamás Dolgos' recent achievement, we prove that in this case the lattice of quasiorders on A is five-generated, as a complete lattice.

Keywords: quasiorder lattice, preorder lattice, accessible cardinal.

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1. Introduction and goal

Reflexive and transitive relations of a set A are called *quasiorders* or, in other fields of mathematics, *preorders*. Let $\mathrm{Quord}(A)$ and $\mathrm{Equ}(A)$ denote the set of all quasiorders and equivalences on A, respectively. Equipped with meet (intersection) and join (transitive hull of union), both of them are algebraic lattices. The lattices $\mathrm{Equ}(A)$ and, more recently, $\mathrm{Quord}(A)$ are natural objects to study; for example, see Czédli [1], Tůma [12], and the other papers mentioned in the rest of this section.

For finite sets, Strietz in [10] and Zádori in [13] gave a four-element generating set for $\mathrm{Equ}(A)$.

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Developing his result in [4] for the $\aleph_0$-case further, Czédli proved in [3] and [5] that if A has at least four elements and there is no inaccessible cardinal m such that $m \leq |A|$, then $\mathrm{Equ}(A)$ is generated by four elements. This means that $\mathrm{Equ}(A)$ has a four-element subset which is included in no proper *complete* sublattice of $\mathrm{Equ}(A)$. A cardinal m is *inaccessible* if $m > \aleph_0$, $k < m$ implies $2^k < m$, and, finally, whenever J is a set of cardinals such that $|J|$ and all $k \in J$ are strictly less than m, then $\sup\{k : k \in J\} < m$. By Kuratowski's result [7], see also [8], ZFC has a model without inaccessible cardinals. Hence the existence of inaccessible cardinals cannot be proved from ZFC, and the scope of our result, Theorem 2.1 includes all sets in an appropriate model of set theory. For more on inaccessible cardinals the reader can resort to standard textbooks, for example, to Levy [8, pages 138–141].

Our first goal is to derive, in a short way, from Czédli's result that $\mathrm{Quord}(A)$ has a five-element generating set. However, the construction of the generating equivalences in [3] and [5] is quite involved and long. In particular, even the $|A| = \aleph_0$ case needs a limit procedure there. This justifies our second goal: to give an easier and more understandable five-element generating set for $\mathrm{Quord}(A)$. For simplicity, we only deal with the case when $|A|$ is at most continuum. Note, however, that combining our approach with Czédli's method [3] and [5] or, alternatively, with Takách [11], one could elaborate an alternative, quite long proof for Theorem 2.1(i).

An earlier counterpart of Theorem 2.1, with three partial orders and their inverses as generators, is due to Chajda and Czédli [2] for all finite and many infinite cardinals, and to Takách [11] for all cardinals below the first inaccessible cardinal. The particular case of Theorem 2.1 for $|A| \leq \aleph_0$ is due to Dolgos [6]. Our construction for the at most $2^{\aleph_0}$-case generalizes Dolgos' approach.

2. The main theorem and two approaches

For $a, b \in A$, let $\langle a, b\rangle$ and $\langle a, b\rangle^e$ denote the smallest quasiorder and the smallest equivalence on A that contain the ordered pair (a, b), respectively. Typically, we use the notation $\langle a, b\rangle$ only for $a \neq b$; then $\langle a, b\rangle$ is an atom in $\mathrm{Quord}(A)$. As usual, Δ stands for the diagonal relation $\{(x, x) \mid x \in A\}$.

Theorem 2.1. *Let A be a set with at least three elements.*

(i) *If there is no inaccessible cardinal m such that $m \leq |A|$, then* $\mathrm{Quord}(A)$ *has a five-element generating set.*

(ii) *If $\aleph_0 \leq |A| \leq 2^{\aleph_0}$, then* $\mathrm{Quord}(A)$ *has a five-element generating set.*

Of course, part (ii) is a particular case of part (i). As mentioned before, these parts will be proved with different methods.

