Discussiones Mathematicae
General Algebra and Applications 31 (2011) 217–229

ON TWO CLASSES OF PSEUDO-BCI-ALGEBRAS

Grzegorz Dymek

Faculty of Mathematics and Natural Sciences
The John Paul II Catholic University of Lublin
Konstantynów 1H, 20–708 Lublin, Poland

e-mail: gdymek@o2.pl

Abstract

The class of p-semisimple pseudo-BCI-algebras and the class of branchwise commutative pseudo-BCI-algebras are studied. It is proved that they form varieties. Some congruence properties of these varieties are displayed.

Keywords: pseudo-BCI-algebra, p-semisimplicity, branchwise commutativity.

2010 Mathematics Subject Classification: 03G25, 06F35.

1. Introduction

The notion of BCI-algebras has been introduced by K. Iséki in 1966 (see [10]). BCI-algebras have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. The name of BCI-algebras originates from the combinatories B, C, I in combinatory logic.

The concept of pseudo-BCI-algebras has been introduced by W.A. Dudek and Y.B. Jun in [1] as an extension of BCI-algebras and it was investigated by several authors in [11] and [12]. Pseudo-BCI-algebras are algebraic models of some extension of a non-commutative version of the BCI-logic (see [5] for details). These algebras have also connections with other algebras of logic such as pseudo-BCK-algebras, pseudo-BL-algebras and pseudo-MV-algebras introduced by G. Georgescu and A. Iorgulescu in [6, 7] and [8], respectively. More about those algebras the reader can find in [9]. In [3] the author introduces the notion of compatible deductive systems, gives characterization of compatibility of closed deductive systems and shows that there is one-to-one correspondence between closed compatible deductive systems and congruence kernels. The concept of p-semisimple pseudo-BCI-algebras is defined and studied by the author in [4].

In this paper we show that the class of p-semisimple pseudo-BCI-algebras forms a variety. We define also the notion of branchwise commutative pseudo-BCI-algebras, give many characterizations of it and prove that the class of branchwise commutative pseudo-BCI-algebras forms a variety. We ilustrate also these notions by many examples. Finally, we present some congruence properties of the varieties of p-semisimple pseudo-BCI-algebras and branchwise commutative pseudo-BCI-algebras. For the convenience of the reader, in Section 2 we give the necessary material needed in the sequel, thus making our exposition self-contained.

2. PRELIMINARIES

One of the convincing arguments for treating pseudo-BCI-algebras as algebras $(X, \leq, \rightarrow, \rightsquigarrow, 1)$ rather than $(X, \leq, *, \circ, 0)$ is that the approach dual to the original one makes obvious the connection with logic – it allows to think of operations $\rightarrow$, $\rightsquigarrow$ as two implications in a suitable (non-commutative) propositional logic. The original approach to pseudo-BCI-algebras the reader can find in [1, 2, 4, 11] and [12]. In this paper we prefer the dual one.

A *pseudo-BCI-algebra* is a structure $\mathcal{X} = (X, \leq, \rightarrow, \rightsquigarrow, 1)$, where $\leq$ is binary relation on a set X, $\rightarrow$ and $\rightsquigarrow$ are binary operations on X and 1 is an element of X such that for all $x, y, z \in X$, we have

(a1) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z), x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$,

(a2) $x \leq (x \rightarrow y) \rightsquigarrow y, x \leq (x \rightsquigarrow y) \rightarrow y$,

(a3) $x \leq x$,

(a4) if $x \leq y$ and $y \leq x$, then $x = y$,

(a5) $x \leq y$ iff $x \rightarrow y = 1$ iff $x \rightsquigarrow y = 1$.

It is obvious that any pseudo-BCI-algebra $(X, \leq, \rightarrow, \rightsquigarrow, 1)$ can be regarded as a universal algebra $(X, \rightarrow, \rightsquigarrow, 1)$ of type $(2, 2, 0)$. Note that a pseudo-BCI-algebra satisfying $x \rightarrow y = x \rightsquigarrow y$ for all $x, y \in X$ is a BCI-algebra.

Every pseudo-BCI-algebra satisfying $x \leq 1$ for all $x \in X$ is a *pseudo-BCK-algebra*. A pseudo-BCI-algebra which is not a pseudo-BCK-algebra will be called *proper*.

