Discussiones Mathematicae
General Algebra and Applications 36 (2016) 15–23
doi:10.7151/dmgaa.1244

ON THE ASSOCIATED PRIME IDEALS OF LOCAL COHOMOLOGY MODULES DEFINED BY A PAIR OF IDEALS

Maryam Jahangiri[1]

Faculty of Mathematical Sciences and Computer
Kharazmi University, Tehran, Iran
and
School of Mathematics
Institute for Research in Fundamental Sciences (IPM)
P.O. Box 19395-5746, Tehran, Iran

e-mail: mjahangiri@ipm.ir
jahangiri.maryam@gmail.com

Zohreh Habibi and Khadijeh Ahmadi Amoli

Payame Noor University
P.O. Box 19395-3697, Tehran, Iran

e-mail: z_habibi@pnu.ac.ir
khahmadi@pnu.ac.ir

Abstract

Let I and J be two ideals of a commutative Noetherian ring R and M be an R-module. For a non-negative integer n it is shown that, if the sets $\mathrm{Ass}_R(\mathrm{Ext}^n_R(R/I, M))$ and $\mathrm{Supp}_R(\mathrm{Ext}^i_R(R/I, H^j_{I,J}(M)))$ are finite for all $i \leq n+1$ and all $j < n$, then so is $\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^n_{I,J}(M)))$. We also study the finiteness of $\mathrm{Ass}_R(\mathrm{Ext}^i_R(R/I, H^n_{I,J}(M)))$ for $i = 1, 2$.

Keywords: local cohomology modules defined by a pair of ideals, spectral sequences, associated prime ideals.

2010 Mathematics Subject Classification: Primary 13D45; Secondary 13E05, 13E10.

[1]Corresponding author.

1. Introduction

Let R be a commutative Noetherian ring, I and J be two ideals of R and M be an R-module. For all $i \in \mathbb{N}_0$ the i-th local cohomology functor with respect to (I, J), denoted by $H^i_{I,J}(-)$, defined by Takahashi *et al.* in [14] as the i-th right derived functor of the (I, J)- torsion functor $\Gamma_{I,J}(-)$, where

$$\Gamma_{I,J}(M) := \{x \in M : I^n x \subseteq Jx \text{ for } n \gg 1\}.$$

This notion coincides with the ordinary local cohomology functor $H^i_I(-)$ when $J = 0$, see [5].

The main motivation for this generalization comes from the study of a dual of ordinary local cohomology modules $H^i_I(M)$ ([12]). Basic facts and more information about local cohomology defined by a pair of ideals can be obtained from [7, 8] and [14].

Hartshorne in [9] proposed the following conjecture: "Let M be a finitely generated R-module and $\mathfrak{a}$ be an ideal of R. Then $\mathrm{Ext}^i_R(R/\mathfrak{a}, H^j_{\mathfrak{a}}(M))$ is finitely generated for all $i \geq 0$ and $j \geq 0$."

Also, Huneke in [10] raised some crucial problems on local cohomology modules. One of them was about the finiteness of the set of associated prime ideals of the local cohomology modules $H^i_I(M)$.

Although there are some counterexamples to these conjectures, see [13], but there are some partial positive answers in some special cases too, see for example [3] or [4].

In this paper, we consider these two problems for local cohomology modules defined by a pair of ideals over not necessary finitely generated modules. In particular, we investigate certain conditions on $H^j_{I,J}(M)$ such that the set of associated prime ideals of $\mathrm{Ext}^i_R(R/I, H^j_{I,J}(M))$ is finite.