***Proof of part* (i) *of Theorem* 2.1.** Pick a pair $(a, b) \in A \times A$ of distinct elements, and denote the sublattice generated by $\langle a, b\rangle$ and the four equivalences introduced in [3] by L. (Remember that the four equivalences in question generate $\mathrm{Equ}(A)$.) It suffices to show that $L = \mathrm{Quord}(A)$. Clearly, $\varrho = \bigvee\{\langle c, d\rangle : (c, d) \in \varrho\}$ for any $\varrho \in \mathrm{Quord}(A)$. Hence it is sufficient to show that $\langle c, d\rangle \in L$ for every $c, d \in A$ with $c \neq d$. Observe the rules

$$\langle p, x\rangle = \langle p, x\rangle^e \wedge (\langle p, q\rangle \vee \langle q, x\rangle^e) \text{ and} \tag{1}$$

$$\langle x, q\rangle = \langle x, q\rangle^e \wedge (\langle x, p\rangle^e \vee \langle p, q\rangle), \tag{2}$$

provided $|\{p, q, x\}| = 3$. Thus, since all equivalences belong to L, we obtain the implication

$$\langle p, q\rangle \in L \Longrightarrow \big(\langle p, x\rangle \in L \text{ and } \langle x, q\rangle \in L\big). \tag{3}$$

Next, assume that $|A| \geq 4$ and pick $u, v \in A$ with $|\{a, b, u, v\}| = 4$. Using (3) twice, we obtain that $\langle a, v\rangle \in L$ and $\langle u, v\rangle \in L$. Similarly, $\langle v, u\rangle \in L$. Changing the role of (a, b, u, v) to that of (u, v, b, a), we obtain that $\langle b, a\rangle \in L$. Furthermore, (3) yields that each atom $\langle u, v\rangle$ with $|\{a, b, u, v\}| = 3$ is also in L. That is, L contains all atoms of $\mathrm{Quo}(A)$ as required.

Finally, let $|A| = 3$, say, $A = \{a, b, c\}$. Using (3), the equality $\langle a, b\rangle^e = \langle a, b\rangle \vee \langle b, a\rangle$, and the fact that $\{\langle a, b\rangle^e, \langle b, c\rangle^e, \langle c, a\rangle^e\}$ generates the five-element $\mathrm{Equ}(A)$, it is straightforward to see that $\{\langle a, b\rangle, \langle b, a\rangle, \langle b, c\rangle^e, \langle c, a\rangle^e\}$ is a four-element generating set of $\mathrm{Quo}(A)$. Adding a fifth element, we obtain a five-element generating set. ■

Next, we give a self-contained proof for part (ii).

***Proof of part* (ii) *of Theorem* 2.1.** Let $A_0 = \{a_0, b_0, a_1, b_1, a_2, b_2, \dots\}$. The subsets $\{a_0, a_1, a_2, \dots\}$ and $\{b_0, b_1, b_2, \dots\}$ are called *rows*, the a-row and the b-row, respectively. For a technical reason, which will be clear soon, we denote a_{3i+10} and b_{3i+11} by e_i and e'_i, respectively; these elements will be black-filled in our figures. In Figure 1, e_i and e'_i are connected by a dotted edge whose role will be explained in due time. Furthermore, sometimes we even use the notation (e_{-1}, e'_{-1}) for (a_7, b_8) in our computations.

We are going to define five quasiorders on A_0, denoted by α_0^0, α_1^0, α_2^0, β^0, and β_*^0; in fact, the first three will be equivalences. (The upper subscripts 0 refer to the fact that they are defined on A_0; later we will also introduce α_0, α_1, α_2, β, and β_*, which will be defined on a larger set A.) Besides (or instead of) their formal definition below, the reader is advised to understand them from Figure 1. For $i \in \{0, 1, 2\}$, we define α_i^0 by the corresponding partition

$$\begin{aligned}&\big\{\{a_{3k+i} : k \in \mathbb{Z}\}\big\} \cup \big\{\{a_{3k+i+1}, a_{3k+i+2}\} : k \in \mathbb{Z}\big\} \cup \\ &\big\{\{b_{3k+i+1} : k \in \mathbb{Z}\}\big\} \cup \big\{\{b_{3k+i+1+1}, b_{3k+i+1+2}\} : k \in \mathbb{Z}\big\}.\end{aligned} \tag{4}$$

Also, let

$$\beta^0 = \langle a_0, a_2 \rangle \vee \langle b_0, b_2 \rangle \vee \langle a_4, b_5 \rangle \vee \langle b_8, a_7 \rangle$$

and, finally, let

$$\beta^0_* = (\beta^0)^{-1}.$$

For $\delta \in \{\alpha^0_0, \alpha^0_1, \alpha^0_2, \beta^0\}$ and $x, y \in A_0$, we have $(x, y) \in \delta$ iff the vertices x and y can be connected by a δ-colored directed path in Figure 1; this is the meaning of the figure. (Almost all edges but (a_0, a_2), (b_0, b_2), (a_4, b_5) and (b_8, a_7) are directed in both ways.) Since β^0_* is the inverse of β^0, the β^0_*-colored edges are not indicated. At present, the dotted edges belong neither to β^0, nor to β^0_*; however, some of these edges (directed upwards or downwards) will be added to β^0 or β^0_* at a later stage of the construction.

