Discussiones Mathematicae
General Algebra and Applications 36 (2016) 101–111
doi:10.7151/dmgaa.1243

POINTED PRINCIPALLY ORDERED REGULAR SEMIGROUPS

T.S. Blyth

Mathematical Institute
University of St Andrews, Scotland
e-mail: tsb@st-and.ac.uk

AND

G.A. Pinto

School of Mathematics, Physics and Technology
College of The Bahamas, Freeport, Commonwealth of The Bahamas
e-mail: gcapinto@gmail.com

Abstract

An ordered semigroup S is said to be *principally ordered* if, for every $x \in S$ there exists $x^\star = \max\{y \in S \mid xyx \leqslant x\}$. Here we investigate those principally ordered regular semigroups that are *pointed* in the sense that the classes modulo Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$ have biggest elements which are idempotent. Such a semigroup is necessarily a semiband. In particular we describe the subalgebra of $(S;^\star)$ generated by a pair of comparable idempotents that are $\mathcal{D}$-related. We also prove that those $\mathcal{D}$-classes which are subsemigroups are ordered rectangular bands.

Keywords: regular semigroup, principally ordered, naturally ordered, Green's relations.

2010 Mathematics Subject Classification: 06F05, 20M17.

An ordered regular semigroup S is said to be *principally ordered* [3] if, for every $x \in S$ there exists $x^\star = \max\{y \in S \mid xyx \leqslant x\}$. The basic properties of the unary operation $x \mapsto x^\star$ in such a semigroup were established in [3] and are listed in [1, Theorem 13.26]. In particular, we recall for the reader's convenience that in such a semigroup the following properties hold and will be used throughout what follows:

(P_1) $(\forall x \in S)$ $x = xx^\star x$;

(P_2) every $\mathcal{L}$-class $[x]_\mathcal{L}$ contains a biggest idempotent, namely $x^\star x$;

(P_3) every $\mathcal{R}$-class $[x]_{\mathcal{R}}$ contains a biggest idempotent, namely $xx^\star$;
(P_4) $(\forall x \in S)x^{\star\star\star} = x^\star$;
(P_5) every $x \in S$ has a biggest inverse, namely $x^\circ = x^\star x x^\star$;
(P_6) $(\forall x \in S)x^\circ \leqslant x^\star$;
(P_7) $(\forall x \in S)\ x \leqslant x^{\star\star} = x^{\circ\star} = x^{\star\circ}$.

The point of departure for our investigation here is the following observation.

Theorem 1. *If S is a principally ordered regular semigroup then the following statements are equivalent*:

(1) *every $\mathcal{L}$-class has a biggest element which is idempotent;*

(2) $(\forall x \in S)x^\star x = \max [x]_{\mathcal{L}}$;

(3) *every $\mathcal{R}$-class has a biggest element which is idempotent*;

(4) $(\forall x \in S)xx^\star = \max [x]_{\mathcal{R}}$;

(5) $(\forall x \in S)x^2 \leqslant x$;

(6) $(\forall x \in S)x^\star \in E(S)$.

Moreover, if S satisfies any of the above conditions then

(7) $(\forall x \in S) \quad \max [x^\star]_{\mathcal{R}} = x^\star = x^{\star\star} = \max [x^\star]_{\mathcal{L}}$;

(8) *S is a semiband and Green's relation $\mathcal{H}$ is equality*;

(9) *$x \in S$ is completely regular if and only if $x \in E(S)$.*

Proof. (1) $\Leftrightarrow$ (2): If (1) holds and $e = e^2 = \max [x]_{\mathcal{L}}$ then, by (P_2), $e = e^\star e = x^\star x$ whence (2) holds. The converse is clear.

(3) $\Leftrightarrow$ (4): This is dual to (1) $\Leftrightarrow$ (2).

(2) $\Rightarrow$ (5): If (2) holds then $x \leqslant x^\star x$ gives, by (P_1), $x^2 \leqslant xx^\star x = x$.

(5) $\Rightarrow$ (2): If (5) holds then $x^3 \leqslant x^2 \leqslant x$, so $x \leqslant x^\star$. Then $xx^\star \leqslant x^{\star 2} \leqslant x^\star$ and consequently $x = xx^\star x \leqslant x^\star x$ for every $x \in S$. Every $y \in [x]_{\mathcal{L}}$ is then such that $y \leqslant y^\star y = x^\star x$, whence it follows that $x^\star x = \max [x]_{\mathcal{L}}$.