More precisely, let $n \in \mathbb{N}_0$ and assume that the sets $\mathrm{Ass}_R(\mathrm{Ext}^n_R(R/I, M))$ and $\mathrm{Supp}_R(\mathrm{Ext}^i_R(R/I, H^j_{I,J}(M)))$ are finite for all $i \leq n+1$ and all $j < n$ then, we use a spectral sequence argument to show that $\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^n_{I,J}(M)))$ is finite, too (Theorem 2.3). Moreover, it is shown that if the sets $\mathrm{Ass}_R(\mathrm{Ext}^{n+1}_R(R/I, M))$ and $\mathrm{Supp}(\mathrm{Ext}^i_R(R/I, H^j_{I,J}(M)))$ are finite for all $i \leq n+2$ and all $j < n$ then, so is $\mathrm{Ass}_R(\mathrm{Ext}^1_R(R/I, H^n_{I,J}(M)))$ (Theorem 2.7).

We also present a necessary and sufficient condition for the finiteness of the set $\mathrm{Ass}_R(\mathrm{Ext}^2_R(R/I, H^n_{I,J}(M)))$ (Theorem 2.8). These, also, generalize some known results concerning ordinary local cohomology modules.

Moreover, we study the $\operatorname{grade}_M \mathfrak{p} (:= \inf\{i \in \mathbb{N}_0 : H^i_{\mathfrak{p}}(M) \neq 0\})$ for $\mathfrak{p} \in \mathrm{Ass}_R(H^t_{I,J}(M))$, where $t = \inf\{i \in \mathbb{N}_0 : H^i_{I,J}(M) \neq 0\}$ (Theorem 2.11).

2. Associated prime ideals

In this section, first, we are going to study the set of associated prime ideals of some Ext-modules of local cohomology modules defined by a pair of ideals.

The following relation between associated prime ideals of modules in an exact sequence, which can be proved easily, is frequently used in our results.

Lemma 2.1. *Let $M \to N \to K \to 0$ be an exact sequence of R-modules. Then* $\mathrm{Ass}(K) \subseteq \mathrm{Supp}(M) \cup \mathrm{Ass}(N)$.

Next lemma describes a convergence of Grothendieck spectral sequences.

Lemma 2.2. *Let M be an R-module. Then the following convergence of spectral sequences exists*

$$\mathrm{Ext}^i_R(R/I, H^j_{I,J}(M)) \overset{i}{\Rightarrow} \mathrm{Ext}^{i+j}_R(R/I, M).$$

Proof. It is easy to see that $\mathrm{Hom}_R(R/I, \Gamma_{I,J}(M)) = \mathrm{Hom}_R(R/I, M)$. Also, for any injective R-module E, $\Gamma_{I,J}(E)$ is an injective R-module, by [14, 3.2] and [5, 2.1.4]. Now, in view of [11, 10.47], the assertion follows. ∎

The following theorem, which concerns with Hartshorne's problem mentioned in the introduction, is one of the main results in this paper.

Theorem 2.3. *Let n be a non-negative integer and M be an R-module such that* $\mathrm{Ass}_R(\mathrm{Ext}^n_R(R/I, M))$ *and* $\mathrm{Supp}_R(\mathrm{Ext}^i_R(R/I, H^j_{I,J}(M)))$ *are finite for all $i \leq n+1$ and all $j < n$. Then so is* $\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^n_{I,J}(M)))$.

Proof. Consider the convergence of spectral sequences in Lemma 2.2 and note that $E^{i,j}_2 = 0$ for all $i < 0$. Therefore, for all $2 \leq r \leq n+1$ there exists an exact sequence

$$0 \to E^{0,n}_{r+1} \to E^{0,n}_r \xrightarrow{d^{0,n}_r} E^{r,n+1-r}_r. \tag{2.1}$$

Since, $E^{r,n+1-r}_r$ is a subquotient of $E^{r,n+1-r}_2 = \mathrm{Ext}^r_R(R/I, H^{n+1-r}_{I,J}(M))$, Supp_R $(E^{r,n+1-r}_r)$ is a finite set. So, the above exact sequence implies that $\sharp\,\mathrm{Ass}_R(E^{0,n}_r) < \infty$ if $\sharp\,\mathrm{Ass}_R(E^{0,n}_{r+1}) < \infty$. Also, from the fact that $E^{i,j}_2 = 0$ for all $j < 0$, we have $E^{0,n}_\infty \cong E^{0,n}_{n+2}$. Therefore, to prove the assertion it is enough to show that $\mathrm{Ass}_R(E^{0,n}_\infty)$ is a finite set.

