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APPLICATIONS OF SADDLE-POINT DETERMINANTS

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Abstract

For a given square matrix $\mathbf{A} \in M_n(\mathbb{R})$ and the vector $\mathbf{e} \in (\mathbb{R})^n$ of ones denote by $(\mathbf{A}, \mathbf{e})$ the matrix $\begin{bmatrix} \mathbf{A} & \mathbf{e} \\ \mathbf{e}^T & 0 \end{bmatrix}$.

This is often called the saddle point matrix and it plays a significant role in several branches of mathematics. Here we show some applications of it in: game theory and analysis. An application of specific saddle point matrices that are hollow, symmetric, and nonnegative is likewise shown in geometry as a generalization of Heron's formula to give the volume of a general simplex, as well as a conditions for its existence.

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1. Introduction

In standard notation $\mathbf{e} \in \mathbb{R}^n$ is the column vector with all components one, and $\mathbf{E} \in M_n(\mathbb{R})$ stands for the matrix with all entries equal to one. So, $\mathbf{E} = \mathbf{e}\mathbf{e}^T$. For a given square matrix $\mathbf{A} \in M_n(\mathbb{R})$ denote by $(\mathbf{A}, \mathbf{e})$ the matrix

$$(\mathbf{A}, \mathbf{e}) = \begin{bmatrix} \mathbf{A} & \mathbf{e} \\ \mathbf{e}^T & 0 \end{bmatrix} \tag{1}$$

often called a saddle point matrix.

If the matrix $\mathbf{A}$ is symmetric, then $(\mathbf{A}, \mathbf{e})$ may be interpreted as the bordered Hessian of a standard quadratic program over the standard simplex, and it is called the Karush-Kuhn-Tucker matrix of the program. It is known to have a large variety of applications (for a review see, for instance, Bomze (1998)). Properties of the determinant of $(\mathbf{A}, \mathbf{e})$ were mentioned by Ostrowski (2007). We quote one of them in the Lemma below.

Lemma. *For any real α, β, the determinant of the matrix $\alpha\mathbf{A} + \beta\mathbf{E}$ can be expressed as follows:*

$$det(\alpha\mathbf{A} + \beta\mathbf{E}) = \alpha^{n-1}[\alpha det(\mathbf{A}) - \beta det(\mathbf{A}, \mathbf{e})] \tag{2}$$

Corollary 1.
a) *From the equality* (2), *putting $\alpha = \beta = 1$, we immediately obtain that the determinant of* $(\mathbf{A}, \mathbf{e})$ *depends on the determinants of the matrices* $\mathbf{A}$ *and* $\mathbf{A}+\mathbf{E}$, *and the relation is expressed by the formula:*

$$det(\mathbf{A}, \mathbf{e}) = det(\mathbf{A}) - det(\mathbf{A} + \mathbf{E}) \tag{3}$$

b) *Assume that* $(\mathbf{A}, \mathbf{e})$ *is nonsingular and let $\alpha = 1$, $\beta = \beta_0 = \frac{det(\mathbf{A})}{det(\mathbf{A},\mathbf{e})}$. Then the matrix $\mathbf{A} + \beta_0\mathbf{E}$ is singular.*

It is easy to see that equality (2) also implies, that a matrix given as a linear combination of the form $det(\mathbf{A}, \mathbf{e})\mathbf{A} + det(\mathbf{A})\mathbf{E}$ is singular for any matrix $\mathbf{A}$.

More properties of saddle point matrices in generalized form (with matrices $\mathbf{A}$ and $\mathbf{B}$)

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{bmatrix}$$

can be found, for example, in Benzi, Golub, and Liesen (2005).

2. The value of a game and optimal strategies

A bimatrix game is a game described by an ordered pair of two payoff matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$, with equal dimensions. When the Row and Column players choose their i-th and j-th pure strategies, respectively, the Row player's payoff is a_{ij} and the Column player's payoff is b_{ij}.

A mixed strategy of the Row (Column) player is a probability vector $\mathbf{x}$ ($\mathbf{y}$) specifying the probability with which each pure strategy is played. If these probabilities are all strictly positive, the strategy is said to be completely mixed. A completely mixed Nash equilibrium is one in which both players strategies are completely mixed. This paper is concerned with games in which the number of pure strategies is the same for both players, so that $\mathbf{A}$ and $\mathbf{B}$ are square, n-by-n matrices ($n \geq 2$). Equilibrium payoffs of such games were analysed by Ostrowski (2006). One of the results could be reformulated in the form given below.

