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SEMIGROUP OF CONTRACTIONS OF WREATH PRODUCTS OF METRIC SPACES*

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Abstract

In this paper semigroups of contractions of metric spaces are considered. The semigroup of contractions of the wreath product of metric spaces is calculated.

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1. Introduction

In articles [1, 2] F. Harary and G. Sabidussi introduced a new construction of composition of graphs. Later this construction was called the wreath product of graphs.

A notion of the wreath product of metric spaces was introduced in [4] analogously to the Sabidussi's and Harary's one.

It is known [4] that the isometry group of the wreath product of metric spaces X and Y is isomorphic as a permutation group to the wreath product of isometry groups of spaces X and Y.

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With every metric space we can associate a few transformations semigroups: the semigroup of partial isometries, the semigroups of 1-Lipschitzian transformations (semigroup of contractions) and the semigroup of partial 1-Lipschitzian transformations. We shall consider semigroups of contractions of metric spaces.

The main result of this report is the following one

Theorem 1. *The semigroup of contractions of wreath product of metric spaces X and Y is isomorphic as a transformation semigroup to the wreath product of semigroups of contractions of spaces X and Y*

$$Ctr(X\, wr Y) \simeq Ctr X \wr Ctr Y.$$

2. Preliminaries

Let (X, d_X) be a metric space. A contraction (or an 1-Lipschitzian transformation) of X is a mapping $f : X \to X$ such that for arbitrary $a, b \in X$ the inequality

$$d_X(f(a), f(b)) \leq d_X(a, b)$$

holds.

Example 1. Let z be some point in X. It is clear that a mapping $f : X \to X$ such that $f(x) = z$ for every point $x \in X$ is a contraction of (X, d_X).

Example 2. Assume that there exists

$$\min_{u,v \in X, u \neq v} \{d_X(u, v)\} = q,$$

then $q > 0$. Let a, b be points of X such that $d_X(a, b) = q$. Define a mapping $f : X \to X$ by the rule:

$$f(a) = a, \qquad f(b) = b$$

and $f(x) \in \{a, b\}$ for other points $x \in X$. Then f is a contraction of (X, d_X).

The set of all contractions of the space (X, d_X) forms a semigroup under composition. We call it the *semigroup of contractions* of metric space (X, d_X) and denote by $Ctr X$.

Example 3. Let (X, d_X) be a metric space with equidistant metric i.e. there exists some positive c such that $d_X(a, b) = c$ for all distinct $a, b \in X$. Then the semigroup of contractions of (X, d_X) is the full transformations semigroup T_X.

Observe, that the isometry group IsX of the space X is the subgroup of the semigroup of contractions $CtrX$ of this space.

Proposition 1. *Let f be a contraction of metric space (X, d_X). If f is one-to-one and the inverse mapping f^{-1} is also a contraction then f is an isometry of the space (X, d_X).*

The proof of this proposition is straightforward.

Metric spaces (X, d_X) and (Y, d_Y) are called isomorphic ([3]) if there exists a scale, that is a strictly increasing continuous function $s : \mathbb{R}^+ \to \mathbb{R}^+$, $s(0) = 0$, such that $d_X = s(d_Y)$.

It is easy to see that if metric spaces (X, d_X) and (Y, d_Y) are isomorphic then their semigroups of contraction $CtrX$ and $CtrY$ are isomorphic.

Assume that there exists a positive number r, such that for arbitrary points $x_1, x_2 \in X$, $x_1 \neq x_2$, the inequality $d_x(x_1, x_2) \geq r$ holds. Additionally assume that the diameter $diamY$ of the space (Y, d_Y) is finite. Then fix a scale $s(x)$ such that

$$diam(s(Y)) < r. \tag{1}$$

Define a metric on the cartesian product $X \times Y$ by the rule:

$$\rho_s((x_1, y_1), (x_2, y_2)) = \begin{cases} d_X(x_1, x_2), & \text{if } x_1 \neq x_2 \\ s(d_Y(y_1, y_2)), & \text{if } x_1 = x_2 \end{cases}. \tag{2}$$

We call $(X \times Y, \rho_s)$ the wreath product of metric spaces X and Y with scale s and denote it by $X wr_s Y$.

