IF-FILTERS OF PSEUDO-BL-ALGEBRAS

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Abstract
Characterizations of IF-filters of a pseudo-BL-algebra are established. Some related properties are investigated. The notation of prime IF-filters and a characterization of a pseudo-BL-chain are given. Homomorphisms of IF-filters and direct product of IF-filters are studied.

Keywords: pseudo-BL-algebra, filter, IF-filter, prime IF-filters, pseudo-BL-chain, homomorphism, direct product.

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1. Introduction
In 1958, Chang [2] gave a notation and a characterization of MV-algebras. In 1998, Hájek [8] introduced BL-algebras, which contain the class of MV-algebras. Georgescu and Iorgulescu [5] and independently Rachůnek [10] introduced pseudo MV-algebras as a noncommutative extension of MV-algebras. Finally, in 2000 there were given a notion of pseudo-BL-algebras, which are a noncommutative extension of BL-algebras. Some important properties of pseudo-BL-algebras were studied in [3, 4] and [7].


In this paper, we introduce a notation of intuitionistic fuzzy filters of pseudo-BL-algebras and study their properties. We introduce prime intuitionistic fuzzy filters and using them we give a characterization of a pseudo-BL-chain. We investigate a homomorphism of intuitionistic fuzzy filters. Finally, we study a direct product of intuitionistic fuzzy filters. We will write shortly IF-filters instead of intuitionistic fuzzy filters.
2. Preliminaries

**Definition 1.** In [6], there were introduced a pseudo-BL-algebra $A$ as an algebra $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following axioms for all $x, y, z \in A$:

(C1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice;
(C2) $(A, \odot, 1)$ is a monoid;
(C3) $x \odot y \leq z \iff x \leq y \rightarrow z \iff y \leq x \rightsquigarrow z$;
(C4) $x \wedge y = (x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y)$;
(C5) $(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$.

**Lemma 1 ([7]).** Let $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ be a pseudo-BL-algebra. Then for all $x, y, z \in A$:

(i) $y \leq x \rightarrow y$ and $y \leq x \rightsquigarrow y$;
(ii) $x \odot y \leq x \wedge y$;
(iii) $x \odot y \leq x$ and $x \odot y \leq y$;
(iv) $x \rightarrow 1 = x \rightsquigarrow 1 = 1$;
(v) $x \leq y \iff x \rightarrow y = x \rightsquigarrow y = 1$;
(vi) $x \rightarrow x = x \rightsquigarrow x = 1$;
(vii) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$ and $x \rightsquigarrow (y \rightsquigarrow z) = (y \odot x) \rightsquigarrow z$.

We will write shortly $A$ instead of $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$.

**Definition 2.** A nonempty subset $F$ of a pseudo-BL-algebra $A$ is called a filter if it satisfies the following two conditions:

(F1) if $x, y \in F$, then $x \odot y \in F$;
(F2) if $x \in F$ and $x \leq y$, then $y \in F$.

A filter $F$ of a pseudo-BL-algebra $A$ is called proper if $F \neq A$. The proper filter $F$ is prime if for all $x, y \in A$

$$x \vee y \in F \; \implies \; (x \in F \; \text{or} \; y \in F).$$

Now, we give definitions of a fuzzy filter and an anti fuzzy filter of a pseudo-BL-algebra $A$ and their some properties.
Recall that a fuzzy set of $A$ is a function $\nu : A \to [0, 1]$. For any fuzzy set $\nu$ and real number $\alpha \in [0, 1]$ there are defined two sets:

$$U(\nu, \alpha) = \{ x \in A : \nu(x) \geq \alpha \}$$

$$L(\nu, \alpha) = \{ x \in A : \nu(x) \leq \alpha \}$$

which are called an upper and a lower $\alpha$-level set of $\nu$.

**Definition 3.** Let $\nu$ be a fuzzy set of pseudo-BL-algebra $A$. A complement of $\nu$ is the fuzzy set $\nu^C$ defined as follows

$$\nu^C(x) = 1 - \nu(x)$$

for any $x \in A$.

A fuzzy set $\mu$ is called:

1. a fuzzy filter, if for all $x, y \in A$
   - (ff1) $\mu(x \odot y) \geq \mu(x) \wedge \mu(y)$;
   - (ff2) $x \leq y \Rightarrow \mu(x) \leq \mu(y)$.

2. an anti fuzzy filter, if for all $x, y \in A$
   - (af1) $\mu(x \odot y) \leq \mu(x) \vee \mu(y)$;
   - (af2) $x \leq y \Rightarrow \mu(y) \leq \mu(x)$.

**Remark 1.** Let $\mu$ and $\nu$ be a fuzzy sets of a pseudo-BL-algebra $A$. Then:

(i) $\mu$ is a fuzzy filter of $A$ iff $\mu^C$ is an anti fuzzy filter of $A$;

(ii) $\nu$ is an anti fuzzy filter of $A$ iff $\nu^C$ is a fuzzy filter of $A$.

**Definition 4** ([11]). Let $F$ be a filter of a pseudo-BL-algebra $A$ and $\alpha, \beta \in [0, 1]$ such that $\alpha > \beta$. Let us define a fuzzy filter $\mu_F(\alpha, \beta)$ as follows

$$\mu_F(\alpha, \beta)(x) = \begin{cases} \alpha & \text{if } x \in F, \\ \beta & \text{otherwise.} \end{cases}$$

**Remark 2** ([13]). A fuzzy set $\mu_F^C(\alpha, \beta)$ is an anti fuzzy filter of $A$.

We denote by $\chi_F$ the characteristic function of $F$ and by $\chi_F^C$ the complement of the characteristic function of $F$.

