ON MAXIMAL IDEALS OF PSEUDO-BCK-ALGEBRAS

ANDRZEJ WALENDZIAK

Institute of Mathematics and Physics
University of Podlasie
3 Maja 54, 08–110 Siedlce, Poland
e-mail: walent@interia.pl

Abstract

We investigate maximal ideals of pseudo-BCK-algebras and give some characterizations of them.

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1. Introduction

K. Iséki and S. Tanaka ([16]) introduced the notion of ideals in BCK-algebras and investigated some interesting and fundamental results. R. Halaš and J. Kühr [11] applied this concept to pseudo-BCK-algebras. (They called ideals as deductive systems.) In this paper, we give some characterizations of maximal ideals in pseudo-BCK-algebras.

2. Preliminaries

The notion of pseudo-BCK-algebras is defined by Georgescu and Iorgulescu [7] as follows:

**Definition 2.1.** A pseudo-BCK-algebra is a structure \((A; \leq, \ast, \circ, 0)\), where “\(\leq\)” is a binary relation on a set \(A\), “\(\ast\)” and “\(\circ\)” are binary operations on \(A\) and “\(0\)” is an element of \(A\), verifying the axioms: for all \(x, y, z \in A\),

\[
\begin{align*}
(p\text{BCK}-1) & \quad (x \ast y) \circ (x \ast z) \leq z \ast y, \quad (x \circ y) \ast (x \circ z) \leq z \circ y, \\
(p\text{BCK}-2) & \quad x \ast (x \circ y) \leq y, \quad x \circ (x \ast y) \leq y, \\
(p\text{BCK}-3) & \quad x \leq x, \\
(p\text{BCK}-4) & \quad 0 \leq x, \\
(p\text{BCK}-5) & \quad (x \leq y \text{ and } y \leq x) \Rightarrow x = y, \\
(p\text{BCK}-6) & \quad x \leq y \Leftrightarrow x \ast y = 0 \Leftrightarrow x \circ y = 0.
\end{align*}
\]

Note that every pseudo-BCK-algebra satisfying \(x \ast y = x \circ y\) for all \(x, y \in A\) is a BCK-algebra.

**Proposition 2.2** ([7]). Let \((A; \leq, \ast, \circ, 0)\) be a pseudo-BCK-algebra. Then for all \(x, y, z \in A\):

\[
\begin{align*}
(a) & \quad x \leq y \text{ and } y \leq z \Rightarrow x \leq z; \\
(b) & \quad x \ast y \leq x, \quad x \circ y \leq x;
\end{align*}
\]
(c) \((x \ast y) \circ z = (x \circ z) \ast y\); 
(d) \(x \ast 0 = x = x \circ 0\);  
(e) \(x \leq y \Rightarrow x \ast z \leq y \ast z, \ x \circ z \leq y \circ z\).

If \((A; \leq, \ast, \circ, 0)\) is a pseudo-BCK-algebra, then \((A; \leq)\) is a poset by (pBCK-3), (pBCK-5), and Proposition 2.2 (a). The underlying order \(\leq\) can be retrieved via (pBCK-6) and hence we may equivalently regard \((A; \leq, \ast, \circ, 0)\) to be an algebra \((A; \ast, \circ, 0)\). J. Kühr [18] showed that pseudo-BCK-algebras as algebras \((A; \ast, \circ, 0)\) of type \(\langle 2, 2, 0 \rangle\) form a quasivariety which is not a variety.

Throughout this paper \(A\) will denote a pseudo-BCK-algebra. For \(x, y \in A\) and \(n \in \mathbb{N}_0\) (\(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\)) we define \(x \ast^n y\) inductively
\[x \ast^0 y = x, \quad x \ast^{n+1} y = (x \ast^n y) \ast y \quad (n = 0, 1, \ldots).
\]
\(x \circ^n y\) is defined in the same way.

**Example 2.3** ([11], Example 2.4). Let \(A = \{0, a, b, c\}\) and define binary operations “\(\ast\)” and “\(\circ\)” on \(A\) by the following tables:

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Then \((A; \ast, \circ, 0)\) is a pseudo-BCK-algebra.