Figure 1. Quasiorders on A_0.

Later we will need $\kappa \le 2^{\aleph_0}$ copies of A_0. Note that Dolgos [6] used only the upper row of a single copy of A_0. When we work in a single row, we often follow his arguments.

Figure 2. A_0 in a concise form.

Figure 3. A part of $\beta \in \mathrm{Quord}(A)$ if $H = \bigl\{\emptyset, \{2,3\}, \{2,4,5\}\bigr\}$.

Starting from the $\aleph_0$-sized graph A_0, we are going to define a more involved graph. (Note at this point that our graphs and their vertex sets are usually denoted in the same way.) Let κ be an arbitrary cardinal such that $\aleph_0 \le \kappa \le 2^{\aleph_0}$. Let $I = \{2, 3, 4, \dots\}$, and take a subset H of $\mathcal{P}(I)$ such that $|H| = \kappa$. For simplicity, assume that $\emptyset \in H$. Next, for $U \in H$, we modify the graph A_0 to obtain a colored graph $A_0(U)$ with vertex set $\{a_0(U), b_0(U), a_1(U), b_1(U), a_2(U), b_2(U), \dots\}$ as follows. When it is not confusing, we drop the parameter U and simply write a_0, b_0, a_1, $b_1, \dots$. In particular, $e_i(U)$ and $e'_i(U)$ are denoted by e_i and e'_i in our figures. However, $A_0(U)$ is given in the figures and it refers to all these elements. Of course, we assume that $A_0(U) \cap A_0(V) = \emptyset$ whenever $U \ne V \in H$. Now, to obtain $A_0(U)$ from A_0, we replace the dotted edges with "real" edges (e'_i, e_i) for $i \in U$ and (e_i, e'_i) for $i \in I \setminus U$. For $U \in H$, the set $A_0(U)$ is called a box. In Figure 3, boxes are grey. For example, the lower grey box in our figure is $A_0(\{2, 4, 5\})$.

Now, we are in the position to define a new colored graph, A, as follows. Its vertex set is the union of the disjoint sets $A_0(U)$, that is, $A = \{A_0(U) : U \in H\}$. Besides that all the previous edges are preserved, we add the β-colored directed edges $(e_0(\emptyset), e_0(U))$ and $(e_1(U), e_1(\emptyset))$ for all $U \in H$. In this way, we obtain our new graph, A; see Figure 3 for the particular case $H = \big\{\emptyset, \{2,3\}, \{2,4,5\}\big\}$. As before, for $\delta \in \{\alpha_0, \alpha_1, \alpha_2, \beta\}$, we let $(x,y) \in \delta$ iff the vertices x and y can be connected by a δ-colored directed path in the graph A. In this way, we have defined four quasiorders, α_0, α_1, α_2, and β on A; the fifth one is $\beta_* := \beta^{-1}$. Notice that if $\delta, \varepsilon \in \{\alpha_0, \alpha_1, \alpha_2, \beta, \beta_*\}$ and $\delta \neq \varepsilon$, then $\delta \wedge \varepsilon = \Delta$. Notice also that all the α_i are *row-preserving*; this means that whenever $(x,y) \in \alpha_i$ for some $i \in \{0,1,2\}$, then there is a unique $U \in H$ such that either $x, y \in \{a_0(U), a_1(U), \dots\}$, or $x, y \in \{b_0(U), b_1(U), \dots\}$. For an equivalence ϱ on A and $x \in A$, the ϱ-block $\{y \in A : (x,y) \in \varrho\}$ will be denoted by x/ϱ.

Now, let L denote the smallest complete sublattice of $\mathrm{Quord}(A)$ such that $\{\alpha_0, \alpha_1, \alpha_2, \beta, \beta_*\} \subseteq L$; our task is to show that $L = \mathrm{Quord}(A)$. As it was pointed out at the beginning of the previous proof, it suffices to show that L contains all atoms $\langle x, y\rangle$, where $x \neq y \in A$.

For $U \in H$ and distinct $x, y \in A(U)$, we introduce the notation

$$\langle x, y\rangle_H := \bigvee_{V \in H} \langle x(V), y(V)\rangle.$$

Let us emphasize that this notation is only permitted if x and y belong to the same copy of A_0, that is, to the same grey box in Figure 3.