**Definition 5.** Let $A$ be a pseudo-BL-algebra and $\nu$ be a fuzzy filter of $A$. Then $\nu$ is called a fuzzy prime filter if

$$\nu(x \vee y) = \nu(x) \vee \nu(y)$$

for all $x, y \in A$. 
Definition 6. Let $A$ be a pseudo-BL-algebra and $\mu$ be an anti fuzzy filter of $A$. Then $\mu$ is called an anti fuzzy prime filter if

$$\mu(x \vee y) = \mu(x) \land \mu(y)$$

for all $x, y \in A$.

For a fuzzy filter $\nu$ of pseudo-BL-algebra $A$ we define a set

$$M_{\nu} = \{x \in A : \nu(x) = \nu(1)\}$$

and similarly, for an anti fuzzy filter $\mu$ we define a set

$$A_{\mu} = \{x \in A : \mu(x) = \mu(1)\}.$$

Remark 3. It is proved in [11] and [13] that a fuzzy filter $\nu$ of $A$ is a fuzzy prime filter (an anti fuzzy filter $\mu$ of $A$ is an anti fuzzy prime filter) iff $M_{\nu}(A_{\mu})$ is a prime filter of $A$.

3. IF-filters

Definition 7. A mapping $B : A \to [0, 1] \times [0, 1]$ such that $B(x) = (\nu_B(x), \mu_B(x))$, in which $\nu_B(x) + \mu_B(x) \leq 1$ for any $x \in A$, is called an IF-set of $A$.

In particular, we use $0_\sim$ and $1_\sim$ to denote the IF-empty set and the IF-whole set in a set $A$ such that $0_\sim(x) = (0; 1)$ and $1_\sim(x) = (1; 0)$ for each $x \in A$, respectively.

For IF-sets $B = (\nu_B, \mu_B)$ and $C = (\nu_C, \mu_C)$ we define a relation $\leq$ as follows:

$$B \leq C \iff (\nu_B(x) < \nu_C(x) \text{ or } (\nu_B(x) = \nu_C(x) \text{ and } \mu_B(x) < \mu_C(x))) \text{ for any } x \in A.$$ 

Now, we give the definition of an IF-filter of a pseudo-BL-algebra. From this place an IF-set $B = (\nu_B, \mu_B)$ will be denoted by $B$.

Definition 8. An IF-set $B$ of pseudo-BL-algebra $A$ is an IF-filter of $A$ if it satisfies the following conditions for all $x, y \in A$:

1. (IF1) $\nu_B(x \circ y) \geq \nu_B(x) \land \nu_B(y)$;
2. (IF2) $\mu_B(x \circ y) \leq \mu_B(x) \lor \mu_B(y)$;
3. (IF3) $x \leq y \Rightarrow (\nu_B(x) \leq \nu_B(y) \text{ and } \mu_B(x) \geq \mu_B(y)).$

Remark 4. An IF-set $B$ of a pseudo-BL-algebra $A$ is an IF-filter of $A$ iff $\nu_B$ is a fuzzy filter and $\mu_B$ is an anti fuzzy filter of $A$. 

It is easy to see, that (IF3) implies

(IF4) \( \nu_B(x) \leq \nu_B(1) \) and \( \mu_B(x) \geq \mu_B(1) \) for every \( x \in A \);

(IF4′) \( \nu_B(0) \leq \nu_B(x) \) and \( \mu_B(0) \geq \mu_B(x) \) for every \( x \in A \).

**Proposition 1.** Let \( B \) be an IF-set of a pseudo-BL-algebra \( A \). Then \( B \) is an IF-filter of \( A \) iff \( B_C = (\nu_B, \mu_B^C) \) and \( cB = (\mu_B^C, \mu_B) \) are IF-filters of \( A \).

**Proof.** \( \Rightarrow \): Let \( B \) be an IF-set of a pseudo-BL-algebra \( A \). By Remark 4 \( \nu_B \) is a fuzzy filter and \( \mu_B \) is an anti fuzzy filter of \( A \). Then \( \nu_B^C \) is an anti fuzzy filter and \( \mu_B^C \) is a fuzzy filter of \( A \). Using Remark 4 once again we obtain that \( B_C = (\nu_B, \nu_B^C) \) and \( cB = (\mu_B^C, \mu_B) \) are IF-filters of \( A \).

\( \Leftarrow \): By Remark 4.

**Example 1.** Let \( F \) be a filter of a pseudo-BL-algebra \( A \) and \( B(F) = (\nu_{B(F)}, \mu_{B(F)}) \) be an IF-set of \( A \) defined as follows

\[
\nu_{B(F)}(x) := \begin{cases} 
\alpha & \text{if } x \in F; \\
\beta & \text{otherwise}
\end{cases}
\]

and \( \mu_{B(F)}(x) := \begin{cases} 
\alpha_1 & \text{if } x \in F; \\
\beta_1 & \text{otherwise}
\end{cases} \)

where \( \alpha, \alpha_1, \beta, \beta_1 \in [0, 1], \alpha > \beta, \alpha_1 < \beta_1 \) and \( \alpha + \alpha_1, \beta + \beta_1 \leq 1 \).

By Definition 4 and Remark 2, \( \nu_{B(F)} \) is a fuzzy filter of \( A \) and \( \mu_{B(F)} \) is an anti fuzzy filter of \( A \). Hence, by Remark 4, \( B(F) \) is an IF-filter of \( A \).