Any pseudo-BCI-algebra $\mathcal{X} = (X, \rightarrow, \rightsquigarrow, 1)$ satisfies the following, for all $x, y, z \in X$, (b1) if $1 \leq x$, then $x = 1$,

(b2) if $x \leq y$, then $y \to z \leq x \to z$ and $y \rightsquigarrow z \leq x \rightsquigarrow z$,

(b3) if $x \leq y$ and $y \leq z$, then $x \leq z$,

(b4) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,

(b5) $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$,

(b6) $x \to y \leq (z \to x) \to (z \to y), x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$,

(b7) if $x \leq y$, then $z \to x \leq z \to y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$,

(b8) $1 \to x = 1 \rightsquigarrow x = x$,

(b9) $((x \to y) \rightsquigarrow y) \to y = x \to y, ((x \rightsquigarrow y) \to y) \rightsquigarrow y = x \rightsquigarrow y$,

(b10) $x \to y \leq (y \to x) \rightsquigarrow 1$,

(b11) $x \rightsquigarrow y \leq (y \rightsquigarrow x) \to 1$,

(b12) $(x \to y) \to 1 = (x \to 1) \rightsquigarrow (y \rightsquigarrow 1)$,

(b13) $(x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \to (y \to 1)$,

(b14) $x \to 1 = x \rightsquigarrow 1$.

If $(X, \leq, \to, \rightsquigarrow, 1)$ is a pseudo-BCI-algebra, then, by (a3), (a4), (b3) and (b1), $(X, \leq)$ is a poset with 1 as a maximal element.

The class of pseudo-BCI-algebras forms a quasivariety:

Proposition 2.1 ([5])**.** *An algebra $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is a pseudo-BCI-algebra if and only if it satisfies the following identities and quasi-identity:*

(i) $(x \to y) \rightsquigarrow [(y \to z) \rightsquigarrow (x \to z)] = 1$,

(ii) $(x \rightsquigarrow y) \to [(y \rightsquigarrow z) \to (x \rightsquigarrow z)] = 1$,

(iii) $1 \to x = x$,

(iv) $1 \rightsquigarrow x = x$,

(v) $x \to y = 1 \;\;\&\;\; y \to x = 1 \;\;\Rightarrow\;\; x = y$.

Since pseudo-BCI-algebras include BCI-algebras, which are not closed under homomorphic images (see [13]), it follows that the quasivariety of pseudo-BCI-algebras is not a variety.

Example 2.2. Let $X = \{a, b, c, d, 1\}$ and define binary operations $\to$ and $\rightsquigarrow$ on X by the following tables:

$\to$	a	b	c	d	1
a	1	b	c	d	1
b	1	1	c	c	1
c	c	c	1	a	c
d	c	c	1	1	c
1	a	b	c	d	1

$\rightsquigarrow$	a	b	c	d	1
a	1	b	c	c	1
b	1	1	c	c	1
c	c	c	1	b	c
d	c	c	1	1	c
1	a	b	c	d	1

Then $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ is a (proper) pseudo-BCI-algebra. Observe that it is not a pseudo-BCK-algebra because $c \not\leq 1$.

Example 2.3 ([11])**.** Let $Y_1 = (-\infty, 0]$ and let $\leq$ be the usual order on Y_1. Define binary operations $\to$ and $\rightsquigarrow$ on Y_1 by

$$x \to y = \begin{cases} 0 & \text{if} \quad x \leq y, \\ \frac{2y}{\pi} \arctan(\ln(\frac{y}{x})) & \text{if} \quad y < x, \end{cases}$$

$$x \rightsquigarrow y = \begin{cases} 0 & \text{if} \quad x \leq y, \\ ye^{-\tan(\frac{\pi x}{2y})} & \text{if} \quad y < x \end{cases}$$

for all $x, y \in Y_1$. Then $\mathcal{Y}_1 = (Y_1, \leq, \to, \rightsquigarrow, 0)$ is a pseudo-BCK-algebra, and hence it is a (non-proper) pseudo-BCI-algebra.

Example 2.4. Let $Y_2 = \mathbb{R}^2$ and define binary operations $\to$ and $\rightsquigarrow$ and a binary relation $\leq$ on Y_2 by

$$\begin{aligned}(x_1, y_1) \to (x_2, y_2) &= (x_2 - x_1, (y_2 - y_1)e^{-x_1}),\\ (x_1, y_1) \rightsquigarrow (x_2, y_2) &= (x_2 - x_1, y_2 - y_1 e^{x_2 - x_1}),\end{aligned}$$

$$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow (x_1, y_1) \to (x_2, y_2) = (0, 0) = (x_1, y_1) \rightsquigarrow (x_2, y_2)$$

for all $(x_1, y_1), (x_2, y_2) \in Y_2$. Then $\mathcal{Y}_2 = (Y_2, \leq, \to, \rightsquigarrow, (0, 0))$ is a (proper) pseudo-BCI-algebra. Notice that $\mathcal{Y}_2$ is not a pseudo-BCK-algebra because there exists $(x, y) = (1, 1) \in Y_2$ such that $(x, y) \not\leq (0, 0)$.