Proposition 2 ([4])**.** *Let s_1 and s_2 be scales such that the inequality* (1) *holds. Then spaces $(X \times Y, \rho_{s_1})$ and $(X \times Y, \rho_{s_2})$ are isomorphic.*

Since the wreath product of metric spaces is unique up to isomorphism we assume in the sequel that the corresponding scale is fixed. Denote the wreath product of metric spaces X and Y by $XwrY$.

For the definition of the wreath product of transformation semigroups see [5].

3. Proof of the main theorem

At first let us prove that an arbitrary element

$$\varphi = [g, h(x)] \in CtrX \wr CtrY$$

defines a contraction of $XwrY$. The definition of the wreath product of transformation semigroup ([5]) implies that φ acts on $X \times Y$. We shall see that φ does not increase the metric ρ_s. Indeed,

$$\rho_s(\varphi(x_1, y_1), \varphi(x_2, y_2)) =$$

$$= \rho_s((x_1^g, y_1^{h(x_1)}), (x_2^g, y_2^{h(x_2)})) = \begin{cases} d_X(x_1^g, x_2^g), & \text{if } x_1^g \neq x_2^g \\ s(d_Y(y_1^{h(x_1)}, y_2^{h(x_2)})), & \text{if } x_1^g = x_2^g. \end{cases}$$

Since $g \in CtrX$, it follows that $d_X(x_1^g, x_2^g) \leq d_X(x_1, x_2)$. Therefore, if $x_1 \neq x_2$ and $x_1^g \neq x_2^g$ then

$$\rho_s(\varphi(x_1, y_1), \varphi(x_2, y_2)) = d_X(x_1^g, x_2^g) \leq d_X(x_1, x_2) = \rho_s((x_1, y_1), (x_2, y_2)).$$

Using (1) and (2), we get that if $x_1 \neq x_2$ and $x_1^g = x_2^g$ then

$$\rho_s(\varphi(x_1, y_1), \varphi(x_2, y_2)) =$$

$$= s(d_Y(y_1^{h(x_1)}, y_2^{h(x_2)})) \leq r \leq d_X(x_1, x_2) = \rho_s((x_1, y_1), (x_2, y_2)).$$

For $g \in CtrX$ the equality $x_1 = x_2$ implies $x_1^g = x_2^g$. Note that t is a contraction of Y iff t is a contraction of $s(Y)$. Then $h(x_1) = h(x_2)$, that is $h(x_1)$ and $h(x_2)$ define the same contraction t of Y. Hence,

$$s(d_Y(y_1^{h(x_1)}, y_2^{h(x_2)})) =$$

$$= s(d_Y(y_1^{h(x_1)}, y_2^{h(x_1)}) \leq s(d_Y(y_1, y_2)) = \rho_s((x_1, y_1), (x_2, y_2)).$$

Therefore we have

$$\rho_s(\varphi(x_1, y_1), \varphi(x_2, y_2)) \leq \rho_s((x_1, y_1), (x_2, y_2)).$$

This means that φ defines a contraction of $X wr Y$.

Now let us prove that for any contraction φ of $X wr Y$ there exist $g \in Ctr X$ and $h(x) \in Ctr Y^X$ such that $[g, h(x)]$ acts on $X \times Y$ as φ does. Let the function φ map some point (x_1, y_1) to a point (x_2, y_2). Using (1) and (2) we obtain that the function φ maps any point of the form $(x_1, \star)$ to a point of the form $(x_2, \star)$. It follows that φ acts as a contraction on each isometric copy $s(Y)_x$, $x \in X$. In each copy $s(Y)_x$ chooses a point y_x. Then φ is a contraction on $\{y_x,\ x \in X\}$. This implies that there exist $g \in Ctr X$ and $h(x) \in Ctr s(Y)^X$, where $[g, h(x)]$ acts on $X \times Y$ as φ does. Since $Ctr(s(Y)) \simeq Ctr Y$, it follows that we can consider $[g, h(x)]$ as an element of $Ctr X \wr Ctr Y$. This completes the proof of the theorem.

4. Corollary

Now let $(X_1, d_1), (X_2, d_2), \ldots, (X_n, d_n), n \geq 2$ be a finite sequence of metric spaces. Assume that the diameters of the spaces $(X_2, d_2), (X_3, d_3), \ldots, (X_n, d_n)$ are finite. Additionally assume that there exists a finite sequence of positive numbers $r_1,\ r_2, \ldots, r_{n-1}$, such that for arbitrary points $a, b \in X_i$, $a \neq b$, the inequalities $d_i(a, b) \geq r_i$ hold, $1 \leq i \leq n-1$.