Theorem 1. *For any $\mathbf{A}, \mathbf{B} \in M_n(\mathbb{R})$, where matrices $(\mathbf{A}, \mathbf{e})$ and $(\mathbf{B}, \mathbf{e})$ are nonnsigular, the bimatrix game $[\mathbf{A}, \mathbf{B}]$ in which players strategies are completely mixed, has the players equilibrium payoffs $\nu(\mathbf{A}), \nu(\mathbf{B})$ equal to*

$$\nu(\mathbf{A}) = -\frac{det(\mathbf{A})}{det(\mathbf{A}, \mathbf{e})}, \quad \nu(\mathbf{B}) = -\frac{det(\mathbf{B})}{det(\mathbf{B}, \mathbf{e})}. \tag{4}$$

The equilibrium strategies of the row player and the column player, respectively, are equal to the vectors $\mathbf{p}$ and $\mathbf{q}$ (respectively) with coefficients given as follow

$$p_i = \frac{det(\mathbf{B}_i), \mathbf{e})}{det(\mathbf{B}, \mathbf{e})}, q_i = \frac{det(\mathbf{A}^i, \mathbf{e})}{det(\mathbf{A}, \mathbf{e})}, \; i = 1, \ldots, n,$$

where $\mathbf{M}_i$ and $\mathbf{M}^i$ denote matrices obtained from $\mathbf{M}$ by replacing the i-th column by the column vector $\mathbf{e}$ and the i-th row by the row vector $\mathbf{e}^T$, $(i = 1, 2, \ldots, n)$, respectively. The necessary and sufficient condition for the existence and uniqueness of a completely mixed Nash equilibrium is that for all pairs i, j,

$$det(\mathbf{A}^i, \mathbf{e}) det(\mathbf{A}^j, \mathbf{e}) > 0 \;\; and \;\; det(\mathbf{B}^i, \mathbf{e}) det(\mathbf{B}^j, \mathbf{e}) > 0.$$

Corollary 2. *Applying Schur complements to the saddle point matrix $(\mathbf{M}, \mathbf{e})$ in the case of nonsingular $\mathbf{M}$ we get*

$$det(\mathbf{M}, \mathbf{e}) = -det(\mathbf{M}) \mathbf{e}^T \mathbf{M}^{-1} \mathbf{e}. \tag{5}$$

Putting (5) in (4) we obtain via saddle point matrices the well-known formula, see e.g. Owen (2013),

$$\nu(\mathbf{A}) = \frac{1}{\mathbf{e}^T\mathbf{A}^{-1}\mathbf{e}} \text{ and } \nu(\mathbf{B}) = \frac{1}{\mathbf{e}^T\mathbf{B}^{-1}\mathbf{e}}.$$

3. The Mean Value Theorem

Let a real-valued function f be continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). The well-known Mean Value Theorem states:

Theorem 2. *If $f(a) = f(b)$, then there exists at least one point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ (the graph of f is parallel to a given line that passes through points $(a, f(a))$ and $(b, f(b))$).*

Proof. Let $F(x)$ be the function of the form

$$F(x) = det(\mathbf{A}, \mathbf{e}),\ \mathbf{A} = \begin{bmatrix} x & f(x) & 0 \\ a & f(a) & 0 \\ b & f(b) & 0 \end{bmatrix}. \tag{6}$$

Function $F(x)$ is also continuous and differentiable on the open interval (a, b). It is easy to see that $F(a) = F(b) = 0$. We may now apply Rolle's Theorem. Therefore the derivative $F'(x) = 0$, and we have

$$F'(x) = det'(\mathbf{A}, \mathbf{e}) = det \begin{bmatrix} 1 & f'(x) & 0 & 1 \\ a & f(a) & 0 & 1 \\ b & f(b) & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = 0. \tag{7}$$

So, there exists at least one point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ and the proof of the Mean Value Theorem is complete. ■

Remark 1. The proof of the Mean Value Theorem (a more descriptive name would be Average Slope Theorem), which generalizes Rolle's Theorem, is accomplished by finding a way to apply Rolle's Theorem. The above is **a non-standard proof of a standard formulation of the Mean Value Theorem**.

Other non-standard proofs of the above theorem can be found in Almeida (2008) (with analysis technics) or in Diener and Loeb (2011) (as an extending a constructive reverse perspective).

4. Volumes of simplicies, generalizations and saddlepoint determinants

In his book (*Metrica*) Hero (or Heron) of Alexandria showed how to calculate surfaces and volumes of a variety geometric objects, including his famous formula for the area of a triangle in terms of sides lenghts a, b, c and (semiparamter $s = \frac{1}{2}(a+b+c)$):

$$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

In terms of (symmetric) saddlepoint determinants, the form of it is:

$$S = \frac{1}{4}\sqrt{det\begin{bmatrix} 0 & a^2 & b^2 & 1 \\ a^2 & 0 & c^2 & 1 \\ b^2 & c^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}}.$$

In a similar way, Tartaglia's formula for the volume of a tetrahedron with side lengths a, b, c, d, e, f may also be presented using a saddle point determinant:

$$\frac{1}{12\sqrt{2}}\sqrt{-det\begin{bmatrix} 0 & a^2 & b^2 & d^2 & 1 \\ a^2 & 0 & c^2 & e^2 & 1 \\ b^2 & c^2 & 0 & f^2 & 1 \\ d^2 & e^2 & f^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}}.$$

Note that in both cases (triangle and tetrahedron) the positivity test of the saddle point determinant is a test for existence of the object with the given side lengths.