Example 2.5. Let $\mathcal{Y}$ be a direct product of pseudo-BCI-algebras $\mathcal{Y}_1$ and $\mathcal{Y}_2$ from Examples 2.3 and 2.4, respectively. Then $\mathcal{Y}$ is a (proper) pseudo-BCI-algebra, where $Y = (-\infty, 0] \times \mathbb{R}^2$ and binary operations $\to$ and $\rightsquigarrow$ and binary relation $\leq$ are defined on Y by

$$(x_1, y_1, z_1) \to (x_2, y_2, z_2) = \begin{cases} (0, y_2 - y_1, (z_2 - z_1)e^{-y_1}) & \text{if } x_1 \leq x_2,\\ (\frac{2x_2}{\pi}\arctan(\ln(\frac{x_2}{x_1})), y_2 - y_1, (z_2 - z_1)e^{-y_1}) & \text{if } x_2 < x_1,\end{cases}$$

$$(x_1, y_1, z_1) \rightsquigarrow (x_2, y_2, z_2) = \begin{cases} (0, y_2 - y_1, z_2 - z_1 e^{y_2 - y_1}) & \text{if } x_1 \leq x_2,\\ (x_2 e^{-\tan(\frac{\pi x_1}{2x_2})}, y_2 - y_1, z_2 - z_1 e^{y_2 - y_1}) & \text{if } x_2 < x_1,\end{cases}$$

$$(x_1, y_1, z_1) \leq (x_2, y_2, z_2) \Leftrightarrow x_1 \leq x_2 \text{ and } y_1 = y_2 \text{ and } z_1 = z_2.$$

Notice that $\mathcal{Y}$ is not a pseudo-BCK-algebra because there exists $(x, y, z) = (0, 1, 1) \in Y$ such that $(x, y, z) \not\leq (0, 0, 0)$.

An element a of a pseudo-BCI-algebra $\mathcal{X}$ is called an *atom* of $\mathcal{X}$ if the following holds, for every $x \in X$,

$$\text{if } a \leq x, \text{ then } x = a.$$

We will denote by $At(X)$ the set of all atoms of $\mathcal{X}$. Obviously, $1 \in At(X)$. It is shown in [2] that $At(X) = \{x \in X : x = (x \to 1) \to 1\}$. Moreover, by (b9) and (b14), $x \to 1 = x \rightsquigarrow 1 = ((x \to 1) \to 1) \to 1$ for any $x \in X$. Hence, it follows that $x \to 1 = x \rightsquigarrow 1 \in At(X)$ for any $x \in X$.

For the pseudo-BCI-algebras $\mathcal{X}$, $\mathcal{Y}_1$, $\mathcal{Y}_2$ and $\mathcal{Y}$ from Examples 2.2, 2.3, 2.4 and 2.5, respectively, we have $At(X) = \{c, 1\}$, $At(Y_1) = \{0\}$, $At(Y_2) = Y_2$, $At(Y) = \{(0, y, z) : y, z \in \mathbb{R}\}$.

Let $\mathcal{X}$ be a pseudo-BCI-algebra. For any $a \in X$ we define a subset $V(a)$ of X as follows

$$V(a) = \{x \in X : x \leq a\}.$$

Note that $V(a)$ is non-empty, because $a \leq a$ gives $a \in V(a)$. Notice also that $V(a) \subseteq V(b)$ for any $a, b \in X$ such that $a \leq b$.

If $a \in At(X)$, then the set $V(a)$ is called a *branch* of $\mathcal{X}$ determined by element a. The following facts are proved in [4]: (1) branches determined by different elements are disjoint, (2) a pseudo-BCI-algebra is a set-theoretic union of branches, (3) comparable elements are in the same branch, (4) elements x and y belong to the same branch if and only if $x \to y \in V(1)$, or equivalently, $x \rightsquigarrow y \in V(1)$.