(4) $\Leftrightarrow$ (5): This is dual to (2) $\Leftrightarrow$ (5).

(5) $\Rightarrow$ (6): Suppose that (5) holds. Then by the above so do (2) and (4). Now if $y \,\mathcal{R}\, x$ then, by (4), we have $y \leqslant xx^\star$ whence $yx \leqslant xx^\star x = x$. It follows by (5) that $xyx \leqslant x^2 \leqslant x$ and so $y \leqslant x^\star$. In particular, on taking $y = xx^\star$ we obtain $xx^\star \leqslant x^\star$ for every $x \in S$. Replacing x by $x^{\star\star}$ in this, we obtain $x^{\star\star}x^\star \leqslant x^\star$ and it follows by (2) that $x^\star = x^{\star\star}x^\star \in E(S)$.

(6) $\Rightarrow$ (2): Clearly, every $e \in E(S)$ is such that $e \leqslant e^\star$. Thus, if (6) holds then $x^\star \leqslant x^{\star\star}$ and $x^{\star\star} \leqslant x^{\star\star\star} = x^\star$. Consequently $x^\star = x^{\star\star}$ for every $x \in S$.

We now observe that

$$y \equiv x(\mathcal{R}) \implies y^\star = x^\star.$$

Indeed, if $y \equiv x(\mathcal{R})$ then, by (6), $xx^\star y^\star x = yy^\star y^\star x = yy^\star x = xx^\star x = x$ whence $x^\star y^\star \leqslant x^\star$. Then $x^\star y^\star x^\star \leqslant x^{\star 2} = x^\star$ and consequently $y^\star \leqslant x^{\star\star} = x^\star$. Interchanging x and y produces the reverse inequality and therefore $y^\star = x^\star$.

Taking in particular $y = xx^\star$ we then have $xx^\star \leqslant (xx^\star)^\star = x^\star$ whence $x = xx^\star x \leqslant x^\star x$ for every $x \in S$. If now $z \in [x]_{\mathcal{L}}$ then it follows that $z \leqslant z^\star z = x^\star x$ and therefore $x^\star x = \max [x]_{\mathcal{L}}$, which is (2).

Suppose now that the above conditions are satisfied.

(7) As shown in (6) $\Rightarrow$ (2), $x^\star = x^{\star\star} \in E(S)$, and therefore $x^\star = x^{\star\star}x^\star$. Then, by (4) and (P_4), $x^\star = \max [x^{\star\star}]_{\mathcal{R}} = \max [x^\star]_{\mathcal{R}}$. Dually, we see that also $x^\star = \max [x^\star]_{\mathcal{L}}$.

(8) Since, by (6), each $x^\star$ is idempotent, we have $x = xx^\star x = xx^\star \cdot x^\star x$ and so every $x \in S$ is a product of two idempotents, whence S is a semiband. Moreover, if $x \,\mathcal{H}\, y$ then $x = xx^\star x = xx^\star \cdot x^\star x = yy^\star \cdot y^\star y = yy^\star y = y$ whence $\mathcal{H}$ reduces to equality.

(9) If $x \in S$ is completely regular then there exists $x' \in V(x)$ such that $xx' = x'x$. Then, by (5), $x' = x'xx' = {x'}^2 x \leqslant x'x$ from which it follows that $x = xx'x \leqslant xx'xx = x^2$ and consequently $x \in E(S)$. The converse is clear. ∎

Definition. We shall say that a principally ordered regular semigroup is *pointed* whenever it satisfies any of the six equivalent properties of Theorem 1.

By way of providing a source of examples, we recall that the *natural order* $\leqslant_n$ on the idempotents of a regular semigroup is defined by

$$e \leqslant_n f \iff e = ef = fe,$$

and that an ordered regular semigroup $(T; \leqslant)$ is said to be *naturally ordered* if the order $\leqslant$ extends the natural order, in the sense that if $e \leqslant_n f$ then $e \leqslant f$. In this case, a fundamental property is that if $e \leqslant f$ then $e = efe$; see, for example, [1, Theorem 13.11].