We claim that

$$\langle a_3, a_2\rangle_H = (\alpha_0 \vee \beta) \wedge \alpha_1 \in L. \tag{5}$$

To show the "$\supseteq$" inclusion, assume that $x \neq y$ and $(x,y) \in (\alpha_0 \vee \beta) \wedge \alpha_1$. Then $(x,y) \in \alpha_1$ and there is a *shortest* path P from x to y in the graph whose edges are colored with α_0 and β. Since α_1 is row-preserving, x and y belong to the same row. Suppose, for a contradiction, that this row is $\{b_0(U), b_1(U), \dots\}$. If P goes entirely within this row, then it is clear by definitions, or by our figures, that either $(x,y) \in \alpha_0 \cup \beta$, or $(x,y) = (b_0(U), b_3(U))$. In both cases, $(x,y) \notin \alpha_1$, which is a contradiction. On the other hand, if P leaves this b-row, then it arrives at some $e_i(V)$ in the next step, where $V \in H$ and $i \in \{-1, 2, 3, 4, \dots\}$. But the only new vertex we can go from $e_i(V)$ via an $(\alpha_0 \cup \beta)$-colored path is the neighboring vertex to the right of $e_i(V)$. Then, in the next step of the path, we must turn back. This contradicts the minimality of P. Therefore, x and y belong to an a-row, $\{a_0(U), a_1(U), \dots\}$. Observe that our path P lies entirely in the same a-row. Really, if not, then P contains a β-colored edge $(e_i(U), e'_i(U))$, $(e_0(\emptyset), e_0(U))$ or $(e_1(U), e_1(\emptyset))$. However, all $e_i(U)$ and all $e'_i(U)$ belong to distinct two-element α_0-classes. All of these α_0-classes have the property that either at most one β-colored edge's eindpoint belongs to the class or if two β-colored edge's eindpoints are in the class, then these edges are directed in the same way. Hence, P can not leave this latter row, which is a contradiction. Thus, P lies in the a-row containing x and y. Since $\alpha_0 \wedge \alpha_1 = \Delta = \alpha_1 \wedge \beta$, both colors, α_0 and β, occur in our path P. Since P is the shortest path and the a-row of x and y contains only one β-colored edge, P contains exactly one β-colored edge, $(a_0(U), a_2(U))$. Therefore, $x \in a_0(U)/\alpha_0$ and $y \in a_2(U)/\alpha_0$. Using (4), we have that $x \in \{a_{3k}(U) : k \in \mathbb{Z}\}$ and $y \in \{a_1(U), a_2(U)\}$. Thus, taking $(x, y) \in \alpha_1$ into account, $(x, y) = (a_3(U), a_2(U)) \in \langle a_3, a_2\rangle_H$. This proves the "$\supseteq$" inclusion in (5); the reverse inclusion is obvious. This proves (5).

Next, we assert that

$$\langle a_0, a_2\rangle_H = (\langle a_3, a_2\rangle_H \vee \alpha_0) \wedge \beta \in L. \tag{6}$$

To see this, let $(x, y) \in (\langle a_3, a_2\rangle_H \vee \alpha_0) \wedge \beta$ such that $x \neq y$. Since both $\langle a_3, a_2\rangle_H$ and α_0 are row-preserving, x and y belong to the same row. In the shortest path connecting x and y, both of the colors α_0 and $\langle a_3, a_2\rangle_H$ occur, because the intersections of these colors with β is Δ. The presence of $\langle a_3, a_2\rangle_H$ yields that we are in an a-row, say, in $A_0(U)$. Since the restriction of β to this a-row is $\langle a_0(U), a_2(U)\rangle$, we obtain that $(x, y) = (a_0(U), a_2(U)) \in \langle a_0, a_2\rangle_H$. This proves the "$\supseteq$" inclusion in (6), while the converse inclusion is evident.

Next, we show that

$$\langle b_0, b_1\rangle_H = (\alpha_2 \vee \beta) \wedge \alpha_1 \in L. \tag{7}$$

Assume that $(x, y) \in (\alpha_2 \vee \beta) \wedge \alpha_1$ and $x \neq y$. Again, since α_1 is row-preserving and $\alpha_2 \wedge \alpha_1 = \Delta = \beta \wedge \alpha_1$, x and y are in the same row and the shortest $(\alpha_2 \cup \beta)$-path P connecting them contains both colors, α_2 and β. As in the argument verifying (5), exactly one edge of this path is β-colored and P does not leave the row of x and y. Suppose, for a contradiction, that we are in an a-row. It follows easily from definitions that either $(x, y) \in \alpha_2 \cup \beta$, or $x \in \{a_0, a_1\}$ and $y \in \{a_{3k+2} : k \in \mathbb{N}_0\}$, but this contradicts $(x, y) \in \alpha_1$. Hence, x and y are in a b-row. So the only β-colored edge in P is $(b_0(U), b_2(U))$. After its β-colored edge, P consists of at most one edge. This gives that $y \in \{b_1(U), b_2(U)\}$. There can be arbitrary many α_2-colored edges before the only β-colored one, but we have that $x \in \{b_{3k} : k \in \mathbb{N}_0\}$. Taking $(x, y) \in \alpha_1$ into account, we conclude that $(x, y) = (b_0(U), b_1(U)) \in \langle b_0, b_1\rangle_H$, as required. The converse inclusion is obvious, so we have proved (7).