The pseudo-BCI-algebra $\mathcal{Y}_1$ from Example 2.3 has only one branch (as the pseudo-BCK-algebra) and the pseudo-BCI-algebra $\mathcal{X}$ from Example 2.2 has two branches: $V(c) = \{c, d\}$ and $V(1) = \{a, b, 1\}$. Every $\{(x, y)\}$ is a branch of the pseudo-BCI-algebra $\mathcal{Y}_2$ from Example 2.4, where $(x, y) \in Y_2$. For the pseudo-BCI-algebra $\mathcal{Y}$ from Example 2.5 the sets $V((0, a_1, a_2)) = \{(x, a_1, a_2) \in Y : x \leq 0\}$ are branches of $\mathcal{Y}$.

A pseudo-BCI-algebra $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ is *p-semisimple* if it satisfies, for all $x \in X$,

$$\text{if } x \leq 1, \text{ then } x = 1.$$

Note that if $\mathcal{X}$ is a p-semisimple pseudo-BCI-algebra, then $V(1) = \{1\}$. Hence, if $\mathcal{X}$ is a p-semisimple pseudo-BCK-algebra, then $X = \{1\}$. Moreover, as it is proved in [4], a pseudo-BCI-algebra $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ is p-semisimple if and only if for all $x, y \in X$, $(x \to y) \rightsquigarrow y = x = (x \rightsquigarrow y) \to y$.

It is not difficult to see that the pseudo-BCI-algebras $\mathcal{X}$, $\mathcal{Y}_1$ and $\mathcal{Y}$ from Examples 2.2, 2.3 and 2.5, respectively, are not p-semisimple, and the pseudo-BCI-algebra $\mathcal{Y}_2$ from Example 2.4 is the p-semisimple algebra.

The class of p-semisimple pseudo-BCI-algebras forms a variety:

Theorem 2.6. *An algebra $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is a p-semisimple pseudo-BCI-algebra if and only if it satisfies the following identities:*

(i) $(x \to y) \rightsquigarrow [(y \to z) \rightsquigarrow (x \to z)] = 1,$

(ii) $(x \rightsquigarrow y) \to [(y \rightsquigarrow z) \to (x \rightsquigarrow z)] = 1,$

(iii) $1 \to x = x,$

(iv) $1 \rightsquigarrow x = x,$

(v) $(x \to y) \rightsquigarrow y = x,$

(vi) $(x \rightsquigarrow y) \to y = x.$

Proof. It is obvious that every p-semisimple pseudo-BCI-algebra satisfies identities (i)–(vi). Conversely, assume that an algebra $\mathcal{X}$ satisfies identities (i)–(vi). By Proposition 2.1 it suffices to prove that if $x \to y = 1$ and $y \to x = 1$, then $x = y$. If $x \to y = 1$ and $y \to x = 1$, then, by (iv) and (v), $x = (x \to y) \rightsquigarrow y = 1 \rightsquigarrow y = y$. Thus, $\mathcal{X}$ is a p-semisimple pseudo-BCI-algebra. ■

3. Branchwise commutativity

A pseudo-BCK-algebra $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ is *commutative* if for all $x, y \in X$, it satisfies the following identities:

$$(x \to y) \rightsquigarrow y \;=\; (y \to x) \rightsquigarrow x, \tag{1}$$

$$(x \rightsquigarrow y) \to y \;=\; (y \rightsquigarrow x) \to x. \tag{2}$$

Note that if we consider identities (1) and (2) in a pseudo-BCI-algebra $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$, then $\mathcal{X}$ becomes a pseudo-BCK-algebra. Indeed, by (a2), (a3) and (b8), $x \leq (x \to 1) \rightsquigarrow 1 = (1 \to x) \rightsquigarrow x = x \rightsquigarrow x = 1$ for each $x \in X$.

In a pseudo-BCI-algebra $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ consider identities (1) and (2) but for x and y belonging to the same branch. Such pseudo-BCI-algebras we will call *branchwise commutative*. Obviously, a commutative pseudo-BCK-algebra is a branchwise commutative pseudo-BCI-algebra.

Now we present several characterizations of branchwise commutative pseudo-BCI-algebras. The following simple lemma will be used in next theorem.