**Proposition 2.** Let \( B = (\nu_B, \mu_B) \) be an IF-filter of a pseudo-BL-algebra \( A \), then for all \( x, y \in A \):

(i) \( \nu_B(x \lor y) \geq \nu_B(x) \land \nu_B(y) \);

(ii) \( \nu_B(x \land y) = \nu_B(x) \land \nu_B(y) \);

(iii) \( \nu_B(x \circ y) = \nu_B(x) \land \nu_B(y) \);

(iv) \( \mu_B(x \land y) = \mu_B(x) \lor \mu_B(y) \);

(v) \( \mu_B(x \circ y) = \mu_B(x) \lor \mu_B(y) \);

(vi) \( \mu_B(x \lor y) \leq \mu_B(x) \lor \mu_B(y) \).

**Proof.** By Lemma 1 (ii) \( x \circ y \leq x \land y \leq x \lor y \). Then, by definition of an IF-filter, \( \nu_B(x) \land \nu_B(y) \leq \nu_B(x \circ y) \leq \nu_B(x \land y) \leq \nu_B(x \lor y) \) and \( \mu_B(x) \lor \mu_B(y) \geq \mu_B(x \circ y) \geq \mu_B(x \land y) \geq \mu_B(x \lor y) \). (i) and (vi) are proved. Applying Lemma 1 (iii), we have \( \nu_B(x) \land \nu_B(y) \leq \nu_B(x \circ y) \leq \nu_B(x \land y) \leq \nu_B(x \lor y) \) and \( \mu_B(x) \lor \mu_B(y) \geq \mu_B(x \circ y) \geq \mu_B(x \land y) \geq \mu_B(x \lor y) \). The proofs for (ii), (iii), (iv) and (v) are finished.
Proposition 3. An IF-set $B$ of a pseudo-BL-algebra $A$ is an IF-filter of $A$ if and only if it satisfies (IF1), (IF2) and

(IF5) $\nu_B(x \lor y) \geq \nu_B(x)$ and $\mu_B(x \lor y) \leq \mu_B(x)$ for all $x, y \in A$.

Proof. $\Rightarrow$: Let us suppose that $B$ is an IF-filter of $A$. Then, by (IF3), $\nu_B(x \lor y) \geq \nu_B(x)$ and $\mu_B(x \lor y) \leq \mu_B(x)$ for all $x, y \in A$.

$\Leftarrow$: Conversely, let $B$ satisfies (IF1), (IF2) and (IF5). We need to show that $B$ satisfies (IF3). Let $x, y \in A$ be such that $x \leq y$. By (IF5) we have $\nu_B(y) = \nu_B(x \lor y) \geq \nu_B(x)$ and $\mu_B(y) = \mu_B(x \lor y) \leq \mu_B(x)$. Hence (IF3) is satisfied. $\blacksquare$

Theorem 1. Let $B$ be an IF-set of a pseudo-BL-algebra $A$. The following are equivalent:

(i) $B$ is an IF-filter;

(ii) $B$ satisfies (IF3) and for all $x, y \in A$

(1) $\nu_B(y) \geq \nu_B(x) \land \nu_B(x \rightarrow y)$,

(2) $\mu_B(y) \leq \mu_B(x) \lor \mu_B(x \rightarrow y)$.

(iii) $B$ satisfies (IF3) and for all $x, y \in A$

(3) $\nu_B(y) \geq \nu_B(x) \land \nu_B(x \twoheadrightarrow y)$,

(4) $\mu_B(y) \leq \mu_B(x) \lor \mu_B(x \twoheadrightarrow y)$.

Proof. Using Remark 4 of this paper, Proposition 3.3 and Corollary 3.4 of [13] and Theorem 3.3 of [11] we have the thesis. $\blacksquare$

Proposition 4. Let $B$ be an IF-set of a pseudo-BL-algebra $A$. The following are equivalent:

(i) $B$ is an IF-filter;

(ii) for all $x, y, z \in A$

(5) $x \rightarrow (y \rightarrow z) = 1 \Rightarrow \nu_B(z) \geq \nu_B(x) \land \nu_B(y)$,

(6) $x \rightarrow (y \rightarrow z) = 1 \Rightarrow \mu_B(z) \leq \mu_B(x) \lor \mu_B(y)$.

(iii) for all $x, y, z \in A$

(7) $x \twoheadrightarrow (y \twoheadrightarrow z) = 1 \Rightarrow \nu_B(z) \geq \nu_B(x) \land \nu_B(y)$,

(8) $x \twoheadrightarrow (y \twoheadrightarrow z) = 1 \Rightarrow \mu_B(z) \leq \mu_B(x) \lor \mu_B(y)$. 
Proof. (i)⇒(ii) Suppose that $B$ is an IF-filter of a pseudo-BL-algebra $A$. Let $x, y, z \in A$ be such that $x \rightarrow (y \rightarrow z) = 1$. By Theorem 1 (ii)
\[(9) \quad \nu_B(y \rightarrow z) \geq \nu_B(x) \land \nu_B(x \rightarrow (y \rightarrow z)) = \nu_B(x) \land \nu_B(1) = \nu_B(x),\]
\[(10) \quad \mu_B(y \rightarrow z) \leq \mu_B(x) \lor \mu_B(x \rightarrow (y \rightarrow z)) = \mu_B(x) \lor \mu_B(1) = \mu_B(x).\]
Aplying Theorem 1 (ii) the second time we obtain
\[(11) \quad \nu_B(z) \geq \nu_B(y) \land \nu_B(y \rightarrow z),\]
\[(12) \quad \mu_B(z) \leq \mu_B(y) \lor \mu_B(y \rightarrow z).\]
(9), (10), (11) and (12) force $\nu_B(z) \geq \nu_B(x) \land \nu_B(y)$ and $\mu_B(z) \leq \mu_B(x) \lor \mu_B(y)$.