Similarly to (6), we obtain the following containment easily:

$$\langle b_0, b_2\rangle_H = (\langle b_0, b_1\rangle_H \vee \alpha_2) \wedge \beta \in L. \tag{8}$$

Since the involutory automorphism $L \to L$, defined by $\varrho \mapsto \varrho^{-1}$, maps β to β_*, it follows that L is closed with respect to this automorphism, that is, for all

$x, y \in A$, $U \in H$, and $u, v \in A_0(U)$,

$$\langle x, y\rangle \in L \iff \langle y, x\rangle \in L \quad \text{and} \quad \langle u, v\rangle_H \in L \iff \langle v, u\rangle_H \in L. \tag{9}$$

Combining (6) and (8) with (9), we obtain that $\langle a_2, a_0\rangle_H \in L$ and $\langle b_2, b_0\rangle_H \in L$. For a subset X of $\mathrm{Quord}(A)$, the smallest complete sublattice including X will be denoted by $[X]$. Our next task is to show that, for all $k \in \mathbb{N}_0$,

$$\langle a_k, a_{k+1}\rangle_H \in \big[\langle a_k, a_{k+2}\rangle_H, \alpha_0, \alpha_1, \alpha_2\big]. \tag{10}$$

Observe that, for every $U \in H$, there exists a unique $i \in \{0, 1, 2\}$ such that the pair $(a_{k+1}(U), a_{k+2}(U))$ is in α_i, and this i depends only on k but not on U. As it is clear from definitions, for all $s, t, j \in \mathbb{N}_0$ and $i \in \{0, 1, 2\}$,

$$(a_s, a_t) \in \alpha_i \iff (a_{s+j}, a_{t+j}) \in \alpha_{i+j}, \tag{11}$$

where the addition in the subscript of α is understood modulo 3. This allows us to assume that i above is 0, that is, $(a_{k+1}(U), a_{k+2}(U)) \in \alpha_0$ for all $U \in H$. This means that $k \equiv 3 \pmod 3$.

To prove (10), it suffices to show that

$$\langle a_k, a_{k+1}\rangle_H = (\alpha_0 \vee \langle a_k, a_{k+2}\rangle_H) \wedge \alpha_2. \tag{12}$$

The "$\subseteq$ " inclusion is obvious. To verify the reverse inclusion, assume that $(x, y) \in (\alpha_0 \vee \langle a_k, a_{k+2}\rangle_H) \wedge \alpha_2$. Since α_2 is row-preserving, there is a $U \in H$ such that x and y are in the same row of $A_0(U)$. Using that $\alpha_0 \wedge \alpha_2 = \Delta$, every $(\alpha_0 \cup \langle a_k, a_{k+2}\rangle_H)$-path P from x to y must contain an $\langle a_k, a_{k+2}\rangle_H$-colored edge. So, since α_0 is also row-preserving, both x and y are in the a-row of $A_0(U)$. Let P above be a shortest path, then it contains an $\langle a_k, a_{k+2}\rangle_H$-colored edge only once. Thinking of the segments of P after this edge, it follows that $y \in \{a_{k+1}(U), a_{k+2}(U)\}$, while the segment before this edge yields that $x \in \{a_i(U) : i \equiv 0 \pmod 3\}$. Now the definition of α_2 gives that $(x, y) = (a_k(U), a_{k+1}(U)) \in \langle a_k, a_{k+1}\rangle_H$, proving (10).