Theorem 2. *If T is a naturally ordered regular semigroup with a biggest idempotent ξ then the semiband $\langle E(T)\rangle$ is a pointed principally ordered regular semigroup.*

Proof. If $\overline{e} = e_1 \cdots e_n \in \langle E(T)\rangle$ then, since ξ is the biggest element of $\langle E(T)\rangle$, we have that $e\xi e = e$ for every $e \in E(T)$, and consequently

$$\overline{e}\,\xi\,\overline{e} = e_1 \cdots e_n \xi e_1 \cdots e_n \leqslant e_1 \xi e_1 \cdots e_n = e_1 \cdots e_n = \overline{e}.$$

It follows that the regular subsemigroup $\langle E(T)\rangle$ is principally ordered with $\overline{e}^\star = \xi$ for every $\overline{e} \in \langle E(T)\rangle$. Furthermore, $\overline{e}^2 = \overline{e}e_1\overline{e} \leqslant \overline{e}\,\xi\,\overline{e} \leqslant \overline{e}$ and it follows by Theorem 1(5) that $\langle E(T)\rangle$ is pointed. ∎

To avoid unnecessary repetition throughout what follows, *S will always denote a pointed principally ordered regular semigroup.*

As we have seen above, a characteristic property of S is that the classes modulo Green's relations $\mathcal{R}$ and $\mathcal{L}$ have biggest elements which are idempotent. We now show that the same is true for Green's relation $\mathcal{D}$.

Theorem 3. *Green's relation $\mathcal{D}$ on S is given by*

$$(x, y) \in \mathcal{D} \iff x^{\circ} = y^{\circ}.$$

Moreover, every $\mathcal{D}$-class has a biggest element which is idempotent. Specifically,

$$(\forall x \in S) \quad x^{\circ} = x^{\circ\circ} = \max\,[x^{\star}x]_{\mathcal{R}} = \max\,[xx^{\star}]_{\mathcal{L}} = \max\,[x]_{\mathcal{D}} \in E(S).$$

Proof. As observed in the proof of Theorem 1, we have $(xx^{\star})^{\star} = x^{\star}$ and therefore, by Theorem 1(4),

$$x^{\circ} = x^{\star}xx^{\star} = x^{\star}x(x^{\star}x)^{\star} = \max\,[x^{\star}x]_{\mathcal{R}} \in E(S),$$

and dually for $\mathcal{L}$. Moreover, by (P_7) and Theorem 1(6,7),

$$x^{\circ\circ} = x^{\circ\star}x^{\circ}x^{\circ\star} = x^{\star\star}x^{\star}xx^{\star}x^{\star\star} = x^{\star}xx^{\star} = x^{\circ}.$$

If now $x\,\mathcal{D}\,y$ then there exists $z \in S$ such that $x\,\mathcal{L}\,z\,\mathcal{R}\,y$. Then $x^{\star}x = z^{\star}z$ and $zz^{\star} = yy^{\star}$. It follows from the above that $x^{\circ} = z^{\circ} = y^{\circ}$. On the other hand, $x\,\mathcal{L}x^{\star}x\,\mathcal{R}\,x^{\circ}$ gives $x\,\mathcal{D}\,x^{\circ}$. Consequently $x\,\mathcal{D}\,y \iff x^{\circ} = y^{\circ}$. Finally, by Theorem 1(2,4) we see that $x \leqslant xx^{\star} \leqslant x^{\star}xx^{\star} = x^{\circ}$ whence it follows that $x^{\circ} = \max\,[x]_{\mathcal{D}} \in E(S)$. ■

Theorem 4. (1) *$x \in S$ is a maximal idempotent if and only if it is a maximal element*;

(2) *S contains at most one maximal element.*

Proof. (1) Suppose that e is a maximal idempotent of S. If $x \in S$ is such that $e \leqslant x$ then we have $e \leqslant x \leqslant x^{\star} \in E(S)$, whence the hypothesis that e is maximal in $E(S)$ gives $e = x$. Thus e is a maximal element of S. Conversely, if $x \in S$ is a maximal element then $x \leqslant x^{\star}$ gives $x = x^{\star}$ whence, by Theorem 1(6), $x \in E(S)$.