Lemma 3.1. *Let $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. If $x \in V(1)$ and $y \in X$, then $y \leq x \to y$ and $y \leq x \rightsquigarrow y$.*

Theorem 3.2. *A pseudo-BCI-algebra $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ is branchwise commutative if and only if for all $x, y \in X$, it satisfies the following quasi-identities:*

$$x \leq y \;\Rightarrow\; y = (y \to x) \rightsquigarrow x, \tag{3}$$

$$x \leq y \;\Rightarrow\; y = (y \rightsquigarrow x) \to x. \tag{4}$$

Proof. Assume that $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ is a branchwise commutative pseudo-BCI-algebra. Let $x \leq y$ for $x, y \in X$. Then x, y belong to the same branch. Hence, by assumption and (b8), $(y \to x) \rightsquigarrow x = (x \to y) \rightsquigarrow y = 1 \rightsquigarrow y = y$ and $(y \rightsquigarrow x) \to x = (x \rightsquigarrow y) \to y = 1 \to y = y$. So, (3) and (4) are satisfied.

Conversely, observe that, by (a2), (3) and (4), we have, for each $x, y \in X$,

$$(x \to y) \rightsquigarrow y = (((x \to y) \rightsquigarrow y) \to x) \rightsquigarrow x, \tag{5}$$

$$(x \rightsquigarrow y) \to y = (((x \rightsquigarrow y) \to y) \rightsquigarrow x) \to x. \tag{6}$$

Let x, y be in the same branch of $\mathcal{X}$. Then, $x \to y, x \rightsquigarrow y \in V(1)$. From Lemma 3.1 it follows that $y \leq (x \to y) \rightsquigarrow y$ and $y \leq (x \rightsquigarrow y) \to y$. Thus, by (b2), we infer that $(y \to x) \rightsquigarrow x \leq (((x \to y) \rightsquigarrow y) \to x) \rightsquigarrow x$ and $(y \rightsquigarrow x) \to x \leq (((x \rightsquigarrow y) \to y) \rightsquigarrow x) \to x$. So, by (5) and (6), respectively, we have $(y \to x) \rightsquigarrow x \leq (x \to y) \rightsquigarrow y$ and $(y \rightsquigarrow x) \to x \leq (x \rightsquigarrow y) \to y$. By replacing x and y we obtain the other inequalities. Therefore, $\mathcal{X}$ is branchwise commutative. ■

Corollary 3.3. *A pseudo-BCI-algebra $\mathcal{X}$ is branchwise commutative if and only if it satisfies the identities* (5) *and* (6)*.*

Corollary 3.4. *Every p-semisimple pseudo-BCI-algebra satisfies* (3) *and* (4)*, so it is branchwise commutative.*

Proposition 3.5. *A pseudo-BCI-algebra $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ is branchwise commutative if and only if for all x and y belonging to the same branch, it satisfies the following identities:*

$$x \to y = ((y \to x) \rightsquigarrow x) \to y,$$
$$x \rightsquigarrow y = ((y \rightsquigarrow x) \to x) \rightsquigarrow y.$$

Proof. If $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ is branchwise commutative, then for x, y from the same branch we have, by (b9), $x \to y = ((x \to y) \rightsquigarrow y) \to y = ((y \to x) \rightsquigarrow x) \to y$ and $x \rightsquigarrow y = ((x \rightsquigarrow y) \to y) \rightsquigarrow y = ((y \rightsquigarrow x) \to x) \rightsquigarrow y$.

Conversely, for $x \leq y$ we have $1 = ((y \to x) \rightsquigarrow x) \to y$ and $1 = ((y \rightsquigarrow x) \to x) \rightsquigarrow y$, i.e., $(y \to x) \rightsquigarrow x \leq y$ and $(y \rightsquigarrow x) \to x \leq y$. These together with (a2) give us (3) and (4). Now, from Theorem 3.2 we obtain branchwise commutativity. ■

Let $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. An important role in our next theorem will be played by subset

$$Z(a) = \{x \in X : a \leq x\}$$

called the *terminal part* determined by $a \in X$.

Theorem 3.6. *Let $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. The following are equivalent:*

(i) *$\mathcal{X}$ is branchwise commutative,*

(ii) *each branch of $\mathcal{X}$ is a semilattice with respect to the join $\vee$ defined by $x \vee y = (x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y$,*

(iii) *$Z(x) \cap Z(y) = Z(x \vee y)$ for all x and y belonging to the same branch.*