Since $\langle a_{k+1}, a_{k+2}\rangle_H = (\alpha_2 \vee \langle a_k, a_{k+2}\rangle_H) \wedge \alpha_0$ follows basically in the same way as (12), we obtain that

$$\langle a_{k+1}, a_{k+2}\rangle_H \in \big[\langle a_k, a_{k+2}\rangle_H, \alpha_0, \alpha_1, \alpha_2\big]. \tag{13}$$

Similarly, we obtain $\langle a_{k+2}, a_{k+3}\rangle_H = \big(\alpha_0 \vee \langle a_{k+2}, a_k\rangle_H\big) \wedge \alpha_1$, whence

$$\langle a_{k+2}, a_{k+3}\rangle_H \in \big[\langle a_{k+2}, a_k\rangle_H, \alpha_0, \alpha_1, \alpha_2\big]. \tag{14}$$

Using the rule

$$(b_{s+1}, b_{t+1}) \in \alpha_i \iff (a_s, a_t) \in \alpha_i, \tag{15}$$

one concludes easily from (10), (13), and (14) that

$$\begin{aligned} \langle b_k, b_{k+1}\rangle_H &\in \big[\langle b_k, b_{k+2}\rangle_H, \alpha_0, \alpha_1, \alpha_2\big],\\ \langle b_{k+1}, b_{k+2}\rangle_H &\in \big[\langle b_k, b_{k+2}\rangle_H, \alpha_0, \alpha_1, \alpha_2\big], \text{ and}\\ \langle b_{k+2}, b_{k+3}\rangle_H &\in \big[\langle b_{k+2}, b_k\rangle_H, \alpha_0, \alpha_1, \alpha_2\big]. \end{aligned} \tag{16}$$

If we combine the generators occurring in (10), (13), and (14), then we obtain a larger subset of $\mathrm{Quord}(A)$ that is closed with respect to the involutory automorphism ϱ mentioned right after (8). Therefore, (10), (13), (14), and (16) yield that

$$\begin{aligned} &\big\{\langle a_{k+1}, a_k\rangle_H, \langle a_{k+2}, a_{k+1}\rangle_H, \langle a_{k+3}, a_{k+2}\rangle_H, \langle a_k, a_{k+1}\rangle_H,\\ &\ \langle a_{k+1}, a_{k+2}\rangle_H, \langle a_{k+2}, a_{k+3}\rangle_H, \langle b_{k+1}, b_k\rangle_H, \langle b_{k+2}, b_{k+1}\rangle_H,\\ &\ \langle b_{k+3}, b_{k+2}\rangle_H, \langle b_k, b_{k+1}\rangle_H, \langle b_{k+1}, b_{k+2}\rangle_H, \langle b_{k+2}, b_{k+3}\rangle_H\big\}\\ &\ \subseteq \big[\langle a_k, a_{k+2}\rangle_H, \langle a_{k+2}, a_k\rangle_H, \langle b_k, b_{k+2}\rangle_H, \langle b_{k+2}, b_k\rangle_H, \alpha_0, \alpha_1, \alpha_2\big] =: \widehat{L}. \end{aligned} \tag{17}$$

Here $\widehat{L}$ denotes the sublattice on the right of "$\subseteq$". We say that two sequences, $(x = p_0, p_1, \dots, p_k = y)$ and $(x = q_0, q_1, \dots, q_n = y)$, are *internally disjoint sequences from x to y* if $\{p_1, \dots, p_{k-1}\} \cap \{q_1, \dots, q_{n-1}\} = \emptyset$. The following lemma is straightforward.

Lemma 2.2. *If $(x = p_0, p_1, \dots, p_k = y)$ and $(x = q_0, q_1, \dots, q_n = y)$ are internally disjoint sequences from x to y, then*

$$(\langle p_0, p_1\rangle \vee \dots \vee \langle p_{k-1}, p_k\rangle) \wedge (\langle q_0, q_1\rangle \vee \dots \vee \langle q_{n-1}, q_n\rangle) = \langle x, y\rangle.$$

We claim that

$$\big\{\langle a_{k+1}, a_{k+3}\rangle_H, \langle b_{k+1}, b_{k+3}\rangle_H\big\} \subseteq \widehat{L}. \tag{18}$$

To see this, consider any $U \in H$ and the equivalence α_i with $(a_k(U), a_{k+3}(U)) \in \alpha_i$. As usual, (11) allows us to assume that $i = 0$, and $\langle a_k, a_{k+3}\rangle_H = (\langle a_k, a_{k+2}\rangle_H \vee \langle a_{k+2}, a_{k+3}\rangle_H) \wedge \alpha_0$ follows easily. So, according to (14),

$$\langle a_k, a_{k+3}\rangle_H \in \widehat{L}. \tag{19}$$

For every $U \in H$, Lemma 2.2 yields that

$$\begin{aligned} \langle a_{k+1}(U), a_{k+3}(U)\rangle = (&\langle a_{k+1}(U), a_k(U)\rangle \vee \langle a_k(U), a_{k+3}(U)\rangle)\\ &\wedge (\langle a_{k+1}(U), a_{k+2}(U)\rangle \vee \langle a_{k+2}(U), a_{k+3}(U)\rangle). \end{aligned} \tag{20}$$