Using the concept of the convergence of spectral sequences, there exists a bounded filtration

$$0 = \varphi^{n+1}H^n \subseteq \varphi^n H^n \subseteq \cdots \subseteq \varphi^1 H^n \subseteq \varphi^0 H^n = \mathrm{Ext}^n_R(R/I, M)$$

of submodules of $\mathrm{Ext}^n_R(R/I, M)$ such that

$$E^{i,n-i}_\infty \cong \varphi^i H^n/\varphi^{i+1}H^n \text{ for all } i = 0, \ldots, n.$$

Therefore, $E^{n,0}_{n+1} \cong E^{n,0}_\infty \cong \varphi^n H^n$ is a subquotient of $E^{n,0}_2 = \mathrm{Ext}^n_R(R/I, \Gamma_{I,J}(M))$. So, by the assumption, $\mathrm{Supp}_R(\varphi^n H^n)$ is a finite set. Now, assume inductively that $\sharp\,\mathrm{Supp}_R(\varphi^i H^n) < \infty$ for all $1 < i \leq n$. Then, since

$$E^{1,n-1}_{n+1} \cong E^{1,n-1}_\infty \cong \varphi^1 H^n/\varphi^2 H^n$$

is a subquotient of $E^{1,n-1}_2 = \mathrm{Ext}^1_R(R/I, H^{n-1}_{I,J}(M))$, we deduce that $\mathrm{Supp}_R(\varphi^1 H^n)$ is finite. But,

$$E^{0,n}_\infty \cong \mathrm{Ext}^n_R(R/I, M)/\varphi^1 H^n$$

and Lemma 2.1 implies that $\sharp\,\mathrm{Ass}_R(E^{0,n}_\infty) < \infty$, as desired. ■

As some immediate consequences of Theorem 2.3, we obtain the following results.

Corollary 2.4. *Let M be a finite R-module. Suppose that there is an integer n such that for all $i < n$ the set $\mathrm{Supp}_R(H^i_{I,J}(M))$ is finite. Then $\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^n_{I,J}(M)))$ is finite.*

Proof. Using the fact that $\mathrm{Supp}_R(\mathrm{Ext}^i_R(R/I, H^j_{I,J}(M))) \subseteq V(I) \cap \mathrm{Supp}_R(H^i_{I,J}(M))$ for all i and j, the result follows from Theorem 2.3. ■

Corollary 2.5. *Let M be a finite R-module. Suppose that $q = \inf\{i : H^i_{I,J}(M)$ is not Artinian$\}$ is an integer, then $\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^q_{I,J}(M)))$ is finite.*

Proof. By [2, IV, p. 275, Proposition 7], $\mathrm{Supp}_R(H^i_{I,J}(M))$ is finite for all $i < q$. Now, the result follows from Corollary 2.4. ■

For an R-module M and an ideal $\mathfrak{a}$ of R, the grade of $\mathfrak{a}$ on M is defined by

$$\operatorname*{grade}_M \mathfrak{a} := \inf\{i \in \mathbb{N}_0 : H^i_\mathfrak{a}(M) \neq 0\},$$

if this infimum exists, and ∞ otherwise. If M is a finite R-module and $\mathfrak{a}M \neq M$, this definition coincides with the length of a maximal M-sequence in $\mathfrak{a}$ (cf. [5, 6.2.7]).