**Example 2.4.** Let \((M; \oplus, \neg, \sim, 0, 1)\) be a pseudo-MV-algebra and we put \(x \circ y = (y^\sim \oplus x^\sim)^\sim\) (\(= (y^\sim \oplus x^\sim)^\sim\) by Proposition 1.7 (1) of [8]). Define

\[x \ast y = x \circ y^\sim \quad \text{and} \quad x \circ y = y^\sim \circ x.
\]

By 4.1.3 of [18], \((M; \ast, \circ, 0)\) is a pseudo-BCK-algebra.
3. **Ideals**

**Definition 3.1.** A subset $I$ of a pseudo-BCK-algebra $A$ is called an ideal of $A$ if it satisfies for all $x, y \in A$:

(i1) $0 \in I$,

(i2) if $x \ast y \in I$ and $y \in I$, then $x \in I$.

We will denote by $\text{Id}(A)$ the set of all ideals of $A$.

**Proposition 3.2.** Let $I \in \text{Id}(A)$. Then for any $x, y \in A$, if $y \in I$ and $x \leq y$, then $x \in I$.

**Proof.** Straightforward.

**Proposition 3.3.** Let $I$ be a subset of $A$. Then $I$ is an ideal of $A$ if and only if it satisfies conditions (I1) and

(ii') for all $x, y \in A$, if $x \circ y \in I$ and $y \in I$, then $x \in I$.

**Proof.** It suffices to prove that if (I2) is satisfied, then (ii') is also satisfied. The proof of the converse of this implication is analogous. Suppose that $x \circ y \in I$ and $y \in I$. From (pBCK-2) we know that $x \ast (x \circ y) \leq y$. Then, by Proposition 3.2, $x \ast (x \circ y) \in I$. Hence, since $x \circ y \in I$, (I2) shows that $x \in I$. \hfill \blacksquare

For every subset $X \subseteq A$, we denote by $(X)$ the ideal of $A$ generated by $X$, that is, $(X)$ is the smallest ideal containing $X$. If $X = \{a\}$, we write $(a)$ for $(\{a\})$. By Lemma 2.2 of [11], $(\emptyset) = \{0\}$ and for every $\emptyset \neq X \subseteq A$,

$$(X) = \{x \in A : (\cdots (x \ast a_1) \ast \cdots) \ast a_n = 0 \text{ for some } a_1, \ldots, a_n \in X\}$$

$$= \{x \in A : (\cdots (x \circ a_1) \circ \cdots) \circ a_n = 0 \text{ for some } a_1, \ldots, a_n \in X\}.$$
Definition 3.4. An ideal $I$ of $A$ is called normal if it satisfies the following condition:

\[(N) \text{ for all } x, y \in A, x \ast y \in I \iff x \circ y \in I.\]

Example 3.5. Let $A$ be the pseudo-BCK-algebra from Example 2.3. Ideals of $A$ are $\{0\}, \{0, a\}, A; \{0, a\}$ is not normal, because $c \circ b = a \in I$ while $c \ast b = b \notin I$.

Example 3.6 ([2], see also [15], 430). Let $A = \{(1, y) \in \mathbb{R}^2 : y \geq 0\} \cup \{(2, y) \in \mathbb{R}^2 : y \leq 0\}$ and $0 = (1, 0), 1 = (2, 0)$. For any $(a, b), (c, d) \in A$, we define operations $\oplus, -, \sim$ as follows:

\[
(a, b) \oplus (c, d) = \begin{cases} 
(ac, bc + d) & \text{if } ac < 2 \text{ or } (ac = 2 \text{ and } bc + d < 0) \\
(2, 0) & \text{otherwise},
\end{cases}
\]

\[
(a, b)^- = \left(\frac{2}{a}, -\frac{b}{a}\right), \quad (a, b)^\sim = \left(\frac{2}{a}, -\frac{2b}{a}\right).
\]

Then $(A, \oplus, -, \sim, 0, 1)$ is a pseudo-MV-algebra. For $x, y \in A$, we set

\[
x \ast y = (y \oplus x^-)^- \quad \text{and} \quad x \circ y = (x^- \oplus y)^-.
\]