Since all the atoms occurring in (20) are row-preserving, we conclude that

$$\begin{aligned} &\langle a_{k+1}, a_{k+3}\rangle_H\\ &= (\langle a_{k+1}, a_k\rangle_H \vee \langle a_k, a_{k+3}\rangle_H) \wedge (\langle a_{k+1}, a_{k+2}\rangle_H \vee \langle a_{k+2}, a_{k+3}\rangle_H). \end{aligned} \tag{21}$$

Hence, using (17) and (19) and (15), which says that the a-rows and b-rows play similar roles, we obtain that

$$\big\{\langle a_{k+1}, a_{k+3}\rangle_H, \langle b_{k+1}, b_{k+3}\rangle_H\big\} \subseteq \widehat{L}. \tag{22}$$

Combining $\widehat{L} \subseteq L$, (6), (8), (9), (17), and (22), we obtain that, for all $i, j \in \mathbb{N}_0$,

$$|i-j| \in \{1,2\} \Longrightarrow \big\{\langle a_i, a_j\rangle_H, \langle b_i, b_j\rangle_H\big\} \subseteq L. \tag{23}$$

Next, let $|i-j| > 2$. In the computation below, (9) allows us to assume, without loss of generality, that $i < j$. If $j - i$ is even, then

$$\begin{aligned}&(a_i, a_{i+2}, a_{i+4}, \dots, a_{j-2}, a_j) \text{ and}\\ &(a_i, a_{i+1}, a_{i+3}, a_{i+5}, \dots, a_{j-5}, a_{j-3}, a_{j-1}, a_j)\end{aligned}$$

are internally disjoint sequences from a_i to a_j in A_0. So, Lemma 2.2 and (23) give that

$$\langle a_i, a_j\rangle_H \quad \text{and} \quad \langle b_i, b_j\rangle_H \quad \text{belong to} \quad L \tag{24}$$

in this case. The same holds for $j - i$ being odd, because then

$$(a_i, a_{i+1}, a_{i+3}, \dots, a_{j-2}, a_j) \text{ and } (a_i, a_{i+2}, a_{i+4}, a_{j-3}, a_{j-1}, a_j)$$

are internally disjoint. Therefore,

$$\text{if } x, y \in A \text{ are in the same row, then } \langle x, y\rangle_H \in L. \tag{25}$$

As a first step to go beyond the limits of a single row, we claim that

$$\begin{aligned}\langle a_5, b_6\rangle_H &= \big(\langle a_5, a_4\rangle_H \vee \beta \vee \langle b_5, b_6\rangle_H\big) \wedge \big(\langle a_5, a_7\rangle_H \vee \beta_* \vee \langle b_8, b_6\rangle_H\big),\\ \langle a_6, b_7\rangle_H &= \big(\langle a_6, a_4\rangle_H \vee \beta \vee \langle b_5, b_7\rangle_H\big) \wedge \big(\langle a_6, a_7\rangle_H \vee \beta_* \vee \langle b_8, b_7\rangle_H\big).\end{aligned} \tag{26}$$

We only deal with the first equality, because the second one is analogous. We say that a β- or β_*-colored edge is *far on the right* if both of its endpoints belong to the set:

$$\{e_i(U), e'_i(U) \mid i \in \mathbb{N}_0,\ U \in H\}.$$

Observe that

$$\begin{aligned}\langle a_5, a_4\rangle_H \vee \beta \vee \langle b_5, b_6\rangle_H = \bigcup_{U\in H} \{&(a_0(U), a_2(U)), (b_0(U), b_2(U)),\\ &(a_4(U), b_5(U)), (b_8(U), a_7(U)), (a_5(U), a_4(U)),\\ &(b_5(U), b_6(U)), (a_5(U), b_5(U)), (a_4(U), b_6(U)),\\ &(a_5(U), b_6(U))\} \cup \{\text{some edges far on the right}\}.\end{aligned} \tag{27}$$

Similarly,

$$\begin{aligned}\langle a_5,a_7\rangle_H\vee\beta_*\vee\langle b_8,b_6\rangle_H=\bigcup_{U\in H}&\{(a_2(U),a_0(U)),(b_2(U),b_0(U)),\\ &(b_5(U),a_4(U)),(a_7(U),b_8(U)),(a_5(U),a_7(U)),\\ &(b_8(U),b_6(U)),(a_5(U),b_8(U)),(a_7(U),b_6(U)),\\ &(a_5(U),b_6(U))\}\cup\{\text{some edges far on the right}\}.\end{aligned}\tag{28}$$

By our construction, no edge far on the right occurs both in (27) and (28). Thus, we obtain (26).

