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A SHORT NOTE ON L_{CBA}—FUZZY LOGIC WITH A NON-ASSOCIATIVE CONJUNCTION

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Abstract

We significantly simplify the axiomatic system L_{CBA} for fuzzy logic with a non-associative conjunction.

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Several investigations in probability theory and the theory of expert systems show that it is important to search for some reasonable generalizations of fuzzy logics having a non-associative conjunction, see [5, 6, 8, 9, 10].

In the paper [1] Botur and Halaš introduced and described a non-associative fuzzy logic L_{CBA} having as an equivalent algebraic semantics lattices with section antitone involutions satisfying the contraposition law, so-called commutative basic algebras. The variety of commutative basic algebras was intensively studied in several recent papers and includes the class of MV-algebras. For more details see the book [2].

In [1] Botur and Halaš introduced a non-associative fuzzy logic L_{CBA} by the following nine axioms:

(i) $x \to (y \to x) = 1$

(ii) $((x \to y) \to y) \to ((y \to x) \to x) = 1$

(iii) $(\neg x \to \neg y) \to (y \to x) = 1$

(iv) $\neg\neg x \to x = 1$

(v) $(x \to y) \to (x \to \neg\neg y) = 1$

(vi) $x \to x = 1$

(vii) $(x = 1 \ \&\ x \to y = 1) \Rightarrow y = 1$

(viii) $x \to y = 1 \Rightarrow (y \to z) \to (x \to z) = 1$
(ix) $(x \to y = 1 \;\&\; y \to x = 1) \Rightarrow x = y$.

In the main theorem we significantly simplify the axiomatic system L_{CBA} for fuzzy logic with a non-associative conjunction. We show that three axioms can be omitted.

Theorem 1. *Identities* (iv), (v), (vi) *follows from identities* (i), (ii), (iii) *and from quasiidentities* (vii), (viii), (ix).

Proof. First, we show that (vi) is redundant. Using (viii) on (i) we easily obtain

(1) $$((y \to x) \to x) \to (x \to x) = 1.$$

Using (viii) on (ii) we obtain

(2) $$(((y \to x) \to x) \to (x \to x)) \to (((x \to y) \to y) \to (x \to x)) = 1.$$

Applying (vii) on (1) and (2) we get

(3) $$((x \to y) \to y)) \to (x \to x) = 1.$$

Further, using (viii) on (i) in the form $y \to ((x \to y) \to y) = 1$ we derive

(4) $$(((x \to y) \to y) \to (x \to x)) \to (y \to (x \to x)) = 1.$$

Applying (vii) on (3) and (4) we get $y \to (x \to x) = 1$. From the last identity, where $y := 1$, using (vii), we obtain (vi).

Now, we show that (iv) is redundant. From (i), using (vi), we easily derive that

(5) $$x \to 1 = 1.$$

Putting $y := 1$ in (ii) and applying (5) we get

$$1 \to ((1 \to x) \to x) = 1,$$

which, by (vii), give us

(6) $$(1 \to x) \to x = 1.$$

From this and from (i) in the form $x \to (1 \to x) = 1$, we obtain

(7) $$1 \to x = x.$$

by (ix). From (iii) and (viii) we infer

(8) $$((y \to x) \to z) \to ((\neg x \to \neg y) \to z) = 1,$$

from (i) and (viii) we obtain

$$((y \to x) \to z) \to (x \to z) = 1, \tag{9}$$

and from (ii) and (ix) we have

$$(x \to y) \to y = (y \to x) \to x. \tag{10}$$

Putting $y := 1$ in (iii) and applying (7) we get

$$(\neg x \to \neg 1) \to x = 1. \tag{11}$$

From this, for $x := \neg 1$, we get immediately $(\neg\neg 1 \to \neg 1) \to \neg 1 = 1$. From (i), where $x := \neg 1$ and $y := \neg\neg 1$, we obtain $\neg 1 \to (\neg\neg 1 \to \neg 1) = 1$. Applying (ix) to the last two equations we conclude

$$\neg\neg 1 \to \neg 1 = \neg 1. \tag{12}$$

Applying (vii) to (11) and to (9), where $y := \neg x$, $x := \neg 1$ and $z := x$, we derive

$$\neg 1 \to x = 1. \tag{13}$$

Now, put $\neg 1$ instead of x and $x \to \neg\neg 1$ instead of y in (10) to obtain

$$(\neg 1 \to (x \to \neg\neg 1)) \to (x \to \neg\neg 1) = ((x \to \neg\neg 1) \to \neg 1) \to \neg 1.$$

The left hand side of the last identity can be reduced to $x \to \neg\neg 1$, using (13) and (7). The right hand side of the last identity is equal to 1, by (9), where $y := x$, $x := \neg\neg 1$ and $z := \neg 1$, using (12). Therefore, we have

$$x \to \neg\neg 1 = 1. \tag{14}$$

From (14) we derive easily that $1 \to \neg\neg 1 = 1$ and from (5) we derive easily that $\neg\neg 1 \to 1 = 1$, whence, by (ix), we conclude

$$\neg\neg 1 = 1. \tag{15}$$

Putting $y := \neg x$, $x := \neg 1$ and $z := x$ in (8) and applying (11) and (vii) we get

$$(\neg\neg 1 \to \neg\neg x) \to x = 1$$

which, by (15) and (5), give us (iv).

Finally, we show that also (v) is redundant. Using (vii) on (iii) in the form $(\neg\neg\neg x \to \neg x) \to (x \to \neg\neg x) = 1$ and on (iv) in the form $\neg\neg\neg x \to x = 1$ we get $x \to \neg\neg x = 1$. This with (iv) give us by (ix)

$$x = \neg\neg x.$$

The last identity together with (vi) in the form $(x \to y) \to (x \to y) = 1$ proves (v). ∎

Among the recent results on simplification of axiomatic systems related to fuzzy logics belong works of Cintula (based on a proof, see [3]) and Lehmke (applying a solver he programmed, see [7]). Both of them have shown the redundancy of one of the axioms of Hájek's axiomatization of BL-logics [4].

Conclusions

We briefly recalled what a non-associative fuzzy logic L_{CBA} is and where it can be useful. Then we significantly simplify the axiomatic system L_{CBA} for fuzzy logic with a non-associative conjunction as given in [1]. We show that three axioms can be omitted. Note that generally, the removal of the redundant axioms from a axiomatic system is desirable because it simplifies definitions and shortens proofs.

Note that although the independence of axioms is a highly desirable property, superfluous axioms still could be of use (as tautologies in the considered theory) when proving results in the considered theory, making proofs more transparent.

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