(ii)⇒(i) Let $B$ be an IF-set of a pseudo-BL-algebra $A$ which satisfies (3). Let $x, y \in A$ be such that $x \leq y$. By Lemma 1 (iv) and (v),
\[x \rightarrow (x \rightarrow y) = 1,\]
hence applying (5) and (6) we have
\[\nu_B(y) \geq \nu_B(x) \land \nu_B(x \rightarrow y),\]
\[\mu_B(y) \leq \mu_B(x) \lor \mu_B(x),\]
that is, (IF3) holds.

Now we prove that (1) and (2) hold. By Lemma 1 (vi), $(x \rightarrow y) \rightarrow (x \rightarrow y) = 1$. Thus, applying (5) and (6) we get
\[\nu_B(y) \geq \nu_B(x \rightarrow y) \land \nu_B(x)\]
\[\mu_B(y) \leq \mu_B(x \rightarrow y) \lor \mu_B(x).\]
Hence by Theorem 1, $B$ is an IF-filter.

(iii)⇔(i) Analogously. \qed

Proposition 5. Let $B$ be an IF-set of a pseudo-BL-algebra $A$. The following are equivalent:

(i) $B$ is an IF-filter;
(ii) for all $x, y, z \in A$
\[(x \circ y) \rightarrow z = 1 \Rightarrow \nu_B(z) \geq \nu_B(x) \land \nu_B(y),\]
\[(x \circ y) \rightarrow z = 1 \Rightarrow \mu_B(z) \leq \mu_B(x) \lor \mu_B(y),\]
(iii) for all $x, y, z \in A$

$$(x \circ y) \rightsquigarrow z = 1 \Rightarrow \nu_B(z) \geq \nu_B(x) \land \nu_B(y),$$

$$(x \circ y) \rightsquigarrow z = 1 \Rightarrow \mu_B(z) \leq \mu_B(x) \lor \mu_B(y).$$

**Proof.** By Proposition 4 and Lemma 1 (vii). ■

Let $B_i = (\nu_{B_i}, \mu_{B_i})$ be IF-filters of a pseudo-BL-algebra $A$ for every $i \in I$. We define fuzzy sets $\bigwedge_{i \in I} \nu_{B_i}$ and $\bigvee_{i \in I} \mu_{B_i}$ as follows:

$$\left( \bigwedge_{i \in I} \nu_{B_i} \right)(x) = \bigwedge \{ \nu_{B_i}(x) : i \in I \},$$

$$\left( \bigvee_{i \in I} \mu_{B_i} \right)(x) = \bigvee \{ \mu_{B_i}(x) : i \in I \}.$$

For any IF-filters $B_i = (\nu_{B_i}, \mu_{B_i})$ for $i \in I$, of a pseudo-BL-algebra $A$ we define the IF-set $\bigcap_{i \in I} B_i$ of $A$ by

$$\bigcap_{i \in I} B_i = \left( \bigwedge_{i \in I} \nu_{B_i}, \bigvee_{i \in I} \mu_{B_i} \right).$$

**Theorem 2.** Let $B_i = (\nu_{B_i}, \mu_{B_i})$ for $i \in I$, be IF-filters of a pseudo-BL-algebra $A$. Then $\bigcap_{i \in I} B_i$ is an IF-filter of $A$.

**Proof.** Let $B_i = (\nu_{B_i}, \mu_{B_i})$ for $i \in I$, be IF-filters of a pseudo-BL-algebra $A$ and

$$B = \bigcap_{i \in I} B_i = (\nu_B, \mu_B).$$

We use Proposition 4 to show that $B$ is an IF-filter of $A$.

Let $x, y, z \in A$ be such that $x \rightarrow (y \rightarrow z) = 1$. Hence

$$\nu_B(z) = \bigwedge_{i \in I} \nu_{B_i}(z) \geq \bigwedge_{i \in I} (\nu_{B_i}(x) \land \nu_{B_i}(y)) = \bigwedge_{i \in I} \nu_{B_i}(x) \land \bigwedge_{i \in I} \nu_{B_i}(y) = \nu_B(x) \land \nu_B(y),$$

$$\mu_B(z) = \bigvee_{i \in I} \mu_{B_i}(z) \leq \bigvee_{i \in I} (\mu_{B_i}(x) \lor \mu_{B_i}(y)) = \bigvee_{i \in I} \mu_{B_i}(x) \lor \bigvee_{i \in I} \mu_{B_i}(y) = \mu_B(x) \lor \mu_B(y).$$

The proof is closed. ■

**Remark 5.** The set of IF-filters of a pseudo-BL-algebra $A$ forms a complete distributive lattice with relation $\leq$. 
Proof. Since [0, 1] is a complete distributive lattice with usual ordering and by Theorem 2, the proof is completed.

Theorem 3. A lattice of IF-filters of a pseudo-BL-algebra \( A \) is bounded.

Proof. It is easily seen that \( 0_\sim \) and \( 1_\sim \) are IF-filters. Since \( 0_\sim \leq B \leq 1_\sim \) for every IF-filter \( B \), then a lattice of IF-filters is bounded.

Theorem 4. The lattice of IF-filters of a pseudo-BL-algebras has no atoms.

Proof. Let \( B \) be an IF-set of a pseudo-BL-algebra \( A \) and \( B \neq 0_\sim \). Let us define an IF-set \( D \) as follows

\[
D = \left( \frac{1}{2} \nu_B, \frac{1}{2} \mu_B \right).
\]

It is obvious that \( D \) is an IF-filter of \( A \) and \( 0_\sim < D < B \). Hence there are no atoms in a lattice of IF-filters of \( A \).