(2) Let e and f be maximal elements of S. By (1), each is then idempotent. Now, by Theorem 1(5), $ef\cdot e\cdot ef = (ef)^2 \leqslant ef$ and gives $e \leqslant (ef)^{\star}$. It follows that $e = (ef)^{\star}$ and likewise $e = (fe)^{\star}$. Similarly, $f = (fe)^{\star} = (ef)^{\star}$ and consequently $e = f$. ■

By [4, Theorem 3.3], a principally ordered regular semigroup is naturally ordered if and only if the assignment $x \mapsto x^{\star}$ is antitone. In this case, as shown in [1, Theorem 13.29], each $(xx^{\star})^{\star}$ is a maximal idempotent. Using this fact in the case where S is pointed, we obtain the following characterisation.

Theorem 5. *The following statements are equivalent*:

(1) S *is naturally ordered*;

(2) S *has a biggest element* ξ *and* $x^\star = \xi$ *for every* $x \in S$.

Proof. (1) ⇒ (2): If (1) holds then each $(xx^\star)^\star = x^\star$ is a maximal idempotent and $x \leqslant x^\star$. Then property (2) follows immediately from Theorem 4.

(2) ⇒ (1): Suppose conversely that (2) holds and let $e, f \in E(S)$ be such that $e \leqslant_n f$. By (2), $e^\star = \xi = f^\star$ and consequently $e = ef = fef \leqslant fe^\star f = ff^\star f = f$. Thus S is naturally ordered. ∎

Corollary. *If* S *is naturally ordered then Green's relations* $\mathcal{D}$ *and* $\mathcal{J}$ *coincide.*

Proof. By Theorem 5, $(x^2)^\star = \xi = x^\star$. Consequently, $x^2 = x^2(x^2)^\star x^2 = x^2 x^\star x^2 = x^3$. Then $x^2 \in E(S)$ and so S is group bound. It follows by [6, Theorem 1.2.20] that $\mathcal{D}$ and $\mathcal{J}$ coincide. ∎

Consider now the subset $S^\star = \{x^\star \mid x \in S\}$. This is related to the subset S° and to the set $C = \{x \in S \mid x^\star = x^\circ\}$ of *compact elements* as follows.

Theorem 6. $S^\star = C \cap S^\circ$.

Proof. The identity $x^{\star\star} = x^{\star\circ}$ shows that $S^\star \subseteq C$. Similarly, $x^\star = x^{\star\star\star} = x^{\star\star\circ} = x^{\star\circ\circ}$ shows that $S^\star \subseteq S^\circ$. Thus $S^\star \subseteq C \cap S^\circ$. Conversely, if $x \in C \cap S^\circ$ then $x^\star = x^\circ$ and $x = x^{\circ\circ}$, whence $x = x^{\circ\circ} = x^{\star\circ} = x^{\star\star} \in S^\star$. ∎

As the following example shows, $S^\star$ is not in general a subsemigroup of S.

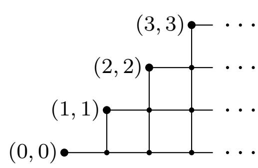
Example 1. Let L be a lattice and consider the cartesian ordered set

$$L^{[2]} = \{(x, y) \in L \times L \mid y \leqslant x\}.$$

With respect to the multiplication defined by

$$(x, y)(a, b) = (x \vee a,\ y \wedge b),$$

it is clear that $L^{[2]}$ is an ordered band. It is readily verified that $L^{[2]}$ is principally ordered with $(x, y)^\star = (x, x)$. By Theorem 1(5), $L^{[2]}$ is pointed with $(L^{[2]})^\star = \{(x, x) \mid x \in L\}$. Now $(L^{[2]})^\star$ is not a subsemigroup, for clearly $(x, y)^\star(a, b)^\star = (x \vee a, x \wedge a)$ and this belongs to $(L^{[2]})^\star$ if and only if $x = a$. The particular case of $\mathbb{N}^{[2]}$ is illustrated as follows:



However, in the presence of an identity element 1 the subset $S^\star$ has a particular description.