Therefore $(A; \ast, \circ, 0)$ is a pseudo-BCK-algebra (see Example 2.4). We have

\[
(a, b) \ast (c, d) = \left((c, d) \oplus \left(\frac{2}{a}, -\frac{2b}{a}\right)\right)^-
\]

and hence

\[
(a, b) \ast (c, d) = \begin{cases} 
\left(\frac{a}{c}, \frac{b - d}{c}\right) & \text{if } a = 2c \text{ or } (a = c \text{ and } d < b) \\
(1, 0) & \text{otherwise}.
\end{cases}
\]
Similarly,

\[(a, b) \circ (c, d) = \begin{cases} \left( \frac{a}{c}, b - \frac{ad}{c} \right) & \text{if } a = 2c \text{ or } (a = c \text{ and } d < b) \\ (1, 0) & \text{otherwise.} \end{cases}\]

It is easy to see that \(I = \{(1, y) : y \geq 0\}\) is an ideal of \(A\). Observe that \(I\) is normal. Indeed,

\[(a, b) * (c, d) \notin I \iff a = 2c \iff (a, b) \circ (c, d) \notin I.\]

**Lemma 3.7.** Let \(I\) be a normal ideal of \(A\). Then

\[x \ast^n a \in I \iff x \circ^n a \in I\]

for all \(x, a \in A\) and \(n \in \mathbb{N}\).

**Proof.** The proof is by induction on \(n\). \(\blacksquare\)

Following [18] (see also [19], p. 357), for any normal ideal \(I\) of \(A\), we define the congruence on \(A\):

\[x \sim_I y \iff x * y \in I \text{ and } y * x \in I.\]

We denote by \(x/I\) the congruence class of an element \(x \in A\) and on the set \(A/I = \{x/I : x \in A\}\) we define the operations:

\[x/I * y/I = (x * y)/I, \ x/I \circ y/I = (x \circ y)/I\]

(* and \(\circ\) are well defined on \(A/I\), because \(\sim_I\) is a congruence on \(A\)). The resulting quotient algebra \((A/I; *, \circ, I)\) becomes a pseudo-BCK-algebra (see Proposition 2.2.4 of [18]), called the *quotient algebra of \(A\) by the normal ideal \(I\)*. It is clear that

\[(1) \quad x/I = 0/I \iff x \in I.\]

**Proposition 3.8.** Let \(I\) be a normal ideal of \(A\) and let \(J \subseteq A/I\). Then \(J \in \text{Id}(A/I)\) if and only if \(J = I_0/I\) for some \(I_0 \in \text{Id}(A)\) such that \(I \subseteq I_0\).
\textbf{Proof.} Suppose that \( J \in \text{Id}(A/I) \). Let \( I_0 = \{ x \in A : x/I \in J \} \). By (1), 
\( I \subseteq I_0 \). Observe that \( I_0 \) is an ideal of \( A \). Indeed, \( 0 \in I_0 \) and let \( x y, y \in I_0 \). Then \( (x y)/I \in J \) and \( y/I \in J \). Hence \( x/I \in J \) and therefore \( x \in I_0 \). Thus \( I_0 \in \text{Id}(A) \). It is easy to see that \( J = I_0/I \).

Conversely, let \( J = I_0/I \) for some \( I_0 \in \text{Id}(A) \) such that \( I \subseteq I_0 \). Of course, 
\( 0/I \in J \). Let \( x = y/I, y/I \in J \). Then \( x y \in I_0 \) and \( y \in I_0 \). Since \( I_0 \) is an ideal of \( A \), we see that \( x \in I_0 \), hence that \( x/I \in J \). Consequently, \( J \in \text{Id}(A/I) \).

\begin{proposition}
Let \( I \) be a normal ideal of \( A \) and let \( a \in A \). Denote by

\[ I_a = \{ x \in A : x \ast^n a \in I \text{ for some } n \in \mathbb{N} \} \]

Then \( I_a = (I \cup \{a\}) \).
\end{proposition}

\textbf{Proof.} We first show that

\begin{equation}
I_a \subseteq (I \cup \{a\})\text{.}
\end{equation}

Let \( x \ast^n a \in I \) for some \( n \in \mathbb{N} \). We have \( (x \ast^n a) \ast (x \ast^n a) = 0 \). Thus

\[ ((\cdots ((x \ast b_1) \ast b_2) \cdots) \ast b_n) \ast b_{n+1} = 0, \]

where \( b_1 = \cdots = b_n = a \) and \( b_{n+1} = x \ast^n a \in I \). Thus \( x \in (I \cup \{a\}) \). This gives (2).