Let \( B \) be an IF-set of a pseudo-BL-algebra \( A \) and \( \alpha, \beta \in [0, 1] \) be such that \( \alpha + \beta \leq 1 \). Then we can define a set

\[
A_B^{(\alpha, \beta)} = \{ x \in A : \nu_B(x) \geq \alpha, \mu_B(x) \leq \beta \}
\]
called an \((\alpha, \beta)\)-level of \( B \).

Let us notice that \( A_B^{(\alpha, \beta)} = U(\nu_B, \alpha) \cap L(\mu_B, \beta) \).

Theorem 5. Let \( B \) be an IF-set of a pseudo-BL-algebra \( A \). If \( B \) is an IF-filter of \( A \), then \( A_B^{(\alpha, \beta)} = \emptyset \) or \( A_B^{(\alpha, \beta)} \) is a filter of \( A \) for all \( \alpha \in [0, \nu_B(1)] \), \( \beta \in [\mu_B(1), 1] \) such that \( \alpha + \beta \leq 1 \).

Proof. By Theorem 3.10 of [13] and Theorem 3.6 of [11] \( \nu_B \) is a fuzzy filter and \( \mu_B \) is an anti fuzzy filter iff \( U(\nu_B, \alpha) \) and \( L(\mu_B, \beta) \) are filters or empty. According to fact that the intersection of filters is a filter and by Remark 4 we have the thesis.

Corollary 1. If \( B \) is an IF-filter of a pseudo-BL-algebra \( A \), then the set

\[
A_b = \{ x \in A : \nu_B(x) \geq \nu_B(b), \mu_B(x) \leq \mu_B(b) \}
\]
is a filter of \( A \) for every \( b \in A \) such that \( \nu_B(b) + \mu_B(b) \leq 1 \).
4. PRIME IF-FILTERS

In this section we introduce and study prime IF-filters and their connection with pseudo-BL-chains.

Definition 9. An IF-filter $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ of a pseudo-BL-algebra $A$ is said to be prime IF-filter if $\nu_{\mathcal{B}}$ and $\mu_{\mathcal{B}}$ are non-constant and satisfies following conditions for all $x, y \in A$:

\[ \nu_{\mathcal{B}}(x \lor y) = \nu_{\mathcal{B}}(x) \lor \nu_{\mathcal{B}}(y) \quad \text{and} \quad \mu_{\mathcal{B}}(x \lor y) = \mu_{\mathcal{B}}(x) \land \mu_{\mathcal{B}}(y). \]

Remark 6. An IF-filter $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ of a pseudo-BL-algebra $A$ is said to be prime IF-filter iff $\nu_{\mathcal{B}}$ is a fuzzy prime filter and $\mu_{\mathcal{B}}$ is an anti fuzzy prime filter of $A$.

Theorem 6. Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ be a non-constant IF-filter of a pseudo-BL-algebra $A$. Then the following are equivalent:

(i) $\mathcal{B}$ is a prime IF-filter of $A$;

(ii) for all $x, y \in A$, if $(\nu_{\mathcal{B}}(x \lor y) = \nu_{\mathcal{B}}(1) \quad \text{and} \quad \mu_{\mathcal{B}}(x \lor y) = \mu_{\mathcal{B}}(1))$, then

\[ (\nu_{\mathcal{B}}(x) = \nu_{\mathcal{B}}(1) \quad \text{or} \quad \nu_{\mathcal{B}}(y) = \nu_{\mathcal{B}}(1)) \quad \text{and} \quad (\mu_{\mathcal{B}}(x) = \mu_{\mathcal{B}}(1) \quad \text{or} \quad \mu_{\mathcal{B}}(y) = \mu_{\mathcal{B}}(1)); \]

(iii) for all $x, y \in A$,

\[ (\nu_{\mathcal{B}}(x \rightarrow y) = \nu_{\mathcal{B}}(1) \quad \text{or} \quad \nu_{\mathcal{B}}(y \rightarrow x) = \nu_{\mathcal{B}}(1)) \quad \text{and} \quad (\mu_{\mathcal{B}}(x \rightarrow y) = \mu_{\mathcal{B}}(1) \quad \text{or} \quad \mu_{\mathcal{B}}(y \rightarrow x) = \mu_{\mathcal{B}}(1)); \]

(iv) for all $x, y \in A$,

\[ (\nu_{\mathcal{B}}(x \multimap y) = \nu_{\mathcal{B}}(1) \quad \text{or} \quad \nu_{\mathcal{B}}(y \multimap x) = \nu_{\mathcal{B}}(1)) \quad \text{and} \quad (\mu_{\mathcal{B}}(x \multimap y) = \mu_{\mathcal{B}}(1) \quad \text{or} \quad \mu_{\mathcal{B}}(y \multimap x) = \mu_{\mathcal{B}}(1)). \]


Theorem 7. Let $A$ be a pseudo-BL-algebra and $\mathcal{B}$ be an IF-filter of $A$. Then $\mathcal{B}$ is a prime IF-filter iff $M_{\nu_{\mathcal{B}}}$ and $A_{\mu_{\mathcal{B}}}$ are prime filters of $A$.

Proof. By Remark 3.

Theorem 8. Let $A$ be a pseudo-BL-algebra, $P$ be a filter of $A$ and $\alpha, \beta \in [0, 1]$ with $\alpha > \beta$. Then $P$ is a prime filter of $A$ if and only if $B(P) = (\mu_P(\alpha, \beta), \mu_P'(1 - \alpha, 1 - \beta))$ define as in Example 1, is a prime IF-filter of $A$.