Corollary 2.6. *Let M be a finite R-module and $t = \inf\{i | H^i_{I,J}(M) \neq 0\}$ be an integer. Then $\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^t_{I,J}(M)))$ is finite. If in addition, $\operatorname*{grade}_M I = t$, then for a maximal M-sequence $x_1, \ldots, x_t$ in I, we have*

$$\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^t_{I,J}(M))) = \{\mathfrak{p} \in \mathrm{Ass}_R(M/(x_1, \ldots, x_t)M) \cap V(I); \operatorname*{grade}_M \mathfrak{p} = t\}.$$

Proof. In view of Theorem 2.3, $\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^t_{I,J}(M)))$ is finite. In the case where $\operatorname{grade}_M I = t$, using [1, 2.4(i)] and [6, 1.2.27], we have

$$\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^t_{I,J}(M))) = \mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^t_I(M))) = \mathrm{Ass}_R(H^t_I(M)).$$

Now, the assertion follows by [15, 3.10]. ■

In the rest of this paper we consider the set of associated prime ideals of some Ext modules of local cohomology modules defined by a pair of ideals.

Theorem 2.7. *Let n be a non-negative integer and M be an R-module such that* $\mathrm{Ass}_R(\mathrm{Ext}^{n+1}_R(R/I, M))$ *and* $\mathrm{Supp}_R(\mathrm{Ext}^i_R(R/I, H^j_{I,J}(M)))$ *are finite for all $i \leq n+2$ and all $j < n$. Then so is* $\mathrm{Ass}_R(\mathrm{Ext}^1_R(R/I, H^n_{I,J}(M)))$.

Proof. Considering the convergence of the spectral sequences of Lemma 2.2, we have to show that $\mathrm{Ass}_R(E_2^{1,n})$ is a finite set. Using similar arguments as used in Theorem 2.3, one can see that it is enough to show that $\mathrm{Ass}_R(E_\infty^{1,n}) = \mathrm{Ass}_R(E_{n+2}^{1,n})$ is a finite set.

By the concept of convergence of spectral sequences, there exists a filtration

$$0 = \varphi^{n+2}H^{n+1} \subseteq \varphi^{n+1}H^{n+1} \subseteq ... \subseteq \varphi^1 H^{n+1} \subseteq \varphi^0 H^{n+1} = \mathrm{Ext}^{n+1}_R(R/I, M)$$

of submodules of $\mathrm{Ext}^{n+1}_R(R/I, M)$ such that $E_\infty^{i,n+1-i} \cong \varphi^i H^{n+1}/\varphi^{i+1}H^{n+1}$ for all $i = 0, \ldots, n+1$. Using the fact that $\sharp\, \mathrm{Supp}_R(E_2^{i,j}) < \infty$ for all $i \leq n+2$ and all $j < n$ one can see that $\mathrm{Supp}_R(\varphi^i H^{n+1})$ is a finite set for all $i = 2, \ldots, n+2$. Also, $\sharp\, \mathrm{Ass}_R(\varphi^1 H^{n+1}) < \infty$. Now, since

$$E_{n+2}^{1,n} \cong E_\infty^{1,n} \cong \varphi^1 H^{n+1}/\varphi^2 H^{n+1},$$

using Lemma 2.1, we have $\sharp\, \mathrm{Ass}_R(E_\infty^{1,n}) < \infty$, and the result follows. ■

The following theorem presents a necessary and sufficient condition for the finiteness of the set $\mathrm{Ass}_R(\mathrm{Ext}^i_R(R/I, H^n_{I,J}(M)))$ when $i = 0, 2$.

Theorem 2.8. *Let n be a non-negative integer and M be an R-module such that the sets* $\mathrm{Supp}_R(\mathrm{Ext}^{n+1}_R(R/I, M))$ *and* $\mathrm{Supp}_R(\mathrm{Ext}^i_R(R/I, H^j_{I,J}(M)))$ *are finite for all $i \leq n+2$ and all $j < n$. Then* $\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^{n+1}_{I,J}(M)))$ *is finite if and only if* $\mathrm{Ass}_R(\mathrm{Ext}^2_R(R/I, H^n_{I,J}(M)))$ *is finite.*

Proof. ($\Leftarrow$) Again, consider the convergence of spectral sequences of Lemma 2.2 and assume that $\mathrm{Ass}_R(E_2^{2,n})$ is finite. Since $E_2^{i,j} = 0$ for all $i < 0$ or $j < 0$, using similar arguments as used in Theorem 2.3, one can see that $E_\infty^{0,n+1} \cong E_{n+3}^{0,n+1}$ and in order to prove that $\sharp\, \mathrm{Ass}_R(E_2^{0,n+1}) < \infty$ we have to show that $\sharp\, \mathrm{Ass}_R(E_\infty^{0,n+1}) < \infty$.