Since \( a \ast a = 0 \in I \), we see that \( a \in I_a \). Let \( x \in I \). Then \( x \ast a \in I \),

because \( x \ast a \subseteq x \). Therefore \( x \in I_a \) and hence \( I_a \) contains \( I \). Suppose now that \( x \ast y \in I_a \) and \( y \in I_a \). It follows that there exist \( k, l \in \mathbb{N} \) such that \( (x \ast y) \ast^k a \in I \) and \( y \ast^l a \in I \). By Lemma 3.7, \( (x \ast y) \ast^k a \in I \). Applying Proposition 2.2 (c) we conclude that

\[ (x \ast y) \ast^k a = ((x \ast^2 a) \ast y) \ast^{k-1} a = ((x \ast^2 a) \ast y) \ast^{k-2} a = \cdots = (x \ast^k a) \ast y. \]

Therefore \( b := (x \ast^k a) \ast y \in I \). Then \( ((x \ast^k a) \ast y) \ast b = 0 \) and hence

\[ ((x \ast^k a) \circ b) \ast y = 0. \]

Thus \( (x \ast^k a) \circ b \leq y \). By Proposition 2.2 (e),

\[ ((x \ast^k a) \circ b) \ast^l a \leq y \ast^l a \in I. \]

Consequently, \( (x \ast^k a) \circ b) \ast^l a \in I \).

\end{proposition}
According to Proposition 2.2 (c) we have \((x \circ^k a) \ast^l a) \circ b \in I\). Since \(b \in I\), we see that \((x \circ^k a) \ast^l a \in I\). Lemma 3.7 now shows that \(x \ast^{k+l} a \in I\), that is, \(x \in I_a\). This proves that \(I_a\) is an ideal of \(A\). Thus

\[
(I \cup \{a\}) \subseteq I_a.
\]

From (2) and (3) we obtain \(I_a = (I \cup \{a\})\).

Proposition 3.9 and Lemma 3.7 give.

**Corollary 3.10.** Let \(I\) be a normal ideal of \(A\) and let \(a \in A\). Then

\[
(I \cup \{a\}) = \{x \in A : x \ast^n a \in I \text{ for some } n \in \mathbb{N}\}
\]

\[
= \{x \in A : x \circ^n a \in I \text{ for some } n \in \mathbb{N}\}.
\]

**Corollary 3.11.** Let \(a \in A\). Then \((a) = \{x \in A : x \ast^n a = 0 \text{ for some } n \in \mathbb{N}\}\).

**Proof.** This follows from Proposition 3.9 when we put \(I = \{0\}\).

Let \(A\) and \(B\) be pseudo-BCK-algebras and let \(f : A \to B\) be a homomorphism. The **kernel** of \(f\) is the set

\[
\text{Ker} f := \{x \in A : f(x) = 0\},
\]

that is, \(\text{Ker} f = f^{-1}(\{0\})\), where \(f^{-1}(X)\) denote the \(f\)-inverse image of \(X \subseteq B\). It is easy to see that the next lemma holds.

**Lemma 3.12.** Let \(f : A \to B\) be a homomorphism and let \(x, y \in A\). If \(f(x) = f(y)\), then \(x \ast y, y \ast x \in \text{Ker} f\).

**Proposition 3.13.** Let \(f : A \to B\) be a homomorphism and let \(I \in \text{Id}(B)\). Then \(f^{-1}(I) \in \text{Id}(A)\).

**Proof.** The proof is straightforward.
Proposition 3.14. Let $f : A \rightarrow B$ be a surjective homomorphism and let $I$ be an ideal of $A$ containing $\ker f$. Then $f(I) \in \text{Id}(B)$.

Proof. Obviously, $0 \in f(I)$. Let $x \in B$, $y \in f(I)$, and let $x * y \in f(I)$. Then there are $a, b \in I$ such that $y = f(a)$ and $x * y = f(b)$. Since $f$ is surjective, $x = f(c)$ for some $c \in A$. We have $f(b) = f(c) * f(a) = f(c * a)$ and hence, by Lemma 3.12, $(c * a) * b \in \ker f \subseteq I$. Since $a, b \in I$, we conclude that $c \in I$. Therefore $x = f(c) \in f(I)$. Consequently, $f(I) \in \text{Id}(B)$.

4. Maximal ideals

Definition 4.1. Let $I$ be a proper ideal of $A$ (i.e., $I \neq A$).

(a) $I$ is called prime if, for all $I_1, I_2 \in \text{Id}(A)$, $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$.