Theorem 9. Let $B$ be an IF-set of a pseudo-BL-algebra $A$ such that $\nu_B$ and $\mu_B$ are non-constant. Then the following are equivalent:

(i) $B$ is a prime IF-filter of $A$;

(ii) for every $\alpha \in [0, 1]$, if $U(\nu_B, \alpha), L(\mu_B, \alpha) \neq \emptyset$ and $U(\nu_B, \alpha), L(\mu_B, \alpha) \neq A$, then $U(\nu_B, \alpha), L(\mu_B, \alpha)$ are prime filters of $A$.


Theorem 10. Let $A$ be a non-trivial pseudo-BL-algebra. The following are equivalent:

(i) $A$ is a pseudo-BL-chain;

(ii) every IF-filter $B$ such that $\nu_B$ and $\mu_B$ are non-constant is a prime IF-filter of $A$;

(iii) every IF-filter $B$ such that $\nu_B$ and $\mu_B$ are non-constant, $\nu_B(1) = 1$ and $\mu_B(1) = 0$ is a prime IF-filter of $A$;

(iv) the IF-filter $\left(\chi_{\{1\}}, \chi_{\{1\}}^C\right)$ is a prime IF-filter of $A$.


5. Homomorphism and IF-filters

Let $A, B$ be pseudo-BL-algebras. Following [3] we define a homomorphism of pseudo-BL-algebras as a mapping $h : A \rightarrow B$ such that the following conditions hold for all $x, y \in A$:

(H1) $h(x \odot y) = h(x) \odot h(y)$;

(H2) $h(x \rightarrow y) = h(x) \rightarrow h(y)$;

(H3) $h(x \hookrightarrow y) = h(x) \hookrightarrow h(y)$;

(H4) $h(0) = 0$.

Recall that if $h : A \rightarrow B$ is a homomorphism of pseudo-BL-algebras, then

(H5) $h(1) = 1$;

(H6) $h(x \land y) = h(x) \land h(y)$;
(H7) \( h(x \lor y) = h(x) \lor h(y) \).

**Definition 10.** Let \( \mathcal{B} \) be an IF-filer of a pseudo-BL-algebra \( B \) and \( f : A \to B \) be a homomorphism of pseudo-BL-algebras. The preimage of \( \mathcal{B} \) is the IF-set \( \mathcal{B}^f = (\nu^f_B, \mu^f_B) \) defined by

\[
\nu^f_B(x) = \nu_B(f(x)) \quad \text{and} \quad \mu^f_B(x) = \mu_B(f(x))
\]

for all \( x \in A \).

**Theorem 11.** Let \( \mathcal{B} \) be an IF-filter of \( B \) and \( f : A \to B \) be a homomorphism of pseudo-BL-algebras. Then \( \mathcal{B}^f \) is an IF-filter of \( A \).

**Proof.** Suppose that \( f : A \to B \) is a homomorphism of pseudo-BL-algebras and \( \mathcal{B} \) be an IF-filter of \( B \). Let \( x, y \in A \). Then

\[
\nu^f_B(x \circ y) = \nu_B(f(x \circ y)) = \nu_B(f(x) \circ f(y)) \\
\geq \nu_B(f(x)) \land \nu_B(f(y)) = \nu^f_B(x) \land \nu^f_B(y)
\]

and

\[
\mu^f_B(x \circ y) = \mu_B(f(x \circ y)) = \mu_B(f(x) \circ f(y)) \\
\leq \mu_B(f(x)) \lor \mu_B(f(y)) = \mu^f_B(x) \lor \mu^f_B(y).
\]

Hence (IF1) and (IF2) hold.

Now let \( x, y \in A \) be such that \( x \leq y \). Therefore,

\[
\nu^f_B(x) = \nu^f_B(x \land y) = \nu_B(f(x \land y)) \\
= \nu_B(f(x) \land f(y)) \leq \nu_B(f(y)) = \nu^f_B(y)
\]

and

\[
\mu^f_B(x) = \mu^f_B(x \land y) = \mu_B(f(x \land y)) \\
= \mu_B(f(x) \land f(y)) \geq \mu_B(f(y)) = \mu^f_B(y).
\]

Thus, (IF3) holds.

Concluding, \( \mathcal{B}^f \) is an IF-filter of \( A \).

**Theorem 12.** Let \( \mathcal{B} \) be an IF-set of \( B \), \( \mathcal{B}^f \) be an IF-filter of \( A \), where \( f : A \to B \) is an epimorphism of pseudo-BL-algebras. Then \( \mathcal{B} \) is an IF-filter of \( A \).
**Proof.** Let \( f : A \to B \) be an epimorphism of pseudo-BL-algebras. Then, for any \( x, y \in B \), there exist \( a, b \in A \) such that \( x = f(a) \) and \( y = f(b) \). Therefore,

\[
\nu_B(x \circ y) = \nu_B((a \circ f(b)) = \nu_B(f(a \circ b))
\]

\[
= \nu_B^f(a \circ b) \geq \nu_B^f(a) \land \nu_B^f(b)
\]

\[
= \nu_B(f(a)) \land \nu_B(f(b)) = \nu_B(x) \land \nu_B(y)
\]

and

\[
\mu_B(x \circ y) = \mu_B((a \circ f(b)) = \mu_B(f(a \circ b))
\]

\[
= \mu_B^f(a \circ b) \leq \mu_B^f(a) \lor \mu_B^f(b)
\]

\[
= \mu_B(f(a)) \lor \mu_B(f(b)) = \mu_B(x) \lor \mu_B(y).
\]

Hence (IF1) and (IF2) hold.