Now, we are in the position to fully extend the validity of (25) as follows:

$$\text{if } x,y\in A \text{ are in the same } A_0(U), \text{ then } \langle x,y\rangle_H\in L.\tag{29}$$

To see this, let $U\in H$ and $x,y\in A_0(U)$ such that $x\neq y$. Apart from x–y symmetry, (25) allows us to assume that $x=a_i(U)$ and $y=b_j(U)$. Since we obtain

$$\langle x,y\rangle_H=\big(\langle x,a_5\rangle_H\vee\langle a_5,b_6\rangle_H\vee\langle b_6,y\rangle_H\big)\wedge\big(\langle x,a_6\rangle_H\vee\langle a_6,b_7\rangle_H\vee\langle b_7,y\rangle_H\big)$$

from Lemma 2.2, (29) follows.

Next, we turn our attention to atoms. As a first step, we will show that, for every $U\in H$,

$$\langle a_1(U),b_1(U)\rangle\in L.\tag{30}$$

To see this, we claim that

$$\begin{aligned}\langle a_1(U),b_1(U)\rangle=\langle a_1,b_1\rangle_H\wedge&\\ &\bigwedge_{i\in U}\big(\langle a_1,e_i'\rangle_H\vee\beta\vee\langle e_i,b_1\rangle_H\big)\\ &\bigwedge_{i\in I\setminus U}\big(\langle a_1,e_i'\rangle_H\vee\beta_*\vee\langle e_i,b_1\rangle_H\big).\end{aligned}\tag{31}$$

The "$\subseteq$" inclusion is evident. To see the reverse inclusion, let $V\in H$, $V\neq U$. This means there is a $j\in I$ such that $j\in V\setminus U$ or $j\in U\setminus V$. Because of symmetry, we can assume that $j\in U$ and $j\notin V$. This means that $\langle a_1,e_j'\rangle_H\vee\beta\vee\langle e_j,b_1\rangle_H$ is a part of the right side of (31). It is clear that $(a_1(U),b_1(U))\in\langle a_1,e_j'\rangle_H\vee\beta\vee\langle e_j,b_1\rangle_H$. However, $(a_1(V),b_1(V))\notin\langle a_1,e_j'\rangle_H\vee\beta\vee\langle e_j,b_1\rangle_H$, because $\langle a_1,e_j'\rangle_H$ and $\langle e_j,b_1\rangle_H$ are box-preserving, $a_1(V)$ and $b_1(V)$ are the only elements of their β-blocks and, since $j\notin V$, $(e_j'(V),e_j(V))\notin\beta$. Hence, (31) holds.

Next, we claim that if $U\in H$ and $\{w,x,y,z\}\subseteq A_0$ such that $|\{w,x,y,z\}|=4$, then

$$\langle w(U),z(U)\rangle\in L\Longrightarrow\langle x(U),y(U)\rangle\in L.\tag{32}$$

Since each quasiorder occurring in the right-hand side of

$$\langle x(U), y(U)\rangle = \langle x, y\rangle_H \wedge \big(\langle x, w\rangle_H \vee \langle w(U), z(U)\rangle \vee \langle z, y\rangle_H\big) \tag{33}$$

is box-preserving, (33) holds and implies (32). Starting from (31) and applying (33) once or twice, we obtain that

$$\text{if } U \in H \text{ and } x, y \in A_0(U) \text{ with } x \neq y, \text{ then } \langle x, y\rangle \in L. \tag{34}$$

Next, we leave a single box similarly as we left a single row around (26)–(29). This justifies to give less details. First we obtain that, for $U \neq V \in H$,

$$\begin{aligned} &\langle a_5(U), a_5(V)\rangle \\ &= \big(\langle a_5(U), e_0(U)\rangle \vee \beta_* \vee \langle e_0(\emptyset), e_1(\emptyset)\rangle \vee \beta_* \vee \langle e_1(V), a_5(V)\rangle\big) \\ &\wedge \big(\langle a_5(U), e_1(U)\rangle \vee \beta \vee \langle e_1(\emptyset), e_0(\emptyset)\rangle \vee \beta \vee \langle e_0(V), a_5(V)\rangle\big) \end{aligned} \tag{35}$$

is in L. Note that the second occurrence of β_* and that of β could be omitted; they only serve a better understanding. Similarly, $\langle a_6(U), a_6(V)\rangle \in L$. Hence, Lemma 2.2 yields easily that for all $x \neq y \in A$, $\langle x, y\rangle \in L$. This proves part (ii) of Theorem 2.1. ∎

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