(b) $I$ is maximal iff whenever $J$ is an ideal such that $I \subseteq J \subseteq A$, then either $J = I$ or $J = A$.

Next lemma is obvious and its proof will be omitted.

Lemma 4.2. Every proper ideal of $A$ can be extended to a maximal ideal.

Lemma 4.3. If $I \in \text{Id}(A)$ is maximal, then $I$ is prime.

Proof. Let $I$ be a maximal ideal of $A$ and let $I = I_1 \cap I_2$ for some $I_1, I_2 \in \text{Id}(A)$. Then $I \subseteq I_1$ and $I \subseteq I_2$. Suppose that $I \neq I_1$. Since $I$ is maximal, we conclude that $I_1 = A$ and hence $I = A \cap I_2 = I_2$. By definition, $I$ is prime.

Theorem 4.4.

(i) For each $t \in T$, let $I_t$ be an ideal of the pseudo-BCK-algebra $(A_t; *_t, c_t, 0_t)$. Then $I := \prod_{t \in T} I_t$ is an ideal of $A := \prod_{t \in T} A_t$. Conversely, if $I$ is an ideal of $A$, then $I_t := \pi_t(I)$, where $\pi_t$ is the $t$-th projection of $A$ onto $A_t$, is an ideal of $A_t$, and $I = \prod_{t \in T} I_t$.

(ii) An ideal $I := \prod_{t \in T} I_t$ is maximal in $A := \prod_{t \in T} A_t$ if and only if there is an unique index $s \in T$ such that $I_s$ is a maximal ideal of $A_s$ and $I_t = A_t$ for any $t \neq s$. 
Proof.

(i) The first part of the assertion is obvious. Suppose now that \( I \) is an ideal of \( A \) and let \( I_t = \pi_t(I) \). Then \( 0_t = \pi_t(0) \in I_t \). Let \( x_t \ast_t y_t \in I_t \) and \( y_t \in I_t \). We define \( x, y \in A \) by:

\[
x(s) = \begin{cases} 
  x_t & \text{for } s = t \\
  0_s & \text{for } s \neq t 
\end{cases}
\]

\[
y(s) = \begin{cases} 
  y_t & \text{for } s = t \\
  0_s & \text{for } s \neq t 
\end{cases}
\]

Since \( I_t = \pi_t(I) \), there exists an element \( z \in I \) such that \( \pi_t(z) = x_t \ast_t y_t \). We have \( (x \ast y)(t) = x(t) \ast_t y(t) = x_t \ast_t y_t = z(t) \) and \( (x \ast y)(s) = 0_s \ast_s 0_s = 0_s \leq z(s) \) for any \( s \neq t \). Therefore \( x \ast y \leq z \) which implies that \( x \ast y \in I \). Similarly there is an element \( v \in I \) such that \( \pi_t(v) = y_t \in I_t \). Obviously, \( y \leq v \) and hence \( y \in I \). This means that \( I_t \) is an ideal of \( A_t \). Since \( \pi_t(I) = I_t \) for all \( t \in T \), we see that \( I = \prod_{t \in T} I_t \).

(ii) Let \( I = \prod_{t \in T} I_t \) be a maximal ideal of \( A \). It is easily seen that there is at least one index \( t \) such that \( I_t \) is a maximal ideal of \( A_t \). Assume that there are two indices \( t_1 \) and \( t_2 \) such that \( I_{t_1} \) and \( I_{t_2} \) are proper ideals of \( A_{t_1} \) and \( A_{t_2} \), respectively. Then \( J := \prod_{t \in T} I_t' \), where \( I_t' = I_t \) if \( t \neq t_1 \) and \( I_{t_1}' = A_{t_1} \), is a proper ideal of \( A \) containing \( I \), which contradicts the maximality of \( I \). Suppose that \( I = \prod_{t \in T} I_t \), where \( I_s \) is a maximal ideal of \( A_s \) and \( I_t = A_t \) for all \( t \neq s \). By (i), \( I \in \text{Id}(A) \). Observe that \( I \) is maximal. Indeed, let \( K \in \text{Id}(A) \) and \( K \supset I \). Then \( \pi_s(K) \supset I_s \) and \( \pi_t(K) = A_t \) for all \( t \neq s \). Since \( I_s \) is maximal in \( A_s \), we see that \( \pi_s(K) = A_s \), and therefore \( \pi_t(K) = A_t \) for all \( t \in T \). Thus \( K = A \) and consequently, \( I \) is a maximal ideal of \( A \).