Now let \( x, y \in B \) be such that \( x \leq y \). Then, there exist \( a, b \in A \) such that \( x = f(a) \) and \( y = f(b) \). Therefore,

\[
\nu_B(x) = \nu_B(x \land y) = \nu_B((a \land f(b)) = \nu_B((a \land b))
\]

\[
= \nu_B^f(a \land b) \leq \nu_B^f(b) = \nu_B(f(b)) = \nu_B(y)
\]

and

\[
\mu_B(x) = \mu_B(x \land y) = \mu_B((a \land f(b)) = \mu_B((a \land b))
\]

\[
= \mu_B^f(a \land b) \geq \mu_B^f(b) = \mu_B(f(b)) = \mu_B(y).
\]

Thus, (IF3) holds.

Concluding, \( B \) is an IF-filter of \( B \).

Now let us denote the set of all filters of pseudo-BL-algebra \( A \) by \( Fil(A) \) and the set of all IF-filters of \( A \) by \( IFil(A) \). Let \( \alpha \in (0, 1) \). We define maps \( f_\alpha : IFil(A) \to Fil(A) \cup \{\emptyset\} \) and \( g_\alpha : IFil(A) \to Fil(A) \cup \{\emptyset\} \) by

\[
f_\alpha(B) = U(\nu_B, \alpha),
\]

\[
g_\alpha(B) = L(\mu_B, \alpha)
\]

for all \( B = (\nu_B, \mu_B) \in IFil(A) \).

**Theorem 13.** For any \( \alpha \in (0, 1) \), the maps \( f_\alpha \) and \( g_\alpha \) are surjective from \( IFil(A) \) onto \( Fil(A) \cup \{\emptyset\} \).
Therefore, $f_\alpha(0,\_)$ is $U(0,\alpha) = \emptyset = L(1,\alpha) = g_\alpha(0,\_).$

Now let $\emptyset \neq F \in \text{Fil}(A).$ Then $(\chi_F, \chi_F^C)$ is an IF-filter of $A.$ Hence,

$$f_\alpha((\chi_F, \chi_F^C)) = U(\chi_F, \alpha) = F = L(\chi_F^C, \alpha) = g_\alpha((\chi_F, \chi_F^C)).$$

Therefore, $f_\alpha$ and $g_\alpha$ are surjective.

6. Direct product of IF-filters

Let us define a direct product $\prod_{i \in I} A_i$ of pseudo-BL-algebras as usually.

**Definition 11.** Let $A$ be a pseudo-BL-algebra. Then we define an IF-relation on $A$ as a mapping $R = (\nu^R, \mu^R) : A \times A \rightarrow [0, 1] \times [0, 1]$ such that $\nu^R(x, y) + \mu^R(x, y) \leq 1$ for all $x, y \in A.$

Now define a direct product of IF-sets of pseudo-BL-algebra $A.$

**Definition 12.** Let $\mathcal{B} = (\nu_B, \mu_B)$ and $\mathcal{G} = (\nu_G, \mu_G)$ be IF-sets of $A.$ We define a direct product $\mathcal{B} \times \mathcal{G}$ by

$$\mathcal{B} \times \mathcal{G} = (\nu_B, \mu_B) \times (\nu_G, \mu_G) = (\nu_B \times \nu_G, \mu_B \times \mu_G),$$

where $(\nu_B \times \nu_G)(x, y) = \nu_B(x) \land \nu_G(y)$ and $(\mu_B \times \mu_G)(x, y) = \mu_B(x) \lor \mu_G(y)$ for all $x, y \in A.$

**Proposition 6.** Let $\mathcal{B} = (\nu_B, \mu_B)$ and $\mathcal{G} = (\nu_G, \mu_G)$ be IF-sets of a pseudo-BL-algebra $A,$ then $\mathcal{B} \times \mathcal{G}$ is an IF-set of $A \times A.$

**Proof.** Let $\mathcal{B}, \mathcal{G}$ be IF-sets of $A.$ Then for every $x \in A$ we have $\nu_B(x) + \mu_B(x) \leq 1$ and $\nu_G(x) + \mu_G(x) \leq 1.$ Suppose that $\nu_B(x) \leq \nu_G(y)$ for some $x, y \in A.$ Then $(\nu_B \times \nu_G)(x, y) = \nu_B(x) \land \nu_G(y) = \nu_B(x).$ Let us consider two cases:

**Case 1.** $\mu_B(x) \leq \mu_G(y)$

Hence $(\mu_B \times \mu_G)(x, y) = \mu_B(x) \lor \mu_G(y) = \mu_G(y)$ and then $(\nu_B \times \nu_G)(x, y) + (\mu_B \times \mu_G)(x, y) = \nu_B(x) + \mu_G(y) \leq \nu_G(y) + \mu_G(y) \leq 1.$

**Case 2.** $\mu_B(x) > \mu_G(y)$

Therefore $(\mu_B \times \mu_G)(x, y) = \mu_B(x) \lor \mu_G(y) = \mu_B(x)$ and then $(\nu_B \times \nu_G)(x, y) + (\mu_B \times \mu_G)(x, y) = \nu_B(x) + \mu_B(x) \leq 1.$ Hence $\mathcal{B} \times \mathcal{G}$ is an IF-set of $A \times A.$

Analogously when $\nu_B(x) > \nu_G(y).$
Now we give a trivial Proposition without a proof:

**Proposition 7.** Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ and $\mathcal{G} = (\nu_{\mathcal{G}}, \mu_{\mathcal{G}})$ be IF-sets of a pseudo-BL-algebra $A$, then

(i) $\mathcal{B} \times \mathcal{G}$ is an IF-relation of $A$;

(ii) $U(\nu_{\mathcal{B}} \times \nu_{\mathcal{G}}; \alpha) = U(\nu_{\mathcal{B}}; \alpha) \times U(\nu_{\mathcal{G}}; \alpha)$ and $L(\mu_{\mathcal{B}} \times \mu_{\mathcal{G}}; \alpha) = L(\mu_{\mathcal{B}}; \alpha) \times L(\mu_{\mathcal{G}}; \alpha)$ for all $\alpha \in [0, 1]$.