There exists a filtration

$$0 = \varphi^{n+2}H^{n+1} \subseteq \varphi^{n+1}H^{n+1} \subseteq \cdots \subseteq \varphi^{1}H^{n+1} \subseteq \varphi^{0}H^{n+1} = \operatorname{Ext}_R^{n+1}(R/I, M)$$

of submodules of $\operatorname{Ext}_R^{n+1}(R/I, M)$ such that $E_\infty^{0,n+1} \cong \operatorname{Ext}_R^{n+1}(R/I, M)/\varphi^1 H^{n+1}$. Since $\sharp \operatorname{Supp}_R(\operatorname{Ext}_R^{n+1}(R/I, M)) < \infty$ we have $\sharp \operatorname{Ass}_R(E_\infty^{0,n+1}) < \infty$, as desired.

($\Rightarrow$) Now, assume that $\operatorname{Ass}_R(\operatorname{Hom}_R(R/I, H_{I,J}^{n+1}(M))) < \infty$ and consider the exact sequence

$$0 \to \operatorname{Ker}\, d_2^{0,n+1} \to E_2^{0,n+1} \xrightarrow{d_2^{0,n+1}} \operatorname{Im} d_2^{0,n+1} \to 0.$$

Since $\operatorname{Ker}\, d_2^{0,n+1} = E_3^{0,n+1}$ and $\sharp \operatorname{Supp}_R(E_3^{0,n+1}) < \infty$, in view of Lemma 2.1, we have $\sharp \operatorname{Ass}_R(\operatorname{Im} d_2^{0,n+1}) < \infty$. Now, using the exact sequence

$$0 \to \operatorname{Im} d_2^{0,n+1} \to E_2^{2,n} \xrightarrow{d_2^{2,n}} E_2^{4,n-1}$$

and the fact that $E_2^{4,n-1} = \operatorname{Ext}_R^4(R/I, H_{I,J}^{n-1}(M))$ has finite support, we have $\sharp \operatorname{Ass}_R(E_2^{2,n}) < \infty$, as desired. ■

Theorem 2.9. *Let n be a non-negative integer and M be an R-module of dimension d, such that $\operatorname{Ass}_R(\operatorname{Ext}_R^{n+d}(R/I, M))$ and $\operatorname{Supp}_R(\operatorname{Ext}_R^i(R/I, H_{I,J}^j(M)))$ are finite for all $i \geq n+1$ and all $j < d$. Then $\operatorname{Ass}_R(\operatorname{Ext}_R^n(R/I, H_{I,J}^d(M)))$ is finite.*

Proof. The method of the proof is similar to the Theorem 2.7, considering [14, 4.7]. ■

In the rest of this paper, we study the grade of prime ideals $\mathfrak{p} \in \operatorname{Ass}_R(H_{I,J}^t(M))$ on M. For that, we shall use the following notations introduced in [14].

$$W(I, J) := \{\mathfrak{p} \in \operatorname{Spec}(R) : I^n \subseteq \mathfrak{p} + J \text{ for some integer } n \geq 1\},$$

and

$$\widetilde{W}(I, J) := \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of } R \text{ and } I^n \subseteq \mathfrak{a} + J \text{ for some integer } n \geq 1\}.$$

The following lemma can be proved using [14, 3.2].

Lemma 2.10. *For any non-negative integer i and any R-module M,*

(i) $\operatorname{Supp}_R(H_{I,J}^i(M)) \subseteq \bigcup_{\mathfrak{a} \in \widetilde{W}(I,J)} \operatorname{Supp}(H_{\mathfrak{a}}^i(M))$.

(ii) $\operatorname{Supp}_R(H_{I,J}^i(M)) \subseteq \operatorname{Supp}_R(M) \cap W(I, J)$.