Proof. (i) $\Rightarrow$ (ii): Assume that $\mathcal{X}$ is a branchwise commutative pseudo-BCI-algebra. Let $a \in At(X)$ and $x, y \in V(a)$. Then, $x, y \leq (x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x$ and $x, y \leq (x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x$. Hence, $(x \to y) \rightsquigarrow y$ and $(x \rightsquigarrow y) \to y$ belong to $V(a)$ and they are upper bounds of $\{x, y\}$. We prove that $(x \to y) \rightsquigarrow y$ and $(x \rightsquigarrow y) \to y$ are both the least upper bounds of $\{x, y\}$. Let z be another upper bound of $\{x, y\}$. Then, $x \leq z$ and $y \leq z$. Hence, $z \in V(a)$. Observe that, by (3), (4) and (b2), we have

$$z = (z \to y) \rightsquigarrow y = (z \rightsquigarrow y) \to y \tag{7}$$

and

$$z \to y \leq x \to y, z \rightsquigarrow y \leq x \rightsquigarrow y. \tag{8}$$

Thus, by (7), (b4), (b9) and (8),

$$\begin{aligned}((x \to y) \rightsquigarrow y) \to z &= ((x \to y) \rightsquigarrow y) \to ((z \to y) \rightsquigarrow y)\\ &= (z \to y) \rightsquigarrow (((x \to y) \rightsquigarrow y) \to y)\\ &= (z \to y) \rightsquigarrow (x \to y) = 1\end{aligned}$$

and

$$\begin{aligned}((x \rightsquigarrow y) \to y) \rightsquigarrow z &= ((x \rightsquigarrow y) \to y) \rightsquigarrow ((z \rightsquigarrow y) \to y)\\ &= (z \rightsquigarrow y) \to (((x \rightsquigarrow y) \to y) \rightsquigarrow y)\\ &= (z \rightsquigarrow y) \to (x \rightsquigarrow y) = 1.\end{aligned}$$

So, $(x \to y) \rightsquigarrow y \leq z$ and $(x \rightsquigarrow y) \to y \leq z$. These mean that $(x \to y) \rightsquigarrow y$ and $(x \rightsquigarrow y) \to y$ are both the least upper bound of $\{x, y\}$. Hence, $x \vee y = (x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y$. Since $x \vee y \in V(a)$ whenever $x, y \in V(a)$, we conclude that $V(a)$ is a semilattice with respect to $\vee$.

(ii) $\Rightarrow$ (iii): Let $a \in At(X)$ and $x, y \in V(a)$. If $z \in Z(x) \cap Z(y)$, then $x \leq z$ and $y \leq z$. Hence, $z \in V(a)$ and $x \vee y \leq z$ because $x \vee y \in V(a)$ is the least upper bound of $\{x, y\}$. Thus, $z \in Z(x \vee y)$, i.e., $Z(x) \cap Z(y) \subseteq Z(x \vee y)$. On the other hand, for any $z \in Z(x \vee y)$, we have, by (a2), $x \leq (x \to y) \rightsquigarrow y = x \vee y \leq z$ whence $z \in Z(x)$. Since x, y are in the same branch, $x \to y \leq 1$. Hence, by (b4), we have $y \to x \vee y = y \to ((x \to y) \rightsquigarrow y) = (x \to y) \rightsquigarrow (y \to y) = (x \to y) \rightsquigarrow 1 = 1$, i.e., $y \leq x \vee y \leq z$. Thus $z \in Z(y)$. So, $z \in Z(x) \cap Z(y)$ whence $Z(x \vee y) \subseteq Z(x) \cap Z(y)$. Therefore, $Z(x) \cap Z(y) = Z(x \vee y)$.

(iii) $\Rightarrow$ (i): For x, y from the same branch we have $Z(x \vee y) = Z(x) \cap Z(y) = Z(y) \cap Z(x) = Z(y \vee x)$. Thus, $x \vee y \in Z(y \vee x)$ and $y \vee x \in Z(x \vee y)$. Hence, $y \vee x \leq x \vee y$ and $x \vee y \leq y \vee x$, i.e., $x \vee y = (x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y = y \vee x = (y \to x) \rightsquigarrow x = (y \rightsquigarrow x) \to x$. Therefore, $\mathcal{X}$ is branchwise commutative. ■

Theorem 3.7. *Let $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. If the following identities hold*

(9) $$(x \to y) \rightsquigarrow x = x,$$

(10) $$(x \rightsquigarrow y) \to x = x$$

for any $a \in At(X)$ and each $x, y \in V(a)$, then $\mathcal{X}$ is branchwise commutative.