Triangles and tetrahedrons are both examples of simplices. Recently in Griffiths (2005) a formula for the volume of the unit simplex in n dimensions was given. This also may be presented as a saddlepoint determinant

$$\frac{1}{n!\sqrt{2^n}}\sqrt{(-1)^{n+1}det\begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & 1 \\ 1 & \dots & \dots & \dots & 0 \end{bmatrix}},$$

Griffiths formula (not presented as above) is unwittingly a special case of the saddlepoint determinant formula for the volume $V(\mathbf{D}_n)$ of a general simplex in

n-space in which d_{ij} is the distance from the vertex i to vertex $j, i, j = 1, \ldots, n$ (an n-dimensional simplex).

$$V(\mathbf{D}_n) = \frac{1}{n!\sqrt{2^n}}\sqrt{(-1)^{n+1}det(\mathbf{D_n}, \mathbf{e})}, \tag{8}$$

$$\mathbf{D}_n = \begin{bmatrix} 0 & d_{12}^2 & d_{13}^2 & \ldots & d_{1n}^2 \\ d_{12}^2 & 0 & d_{23}^2 & \ldots & d_{2n}^2 \\ d_{13}^2 & d_{23}^2 & 0 & \ldots & d_{3n}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{1n}^2 & d_{2n}^2 & d_{3n}^2 & \ldots & 0 \end{bmatrix}. \tag{9}$$

The volume of n-dimensional simplex was discussed almost ninety years ago (and somehow forgotten for years) by Menger (1928).

Volumes of an n-dimensional hypertetrahedron – examples.

For

$$\mathbf{D}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{D}_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \mathbf{D}_3 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \mathbf{D}_4 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix},$$

we get

$$det(\mathbf{D}_1) = -1, \;\; det(\mathbf{D}_2) = 2, \;\; det(\mathbf{D}_3) = -3, \;\; det(\mathbf{D}_4) = 4,$$

$$V(\mathbf{D}_1) = 1, \;\; V(\mathbf{D}_2) = 0.43, \;\; V(\mathbf{D}_3) = 0.12, \;\; V(\mathbf{D}_4) = 0.02,$$

$$det(2\mathbf{D}_1) = -4, \;\; det(2\mathbf{D}_2) = 16, \;\; det(2\mathbf{D}_3) = -48, \;\; det(2\mathbf{D}_4) = 128,$$

$$V(2\mathbf{D}_1) = 1.41, \;\; V(2\mathbf{D}_2) = 0.87, \;\; V(2\mathbf{D}_3) = 0.33, \;\; V(2\mathbf{D}_4) = 0.09,$$

$$det(3\mathbf{D}_1) = -9, \;\; det(3\mathbf{D}_2) = 54, \;\; det(3\mathbf{D}_3) = -243, \;\; det(3\mathbf{D}_4) = 972,$$

$$V(3\mathbf{D}_1) = 1.73, \;\; V(3\mathbf{D}_2) = 1.30, \;\; V(3\mathbf{D}_3) = 0.61, \;\; V(3\mathbf{D}_4) = 0.21,$$

$$det(4\mathbf{D}_1) = -16, \;\; det(4\mathbf{D}_2) = 128, \;\; det(4\mathbf{D}_3) = -768, \;\; det(4\mathbf{D}_4) = 4096,$$

$$V(4\mathbf{D}_1) = 2, \;\; V(4\mathbf{D}_2) = 1.73, \;\; V(4\mathbf{D}_3) = 0.94, \;\; V(4\mathbf{D}_4) = 0.37.$$

For

$$\mathbf{F}_1 = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}, \mathbf{F}_2 = \begin{bmatrix} 0 & 3 & 4 \\ 3 & 0 & 5 \\ 4 & 5 & 0 \end{bmatrix}, \mathbf{F}_3 = \begin{bmatrix} 0 & 3 & 4 & 3 \\ 3 & 0 & 5 & 3 \\ 4 & 5 & 0 & 3 \\ 3 & 3 & 3 & 0 \end{bmatrix}, \mathbf{F}_4 = \begin{bmatrix} 0 & 3 & 4 & 3 & 3 \\ 3 & 0 & 5 & 3 & 3 \\ 4 & 5 & 0 & 3 & 3 \\ 3 & 3 & 3 & 0 & 3 \\ 3 & 3 & 3 & 3 & 0 \end{bmatrix},$$

we get

$$det(\mathbf{F}_1) = -9, \quad det(\mathbf{F}_2) = 120, \quad det(\mathbf{F}_3) = -396, \quad det(\mathbf{F}_4) = 1296,$$

$$V(\mathbf{F}_1) = 1.73, \quad V(\mathbf{F}_2) = 1.66, \quad V(\mathbf{F}_3) = 0.71, \quad V(\mathbf{F}_4) = 0.23.$$

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