In [15, 3.6] the authors study the $\operatorname{grade}_M \mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Ass}_R(H^t_{I,J}(M))$, where

$$t = \inf \{i \in \mathbb{N}_0 : H^i_{I,J}(M) \neq 0\}$$

in the case where M is a finitely generated R-module. But their proof is not correct. Actually, they use the equality $\operatorname{Supp}_R(M_x) = \{\mathfrak{p} \in \operatorname{Supp}_R(M) : x \notin \mathfrak{p}\}$ which is not true. Here, we also made a correction to this result for not necessary finite modules.

Theorem 2.11. *Let M be an R-module and $t = \inf\{i \in \mathbb{N}_0 : H^i_{I,J}(M) \neq 0\}$ be a non-negative integer. Then for all $\mathfrak{p} \in \operatorname{Ass}_R(H^t_{I,J}(M))$, $\operatorname{grade}_M \mathfrak{p} = t$.*

Proof. We use induction on t. Let $t = 0$ and $\mathfrak{p} \in \operatorname{Ass}_R(\Gamma_{I,J}(M))$. Then $\mathfrak{p} = (0 :_R x)$ for some $x \in \Gamma_{I,J}(M)$. Hence $x \in \Gamma_{\mathfrak{p}}(M)$ and so $\Gamma_{\mathfrak{p}}(M) \neq 0$.

Now suppose that $t > 0$ and the case $t-1$ is settled. Let $\mathfrak{p} \in \operatorname{Ass}_R(H^t_{I,J}(M))$ and consider the exact sequence $0 \to M \to E \to L \to 0$, where $E = E_R(M)$ is the injective envelope of M. Therefore, using [15, 2.2], $H^i_{I,J}(L) \cong H^{i+1}_{I,J}(M)$ for all $i \geq 0$ and we get

$$\inf \{i \in \mathbb{N}_0 : H^i_{I,J}(L) \neq 0\} = \inf \{i \in \mathbb{N}_0 : H^i_{I,J}(M) \neq 0\} - 1 = t - 1$$

and that $\mathfrak{p} \in \operatorname{Ass}_R(H^{t-1}_{I,J}(L))$. Thus, by inductive hypothesis, $\operatorname{grade}_L \mathfrak{p} = t-1$. Now, consider the long exact sequence

$$H^{i-1}_{\mathfrak{p}}(M) \to H^{i-1}_{\mathfrak{p}}(E) \to H^{i-1}_{\mathfrak{p}}(L) \to H^{i}_{\mathfrak{p}}(M).$$

If $t > 1$, then $H^i_{\mathfrak{p}}(M) \cong H^{i-1}_{\mathfrak{p}}(L) = 0$ for all $i < t$ and $H^t_{\mathfrak{p}}(M) \cong H^{t-1}_{\mathfrak{p}}(L) \neq 0$. Thus $\operatorname{grade}_M \mathfrak{p} = t$.

Let $t = 1$. Then $\Gamma_{\mathfrak{p}}(L) \neq 0$. By the above exact sequence, it is enough to show that $\Gamma_{\mathfrak{p}}(E) = 0$. On the contrary, assume that $\Gamma_{\mathfrak{p}}(E) \neq 0$. Then there exists a non-zero element $x \in E$ and $n \in \mathbb{N}$ such that $\mathfrak{p}^n x = 0$. We may assume that $\mathfrak{p}^n x = 0$ and $\mathfrak{p}^{n-1} x \neq 0$. So, there exists $r \in \mathfrak{p}^{n-1}$ such that $rx \neq 0$. Thus $\mathfrak{p} \subseteq (0 :_R rx)$. On the other hand, by Lemma 2.10,

$$\mathfrak{p} \in \operatorname{Ass}_R(H^1_{I,J}(M)) \subseteq \operatorname{Supp}_R(H^1_{I,J}(M)) \subseteq \bigcup_{\mathfrak{a} \in \widetilde{W}(I,J)} \operatorname{Supp}_R(H^1_{\mathfrak{a}}(M)).$$