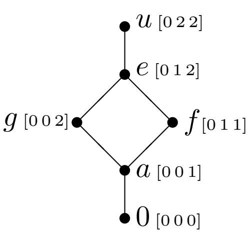
Theorem 7. *If S has an identity element* 1, *then* $S^\star = \{x \in S \mid 1 \leqslant x\}$ *and is a join semilattice in which* $x \vee y = xy$.

Proof. If $x \in S$, then since $x1x = x^2 \leqslant x$ we have $1 \leqslant x^\star$. Conversely, let $1 \leqslant x$. Then

$$x^\star = \begin{cases} 1x^\star \leqslant xx^\star \leqslant x^\star x^\star = x^\star & \text{whence } x^\star = xx^\star; \\ x^\star 1 \leqslant x^\star x \leqslant x^\star x^\star = x^\star & \text{whence } x^\star = x^\star x. \end{cases}$$

Hence $x^\star \,\mathcal{H}\, x$ and so $x^\star = x$ by Theorem 1(8). Thus we see that $S^\star = \{x \in \langle E(S)\rangle \mid 1 \leqslant x\}$ and is a sub-band. Now if $x, y \in S^\star$ then $x = x1 \leqslant xy$ and $y = 1y \leqslant xy$, so that xy is an upper bound for $\{x, y\}$. Furthermore, if $z \in S$ is any upper bound for $\{x, y\}$ then necessarily $z \in S^\star$ whence $xy \leqslant z^2 = z$. Consequently, $S^\star$ is a join semilattice in which $x \vee y = xy$. ■

Example 2. Let **3** denote the 3-element chain $0 < 1 < 2$ and consider the ordered regular semigroup consisting of those isotone mappings f on **3** which are such that $f(0) = 0$. Equivalently, this is the semigroup Res **3** of residuated mappings on **3** [2]. It has the following Hasse diagram and Cayley table, in which $[0\,a\,b]$ denotes the mapping f such that $f(0) = 0$, $f(1) = a$, $f(2) = b$.



	u	e	f	g	a	0
u	u	u	u	g	g	0
e	u	e	f	g	a	0
f	f	f	f	a	a	0
g	u	g	0	g	0	0
a	f	a	0	a	0	0
0	0	0	0	0	0	0

This semiband is principally ordered and pointed, with identity element e. Here we have $x^\star = u$ for $x \neq e$ and $e^\star = e$, so that $S^\star = \{e, u\}$.

Example 3. Consider, for $n \geqslant 2$, the ordered semigroup $\mathbf{B}_n$ of $n \times n$ matrices with entries in a boolean algebra **B**. For the basic operations in **B** we use the notation $a + b$ (for $a \vee b$) and ab (for $a \wedge b$).

As shown in [1], this semigroup is regular if and only if $n = 2$. Moreover, as is established in [5], $\mathbf{B}_2$ is principally ordered with

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\star = \begin{bmatrix} b' + c' + d & a' + d' + b \\ a' + d' + c & b' + c' + a \end{bmatrix}.$$

The set of idempotents is

$$E(\mathbf{B}_2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid b + c \leqslant a + d,\ bc \leqslant ad \right\},$$

and the regular subsemigroup they generate is

$$\langle E(\mathbf{B}_2) \rangle = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid bc \leqslant ad \right\}.$$

The semiband $\langle E(\mathbf{B}_2) \rangle$ is also principally ordered and pointed. This follows from Theorem 1 and the observation that $bc \leqslant ad$ gives $b' + c' \geqslant a' + d'$ whence

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\star} = \begin{bmatrix} 1 & a' + d' + b \\ a' + d' + c & 1 \end{bmatrix} \in E(\mathbf{B}_2).$$

Since $\mathbf{B}_2$ has an identity element it follows from Theorem 7 that $\langle E(\mathbf{B}_2) \rangle^{\star}$ is the join semilattice

$$\langle E(\mathbf{B}_2) \rangle^{\star} = \{ X \in \mathbf{B}_2 \mid I_2 \leqslant X \} = \left\{ \begin{bmatrix} 1 & x \\ y & 1 \end{bmatrix} \mid x, y \in \mathbf{B} \right\}.$$