**Theorem 14.** Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ and $\mathcal{G} = (\nu_{\mathcal{G}}, \mu_{\mathcal{G}})$ be IF-filters of a pseudo-BL-algebra $A$. Then $\mathcal{B} \times \mathcal{G}$ is an IF-filter of $A \times A$.

**Proof.** Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ and $\mathcal{G} = (\nu_{\mathcal{G}}, \mu_{\mathcal{G}})$ be IF-filters of a pseudo-BL-algebra $A$. Suppose that $x, y \in A$. Then by (IF1) and (IF2), $\nu_{\mathcal{B}}(x \circ y) \geq \nu_{\mathcal{B}}(x) \land \nu_{\mathcal{B}}(y)$, $\nu_{\mathcal{G}}(x \circ y) \geq \nu_{\mathcal{G}}(x) \land \nu_{\mathcal{G}}(y)$ and $\mu_{\mathcal{B}}(x \circ y) \leq \mu_{\mathcal{B}}(x) \lor \mu_{\mathcal{B}}(y)$, $\mu_{\mathcal{G}}(x \circ y) \leq \mu_{\mathcal{G}}(x) \lor \mu_{\mathcal{G}}(y)$. Let $(x_1, x_2), (y_1, y_2) \in A \times A$. Then,

\[
(\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})((x_1, x_2) \circ (y_1, y_2)) = (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(x_1 \circ y_1, x_2 \circ y_2)
= \nu_{\mathcal{B}}(x_1 \circ y_1) \land \nu_{\mathcal{G}}(x_2 \circ y_2)
\geq \nu_{\mathcal{B}}(x_1) \land \nu_{\mathcal{B}}(y_1) \land \nu_{\mathcal{G}}(x_2) \land \nu_{\mathcal{G}}(y_2)
= (\nu_{\mathcal{B}}(x_1) \land \nu_{\mathcal{G}}(x_2)) \land (\nu_{\mathcal{B}}(y_1) \land \nu_{\mathcal{G}}(y_2))
= (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(x_1, x_2) \land (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(y_1, y_2).
\]

Similarly, we can prove that $(\mu_{\mathcal{B}} \times \mu_{\mathcal{G}})((x_1, x_2) \circ (y_1, y_2)) \leq (\mu_{\mathcal{B}} \times \mu_{\mathcal{G}})(x_1, x_2) \lor (\mu_{\mathcal{B}} \times \mu_{\mathcal{G}})(y_1, y_2)$.

It is proved that (IF1) and (IF2) hold.

Now let $(x_1, x_2), (y_1, y_2) \in A \times A$ be such that $(x_1, x_2) \leq (y_1, y_2)$. Then

\[
(\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(x_1, x_2) = (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})((x_1, x_2) \land (y_1, y_2))
= (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(x_1 \land y_1, x_2 \land y_2)
= \nu_{\mathcal{B}}(x_1 \land y_1) \land \nu_{\mathcal{G}}(x_2 \land y_2)
\leq \nu_{\mathcal{B}}(y_1) \land \nu_{\mathcal{G}}(y_2)
= (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(y_1, y_2).
\]

and similarly $(\mu_{\mathcal{B}} \times \mu_{\mathcal{G}})(x_1, x_2) \geq (\mu_{\mathcal{B}} \times \mu_{\mathcal{G}})(y_1, y_2)$.

The proof is completed. ■

**Theorem 15.** Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ be IF-set of a pseudo-BL-algebra $A$. Then $\mathcal{B}$ is an IF-filter of $A$ if and only if $\mathcal{B} \times \mathcal{B}$ is an IF-filter of $A \times A$. 
Proof. $\Rightarrow$: By Theorem 14.
$\Leftarrow$: Let $B \times B$ be an IF-filter of $A \times A$. Let $(x_1, x_2), (y_1, y_2) \in A \times A$. Hence
\[
\nu_B(x_1 \circ y_1) \land \nu_B(x_2 \circ y_2) = (\nu_B \times \nu_B)(x_1 \circ y_1, x_2 \circ y_2) = (\nu_B \times \nu_B)((x_1, x_2) \circ (y_1, y_2)) \geq (\nu_B \times \nu_B)(x_1, x_2) \land (\nu_B \times \nu_B)(y_1, y_2) = \nu_B(x_1) \land \nu_B(x_2) \land \nu_B(y_1) \land \nu_B(y_2).
\]
Putting $x_1 = x_2$ and $y_1 = y_2$ we have
\[
\nu_B(x_1 \circ y_1) \geq \nu_B(x_1) \land \nu_B(x_1) \land \nu_B(y_1) = \nu_B(x_1) \land \nu_B(y_1).
\]
Similarly, $\mu_B(x_1 \circ y_1) \leq \mu_B(x_1) \lor \mu_B(y_1)$.

Let $x, y \in A$ be such that $x \leq y$. Then by (IF3),
\[
\nu_B(x) = (\nu_B \times \nu_B)(x, x) \leq (\nu_B \times \nu_B)(y, y) = \nu_B(y).
\]
Analogously, $\mu_B(x) \geq \mu_B(y)$.

Hence $B = (\nu_B, \mu_B)$ is an IF-filter of $A$. $\blacksquare$

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References


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