So that there exists $\mathfrak{a} \in \widetilde{W}(I,J)$ such that $\mathfrak{a} \subseteq \mathfrak{p}$. Let $m \in \mathbb{N}$ with $I^m \subseteq \mathfrak{a} + J \subseteq \mathfrak{p} + J \subseteq (0 :_R rx) + J$. Hence $rx \in \Gamma_{I,J}(M)$ which contradicts with hypothesis and the choice of rx. Therefore $\Gamma_{\mathfrak{p}}(E) = 0$ and so $\operatorname{grade}_M \mathfrak{p} = 1$. ■

Acknowledgments

The authors would like to thank the referee for his/her careful reading of the paper. Also, the first author was supported by a grant from IPM (No. 92130111).

References

[1] M. Aghapournahr, Kh. Ahmadi-Amoli and M.Y. Sadeghi, *The concept of* (I, J)*-cohen Macaulay modules*, J. Algebraic Syst. **3** (1) (2015) 1–10.

[2] N. Bourbaki, Commutative Algebra, Translated from French (Hermann, Paris, 1972).

[3] M. Brodmann, *Asymptotic behaviour of cohomology: tameness, supports and associated primes*, S. Ghorpade, H. Srinivasan, J. Verma (Eds.), "Commutative Algebra and Algebraic Geometry" Proceedings, Joint International Meeting of the AMS and the IMS on Commutative Algebra and Algebraic Geometry, Bangalore/India, December 17–20, 2003, Contemporary Mathematics **390** (2005) 31–61. doi:10.1090/conm/390/07292

[4] M.P. Brodmann and A. Lashgari Faghani, *A finiteness result for associated primes of local cohomology modules*, Proc. Amer. Math. Soc. **128** (10) (2000) 2851–2853. doi:10.1090/S0002-9939-00-05328-4

[5] M.P. Brodmann and R.Y. Sharp, Local cohomology: An algebraic introduction with geometric applications (Cambridge University Press, 1998). doi:10.1017/CBO9780511629204

[6] W. Bruns and J. Herzog, Cohen-Macaulay Rings (Cambridge University Press, revised ed., 1998). doi:10.1017/CBO9780511608681

[7] L. Chu, *Top local cohomology modules with respect to a pair of ideals*, Proc. Amer. Math. Soc. **139** (2011) 777–782. doi:10.1090/S0002-9939-2010-10471-9

[8] L. Chu and Q. Wang, *Some results on local cohomology modules defined by a pair of ideals*, J. Math. Kyoto Univ. **49** (2009) 193–200.

[9] R. Hartshorne, *Affine duality and cofiniteness*, Invent. Math. **9** (1970) 145–164. doi:10.1007/BF01404554

[10] C. Huneke, *Free resolutions in commutative algebra and algebraic geometry*, Res. Notes Math. 2, Jones and Bartlett (Boston, MA, 1992) 93–108.

[11] J. Rotman, An Introduction to Homological Algebra (Academic Press, Second Edition, 2009). doi:10.1007/b98977

[12] P. Schenzel, *Explicit computations around the Lichtenbaum-Hartshorne vanishing theorem*, Manuscripta Math. **78** (1) (1993) 57–68. doi:10.1007/BF02599300

[13] A. Singh, *P-torsion elements in local cohomology modules*, (English summary) Math. Res. Lett. **7** (2000) 165–176. doi:10.4310/MRL.2000.v7.n2.a3

[14] R. Takahashi, Y. Yoshino and T. Yoshizawa, *Local cohomology based on a nonclosed support defined by a pair of ideals*, J. Pure Appl. Algebra. **213** (2009) 582–600. doi:10.1016/j.jpaa.2008.09.008

[15] A. Tehranian and A. Pour Eshmanan Talemi, *Cofinitness of local cohomology based on a non-closed spport defiend by a pair of ideals*, Bull. Iranian Math. Soc. **36** (2) (2010) 145–155.

Recived 14 June 2015
Revised 30 November 2015