Proof. Let $x, y \in X$. Since, by (a2), x and $(x \to y) \rightsquigarrow y$ are comparable, we have that $x, (x \to y) \rightsquigarrow y \in V(b)$ for some $b \in At(X)$. Hence, by (9) and (10),

$$(((x \to y) \rightsquigarrow y) \to x) \rightsquigarrow ((x \to y) \rightsquigarrow y) = (x \to y) \rightsquigarrow y,$$

$$(((x \rightsquigarrow y) \to y) \rightsquigarrow x) \to ((x \rightsquigarrow y) \to y) = (x \rightsquigarrow y) \to y.$$

Hence, by (a2) and (b7),

$$\begin{aligned}(x \to y) \rightsquigarrow y &\le (((x \to y) \rightsquigarrow y) \to x) \rightsquigarrow x \\ &\le (((x \to y) \rightsquigarrow y) \to x) \rightsquigarrow ((x \to y) \rightsquigarrow y) \\ &= (x \to y) \rightsquigarrow y\end{aligned}$$

and

$$\begin{aligned}(x \rightsquigarrow y) \to y &\le (((x \rightsquigarrow y) \to y) \rightsquigarrow x) \to x \\ &\le (((x \rightsquigarrow y) \to y) \rightsquigarrow x) \to ((x \rightsquigarrow y) \to y) \\ &= (x \rightsquigarrow y) \to y,\end{aligned}$$

so that (5) and (6) are satisfied. Now, from Corollary 3.3, the proof is complete. ■

Remark. A pseudo-BCI-algebra satisfying (9) and (10) for x and y belonging to the same branch can be called *branchwise implicative* (like in the theory of BCK/BCI-algebras). So, Theorem 3.7 says that a branchwise implicative pseudo-BCI-algebra is branchwise commutative. Branchwise implicativity of pseudo-BCI-algebras is the notion to further research.

The class of branchwise commutative pseudo-BCI-algebras forms a variety:

Theorem 3.8. *An algebra $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is a branchwise commutative pseudo-BCI-algebra if and only if it satisfies the following identities:*

(i) $(x \to y) \rightsquigarrow [(y \to z) \rightsquigarrow (x \to z)] = 1,$

(ii) $(x \rightsquigarrow y) \to [(y \rightsquigarrow z) \to (x \rightsquigarrow z)] = 1,$

(iii) $1 \to x = x,$

(iv) $1 \rightsquigarrow x = x,$

(v) $(x \to y) \rightsquigarrow y = (((x \to y) \rightsquigarrow y) \to x) \rightsquigarrow x,$

(vi) $(x \rightsquigarrow y) \to y = (((x \rightsquigarrow y) \to y) \rightsquigarrow x) \to x.$

Proof. If $\mathcal{X}$ is a branchwise commutative pseudo-BCI-algebra, then, obviously, it satisfies identities (i)–(vi). Conversely, assume that an algebra $\mathcal{X}$ satisfies identities (i)–(vi). From Proposition 2.1 it is sufficient to show that if $x \to y = 1$ and $y \to x = 1$, then $x = y$. Assume that $x \to y = 1$ and $y \to x = 1$. Then, by (iv) and (v), $y = 1 \rightsquigarrow y = (x \to y) \rightsquigarrow y = (((x \to y) \rightsquigarrow y) \to x) \rightsquigarrow x = ((1 \rightsquigarrow y) \to x) \rightsquigarrow x = (y \to x) \rightsquigarrow x = 1 \rightsquigarrow x = x$. Thus, $\mathcal{X}$ is a branchwise commutative pseudo-BCI-algebra. ■

Corollary 3.9. *The variety of p-semisimple pseudo-BCI-algebras is a subvariety of the variety of branchwise commutative pseudo-BCI-algebras.*

Examples.
1. The pseudo-BCI-algebra $\mathcal{X}$ from Example 2.2 is not branchwise commutative because it does not satisfy the identity (3).
2. The pseudo-BCK-algebra $\mathcal{Y}_1$ from Example 2.3 satisfies the identities (3) and (4). So, it is (branchwise) commutative.
3. Since the pseudo-BCI-algebra $\mathcal{Y}_2$ from Example 2.4 is p-semisimple, it is also branchwise commutative.
4. By simple calculation it is seen that the pseudo-BCI-algebra $\mathcal{Y}$ from Example 2.5 is branchwise commutative. It is an example of proper pseudo-BCI-algebra which is branchwise commutative but is not p-semisimple. This implies that the variety of p-semisimple pseudo-BCI-algebras is a proper subvariety of the variety of branchwise commutative pseudo-BCI-algebras.