We can also identify the compact elements of $\langle E(\mathbf{B}_2) \rangle$. For this we recall from (P_5) that every $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathbf{B}_2$ has a biggest inverse, namely

$$X^{\circ} = X^{\star} X X^{\star} = \begin{bmatrix} b'(a+c) + c'(a+b) + d & a'(c+d) + d'(a+c) + b \\ a'(b+d) + d'(a+b) + c & b'(c+d) + c'(b+d) + a \end{bmatrix}.$$

In particular, if $X \in \langle E(\mathbf{B}_2) \rangle$ then the inequality $bc \leqslant ad$ gives $d = d + bc$ and $a = a + bc$, so that we obtain

$$X^{\circ} = \begin{bmatrix} a+b+c+d & a'(c+d) + d'(a+c) + b \\ a'(b+d) + d'(a+b) + c & a+b+c+d \end{bmatrix}.$$

Thus, if $X \in \langle E(\mathbf{B}_2) \rangle$ is compact then necessarily $a + b + c + d = 1$. Conversely, if the property $a + b + c + d = 1$ holds then

$$\begin{aligned} a'(c+d) + d'(a+c) + b &\geqslant a'(a+b)' + d'(b+d)' + b \\ &= a'b' + b'd' + b \\ &= a' + d' + b. \end{aligned}$$

Clearly, the reverse inequality holds, so that $a'(c+d) + d'(a+c) + b = a' + d' + b$. Likewise, we see that $a'(b+d) + d'(a+b) + c = a' + d' + c$ and consequently $X^{\circ} = X^{\star}$. Hence the set of compact elements of $\langle E(\mathbf{B}_2) \rangle$ is

$$C = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \langle E(\mathbf{B}_2) \rangle \mid a + b + c + d = 1 \right\}.$$

We now turn attention to the $\mathcal{D}$-classes of S. For idempotents e, f with $e \leqslant f$ and $(e, f) \in \mathcal{D}$ we first focus on the structure of the subalgebra of $(S; {}^\star)$ generated by $\{e, f\}$. In this connection the following observation is important.

Theorem 8. *Any two comparable $\mathcal{D}$-related idempotents of S are mutually inverse.*

Proof. Let $e, f \in E(S)$ be such that $e \leqslant f$ and $e \,\mathcal{D}\, f$. Then, by Theorem 3, $e^\circ = f^\circ$. Consequently, by Theorem 1(7) and (P_7), $e^\star = e^{\star\star} = e^{\circ\star} = f^{\circ\star} = f^{\star\star} = f^\star$. Moreover, the idempotents e° and $e^\star$ are such that $e^\circ e^\star = e^\circ = e^\star e^\circ$, whence $e^\circ \leqslant_n e^\star$.

We first observe that $e = eee \leqslant efe \leqslant ef^\star e = ee^\star e = e$ so that $e = efe$.

Consider now $fee^\star$. That $fee^\star \in E(S)$ follows from the inequalities

$$fee^\star = fee^\star eee^\star \leqslant fee^\star \cdot fee^\star \leqslant fff^\star fee^\star = fee^\star.$$

Now $fee^\star \cdot e^\circ = fee^\circ = fee^\star$ and

$$e^\circ \cdot fee^\star \begin{cases} \leqslant & f^\circ ffe^\star = f^\circ ff^\star = f^\circ ff^\circ = f^\circ = e^\circ; \\ \geqslant & e^\circ ee^\star = e^\circ ee^\circ = e^\circ, \end{cases}$$

so that $e^\circ \cdot fee^\star = e^\circ$. Consequently $fee^\star \,\mathcal{L}\, e^\circ = f^\circ \,\mathcal{L}\, ff^\star$.

Furthermore, $fee^\star \cdot ff^\star = fef^\star ff^\star = fef^\circ = fee^\circ = fee^\star$ and $ff^\star \cdot fee^\star = fee^\star$ show that $fee^\star \leqslant_n ff^\star$. Since these idempotents are also $\mathcal{L}$-equivalent it follows that $fee^\star = ff^\star$.

Using the above observations, we see that $fef \cdot ff^\star = feff^\star = fefee^\star = fee^\star = ff^\star$ whence $fef \,\mathcal{R}\, ff^\star \,\mathcal{R}\, f$. Since $fef \in E(S)$ with $fef \leqslant_n f$ it follows that $f = fef$.