The following two theorems give the homomorphic properties of maximal ideals.

**Theorem 4.5.** Let \( f : A \rightarrow B \) be a surjective homomorphism and let \( I \) be a maximal ideal of \( A \) containing \( \text{Ker} f \). Then \( f(I) \) is a maximal ideal of \( B \).
Proof. By Proposition 3.14, $f(I) \in \text{Id}(B)$. Let $x \in A - I$ and suppose that $f(I) = B$. Then $f(x) = f(y)$ for some $y \in I$. Applying Lemma 3.12 we conclude that $x * y \in I$, and hence $x \in I$, a contradiction. Therefore $f(I) \neq B$. We take a proper ideal $J$ of $B$ such that $J \supseteq f(I)$. From Proposition 3.13 we deduce that $f^{-1}(J) \in \text{Id}(A)$. It is easy to see that $I \subseteq f^{-1}(J) \subset A$. Since $I$ is maximal, $f^{-1}(J) = I$. Consequently, $f(I) = f(f^{-1}(J)) = J$. Thus $f(I)$ is a maximal ideal of $B$.

Theorem 4.6. Let $f : A \to B$ be a surjective homomorphism and let $J$ be a maximal ideal of $B$. Then $f^{-1}(J)$ is a maximal ideal of $A$.

Proof. From Proposition 3.13 it follows that $I := f^{-1}(J) \in \text{Id}(A)$. It is easily seen that $I \neq A$. By Lemma 4.2 there is a maximal ideal $I'$ of $A$ containing $I$. We have

$$I = f^{-1}(J) \supseteq f^{-1}([0]) = \text{Ker}f.$$ 

Since $I' \supseteq I \supseteq \text{Ker}f$, Theorem 4.5 shows that $f(I')$ is a maximal ideal of $B$. Obviously, $f(I') \supseteq f(f^{-1}(J)) = J$ and hence $f(I') = J$. Then $I' \subseteq f^{-1}(f(I')) = f^{-1}(J) = I \subseteq I'$, that is, $f^{-1}(J) = I'$. Thus $f^{-1}(J)$ is a maximal ideal of $A$.

Theorem 4.7. For every proper normal ideal $I$ of a pseudo-BCK-algebra $A$, the following conditions are equivalent:

(a) $I$ is a maximal ideal of $A$;

(b) for any $x \in A$, $y \in A - I$, $x *^n y \in I$ for some $n \in \mathbb{N}$;

(c) for any $x \in A$, $y \in A - I$, $x \circ^n y \in I$ for some $n \in \mathbb{N}$;

(d) $|\text{Id}(A/I)| = 2$.

Proof. (a) $\Rightarrow$ (b): Let $x \in A$. Suppose that $I$ is a maximal ideal of $A$ and let $y \in A - I$. Then $(I \cup \{y\}) = A$ and hence $x \in (I \cup \{y\})$. By Proposition 3.9, $x *^n y \in I$ for some $n \in \mathbb{N}$.

(b) $\iff$ (c): The equivalence of (b) and (c) follows from the fact that $I$ is a normal ideal.
(c) ⇒ (a): Let \( J \) be an ideal of \( A \) containing \( I \). Suppose that \( J \neq I \) and let \( y \in J - I \). For every \( x \in A \), by assumption, \( x \circ^n y \in I \) for some \( n \in \mathbb{N} \). Then \( x \circ^n y \in J \) and hence \( x \in J \), because \( y \in J \). Therefore \( J = A \).

(a) ⇒ (d): Let \( I \) be a normal and maximal ideal of \( A \), and let \( J \) be an ideal of \( A/I \). By Proposition 3.8, \( J = I_0/I \) for some \( I_0 \in \text{Id}(A) \) such that \( I \subseteq I_0 \). Since \( I \) is maximal, \( I_0 = I \) or \( I_0 = A \). Consequently, \( J = \{0/I\} \) or \( J = A/I \).

(d) ⇒ (a): Let \( I_0 \) be a proper ideal of \( A \) containing \( I \). From Proposition 3.8 it follows that \( J = I_0/I \) is an ideal of \( A/I \). Therefore \( J = \{0/I\} \), that is, \( I_0 = I \), which proves that \( I \) is maximal.

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References


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