Finally, we present some congruence properties of varieties of p-semisimple pseudo-BCI-algebras and branchwise commutative pseudo-BCI-algebras. But first we have to give some definitions and facts. Let $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. We say that a subset D of X is a *deductive system* of $\mathcal{X}$ if it satisfies: (i) $1 \in D$, and (ii) for all $x, y \in X$, if $x \in D$ and $x \to y \in D$, then $y \in D$. Moreover, a deductive system D is called: (i) *closed* if it is closed under operations $\to$ and $\rightsquigarrow$, i.e., if it is a subalgebra of $\mathcal{X}$, and (ii) *compatible* if for all $x, y \in X$, $x \to y \in D$ iff $x \rightsquigarrow y \in D$. By $\mathcal{DS}_{cc}(X)$ denote the set of all closed compatible deductive systems of a pseudo-BCI-algebra $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$.
Recall also the universal algebraic notion of weak regularity. Let $\mathcal{V}$ be a variety of algebras of type F with a constant 1. An algebra $X \in \mathcal{V}$ is called *weakly regular* if for every $\Theta, \Phi \in \mathrm{Con}(X)$, if $[1]_\Theta = [1]_\Phi$, then $\Theta = \Phi$. A variety $\mathcal{V}$ is *weakly regular* if every $X \in \mathcal{V}$ is weakly regular.

From [3] we have the following facts.

Proposition 3.10. *Let $\mathcal{V}$ be an arbitrary variety of pseudo-BCI-algebras. Then*

(i) *$\mathcal{V}$ is weakly regular,*

(ii) *for every $\mathcal{X} = (X, \to, \rightsquigarrow, 1)$ in $\mathcal{V}$, $\mathrm{Con}(X) \cong \mathcal{DS}_{cc}(X)$.*

Hence, we have the following theorem.

Theorem 3.11. *If $\mathcal{V}$ is a variety of p-semisimple pseudo-BCI-algebras or $\mathcal{V}$ is a variety of branchwise commutative pseudo-BCI-algebras, then*

(i) *$\mathcal{V}$ is weakly regular,*

(ii) *for every* $\mathcal{X} = (X, \rightarrow, \rightsquigarrow, 1)$ *in* $\mathcal{V}$, $\mathrm{Con}(X) \cong \mathcal{DS}_{cc}(X)$.

Acknowledgements

The author would like to express his thanks to the referee for remarks and help.

References

[1] W.A. Dudek and Y.B. Jun, *Pseudo-BCI algebras*, East Asian Math. J. **24** (2008), 187–190.

[2] G. Dymek, *Atoms and ideals of pseudo-BCI-algebras*, submitted.

[3] G. Dymek, *On compatible deductive systems of pseudo-BCI-algebras*, submitted.

[4] G. Dymek, *p-semisimple pseudo-BCI-algebras*, J. Mult.-Val. Log. Soft Comput., to appear.

[5] G. Dymek and A. Kozanecka-Dymek, *Pseudo-BCI-logic*, submitted.

[6] G. Georgescu and A. Iorgulescu, Pseudo-BCK algebras: an extension of BCK-algebras, Proceedings of DMTCS'01: Combinatorics, Computability and Logic, Springer, London, 2001, 97–114.

[7] G. Georgescu and A. Iorgulescu, Pseudo-BL algebras: a noncommutative extension of BL-algebras, Abstracts of The Fifth International Conference FSTA 2000, Slovakia, February 2000, 90–92.

[8] G. Georgescu and A. Iorgulescu, Pseudo-MV algebras: a noncommutative extension of MV-algebras, The Proceedings The Fourth International Symposium on Economic Informatics, INFOREC Printing House, Bucharest, Romania, May (1999), 961–968.

[9] A. Iorgulescu, Algebras of logic as BCK algebras, Editura ASE, Bucharest, 2008.

[10] K. Iséki, *An algebra related with a propositional calculus*, Proc. Japan. Academy **42** (1966), 26–29. doi:10.3792/pja/1195522171

[11] Y.B. Jun, H.S. Kim and J. Neggers, *On pseudo-BCI ideals of pseudo BCI-algebras*, Mat. Vesnik **58** (2006), 39–46.

[12] K.J. Lee and C.H. Park, *Some ideals of pseudo-BCI algebras*, J. Appl. Math. & Informatics **27** (2009), 217–231.

[13] A. Wroński, *BCK-algebras do not form a variety*, Math. Japon. **28** (1983), 211–213.

Received 3 June 2011
Revised 25 July 2011