Thus e and f are mutually inverse. ■

Corollary. *The following statements are equivalent*:

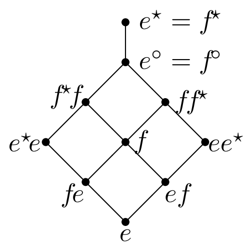
(1) *S is completely simple*;

(2) *S is compact and naturally ordered.*

Proof. (1) ⇒ (2): If S is completely simple then, since $\leqslant_n$ reduces to equality, S is trivially naturally ordered. Since the idempotents x°, $x^\star$ are such that $x^\circ \leqslant_n x^\star$, it follows that $x^\circ = x^\star$ and therefore S is compact.

(2) ⇒ (1): Suppose that (2) holds and that $e, f \in E(S)$ are such that $e \leqslant_n f$. By Theorem 5, S has a biggest element ξ and $f^\star = \xi = e^\star$. Compactness now gives $f^\circ = e^\circ$ whence, by Theorem 3, $(e, f) \in \mathcal{D}$. Since also the natural order implies that $e \leqslant f$, it follows by Theorem 8 that the idempotents e and f are mutually inverse. Consequently, $f = fef = e$. Thus $\leqslant_n$ reduces to equality and S is completely simple. ■

Theorem 9. *Let e, f be idempotents of S such that $e \leqslant f$ and $e\,\mathcal{D}\,f$. If T is the subalgebra of $(S;{}^\star)$ generated by $\{e, f\}$ then T is a band having at most* 10 *elements. In the case where T has precisely* 10 *elements it is represented by the Hasse diagram*



in which elements joined by lines of positive gradient are $\mathcal{R}$-related, those joined by lines of negative gradient are $\mathcal{L}$-related, and the vertical line also indicates $\leqslant_n$.

Proof. Since $e\,\mathcal{D}\,f$ it follows from Theorem 3 that $e^\circ = f^\circ$ whence $e^\star = f^\star$. The elements of T are then finite products of the elements e, f and $e^\star[= f^\star]$. Moreover, since $e \leqslant f$, every $x \in T$ is such that $e \leqslant x \leqslant e^\star$. By Theorem 8, e and f are mutually inverse, so for every $x \in T$ we have

$$f = fef \leqslant fxf \leqslant fe^\star f = ff^\star f = f$$

whence $f = fTf$. In a similar way we see that $e = eTe$ and likewise

$$ee^\star = eTe^\star,\;\; e^\star e = e^\star Te,\;\; ff^\star = fTf^\star,\;\; f^\star f = f^\star Tf,\;\; ef = eTf,\;\; fe = fTe.$$

For example, $ee^\star = eee^\star \leqslant exe^\star \leqslant ee^\star e^\star = ee^\star$ gives $ee^\star = eTe^\star$. It follows from this that $ee^\star = efe^\star = eff^\star$ whence $ee^\star f = ef$ and then $ef = eef \leqslant exf \leqslant ee^\star f = ef$ and consequently $ef = eTf$. Finally, it is readily seen from the above that $e^\star Te^\star = \{e^\circ, e^\star\}$. It now follows from these observations that T is a band which consists of at most 10 elements, has precisely two $\mathcal{D}$-classes, and is as described in the above Hasse diagram. ■

Example 4. In the semigroup $\langle E(\mathbf{B}_2)\rangle$ of Example 3, let $|\mathbf{B}| \geqslant 8$ and consider the idempotents

$$e = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} a & b \\ b & b \end{bmatrix} \quad \text{where } 0 < b < a < 1.$$

Simple calculations which use the expressions for $X^\star$ and X° given in Example 3 reveal that $e^\star = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = f^\star$, and that $e^\circ = \begin{bmatrix} a & a \\ a & a \end{bmatrix} = f^\circ$ whence $e\,\mathcal{D}\,f$ with $e < f$.

Furthermore,

$$ee^\star = \begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix},\ ff^\star = \begin{bmatrix} a & a \\ b & b \end{bmatrix},\ e^\star e = \begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix},\ f^\star f = \begin{bmatrix} a & b \\ a & b \end{bmatrix},\ ef = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix},\ fe = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}.$$

Consequently we have a copy of the band depicted in Theorem 9.

We now proceed to describe the structure of those $\mathcal{D}$-classes that are subsemigroups of S (which is the case for D_e in Theorem 9, but not so for D_u in Example 2 since $gf = 0$).

Theorem 10. *Given $e \in E(S)$, suppose that D_e is a subsemigroup of S. Then L_{e° is a left zero semigroup, R_{e° is a right zero semigroup, and D_e is isomorphic to the ordered rectangular band $L_{e^\circ} \times R_{e^\circ}$.*

Proof. We observe first that, since $x^\circ = e^\circ$ for every $x \in D_e$,

$$x \in L_{e^\circ} \iff x^\circ x = e^\circ \iff x = xx^\circ \in D_e;$$
$$x \in R_{e^\circ} \iff xx^\circ = e^\circ \iff x = x^\circ x \in D_e.$$

If therefore $x, y \in L_{e^\circ}$ we have $xy = xx^\circ y = xe^\circ y = xy^\circ y = xe^\circ = xx^\circ = x$ and consequently L_{e° is a left zero semigroup. Likewise, R_{e° is a right zero semigroup. Then

$$L_{e^\circ} \times R_{e^\circ} = \{(xe^\circ,\, e^\circ y) \mid x, y \in D_e\}$$

is a rectangular band. Consider therefore the mapping $\vartheta : D_e \to L_{e^\circ} \times R_{e^\circ}$ given by the prescription $\vartheta(x) = (xe^\circ,\, e^\circ x)$, which is clearly isotone.

Now if $(a, b) \in L_{e^\circ} \times R_{e^\circ}$ then, since $ab \in D_e$ by the hypothesis with

$$\vartheta(ab) = (abe^\circ, e^\circ ab) = (abb^\circ, a^\circ ab) = (ae^\circ, e^\circ b) = (a, b),$$

we see that ϑ is surjective. Moreover,

$$\begin{aligned} \vartheta(x) \leqslant \vartheta(y) &\iff xe^\circ \leqslant ye^\circ,\ e^\circ x \leqslant e^\circ y \\ &\iff x = xe^\circ x \leqslant ye^\circ y = y. \end{aligned}$$

It follows from these observations that ϑ is an order isomorphism.

We now observe that if e, f are $\mathcal{D}$-equivalent idempotents such that $e \leqslant_n f$ then $e = ef = fe \leqslant fe^\circ = ff^\circ$ and consequently $e = ef \leqslant ff^\circ f = f$. Thus D_e is a naturally ordered regular semigroup with a biggest idempotent e°. Since $(xy)^\circ = e^\circ = y^\circ x^\circ$ for all $x, y \in D_e$, it follows by [1, Theorem 13.18] that e° is a middle unit of D_e. Using this fact, we see that

$$\vartheta(x)\vartheta(y) = (xe^\circ,\, e^\circ x)(ye^\circ,\, e^\circ y) = (xe^\circ ye^\circ,\, e^\circ xe^\circ y) = (xye^\circ,\, e^\circ xy) = \vartheta(xy),$$

whence we conclude that ϑ defines an ordered semigroup isomorphism $D_e \simeq L_{e^\circ} \times R_{e^\circ}$. ■

REFERENCES

[1] T.S. Blyth, Lattices and Ordered Algebraic Structures (Springer, 2005). doi:10.1007/b139095

[2] T.S. Blyth and M.F. Janowitz, Residuation Theory (Pergamon, 1972).

[3] T.S. Blyth and G.A. Pinto, *Principally ordered regular semigroups*, Glasgow Math. J. **32** (1990) 349–364. doi:10.1017/S0017089500009435

[4] T.S. Blyth and G.A. Pinto, *Idempotents in principally ordered regular semigroups*, Communications in Algebra **19** (1991) 1549–1563. doi:10.1080/00927879108824220

[5] T.S. Blyth and M.H. Almeida Santos, *On weakly multiplicative inverse transversals*, Proc. Edinburgh Math. Soc. **37** (1993) 93–99. doi:10.1017/S001309150001871X

[6] P.M. Higgins, Techniques of Semigroup Theory (Oxford Science Publiications, 1992). doi:10.1007/BF02573500

Received 11 January 2016
Revised 20 May 2016