Proposition 3 [4]**.** *Let (X_1, d_1), (X_2, d_2) and (X_3, d_3) be metric spaces as above. Then the spaces $(X_1 wr X_2) wr X_3$ and $X_1 wr (X_2 wr X_3)$ are isomorphic.*

Using proposition (3) we introduce the n-iterated wreath product of metric spaces.

First fix a finite sequence of scales $s_i(x)$, $2 \leq i \leq n$ such that

(3) $diam(s_2(X_2)) < r_1,$

$$diam(s_3(X_3)) < s_2(r_2),$$

$$\cdots\cdots\cdots$$

$$diam(s_n(X_n)) < s_{n-1}(r_{n-1}).$$

Define a metric $\rho_{s_2,\ldots,s_n}$ on the cartesian product $X_1 \times X_2 \times \ldots \times X_n$ by the rule:

$$\rho_{s_2,\ldots,s_n}((a_1,\ldots,a_n),(b_1,\ldots,b_n)) = \tag{4}$$

$$= \begin{cases} d_1(a_1,b_1), & \text{if } a_1 \neq b_1; \\ s_1(d_2(a_2,b_2)), & \text{if } a_1 = b_1 \text{ and } a_2 \neq b_2; \\ s_2(d_3(a_3,b_3)), & \text{if } a_1 = b_1,\ a_2 = b_2,\ a_3 \neq b_3; \\ \ldots \quad \ldots \quad \ldots & \\ s_{n-1}(d_n(a_n,b_n)), & \text{if } a_1 = b_1, \ldots, a_{n-1} = b_{n-1}. \end{cases}$$

where $a_1,\ b_1 \in X_1, \ldots, a_n,\ b_n \in X_n$.

We call $(X_1 \times X_2 \times \ldots \times X_n, \rho_{s_2,\ldots,s_n})$ the *n-iterated wreath product* of the spaces $(X_1,d_1),(X_2,d_2),\ldots,(X_n,d_n)$ with respect to the sequence of scales $s_i(x)$, $2 \leq i \leq n$, and denote it by

$$X_1 wr_{s_1} X_2 wr_{s_2} X_3 wr_{s_3} \ldots wr_{s_n} X_n.$$

Let $X_1, X_2, \ldots, X_n$ be metric spaces as above. Fix two finite sequences of scales s_i, $1 \leq i \leq n$ and g_j, $1 \leq j \leq n$ such that the inequalities (3) hold for each of this sequences. Then from Propositions 3 and 2 we obtain

Proposition 4. *The spaces*

$$X_1 wr_{s_1} X_2 wr_{s_2} X_3 wr_{s_3} \ldots wr_{s_n} X_n$$

and

$$X_1 wr_{g_1} X_2 wr_{g_2} X_3 wr_{g_3} \ldots wr_{g_n} X_n$$

are isomorphic.

Since the n-iterated wreath product of metric spaces is unique up to isomorphism we assume in the sequel that the corresponding sequence of scales s_i, $1 \leq i \leq n$ is fixed. Denote the n-iterated wreath product of metric spaces $X_1, X_2, \ldots, X_n$ by

$$X_1 wr X_2 wr X_3 wr \ldots wr X_n.$$

Note, that we can consider the space $X_1 wr X_2 wr X_3 wr \dots wr X_n$ as the space

$$(\dots(((X_1 wr X_2) wr X_3) wr) \dots wr X_n).$$

From Proposition 3 and Theorem 1 it follows:

Theorem 2. *The semigroup of contractions of the n-iterated wreath product of the metric spaces $X_1, X_2, \dots, X_n$ is isomorphic as a transformation semigroup to the n-iterated wreath product of semigroups of contractions of the spaces $X_1, X_2, \dots, X_n$*

$$Ctr(X_1 wr X_2 wr \dots wr X_n) \simeq Ctr X_1 \wr Ctr X_2 \wr \dots \wr Ctr X_n.$$

5. Example

Let $k_1, k_2, \dots, k_n$ be a finite sequence of natural numbers and let (Y_{k_i}, d_i) be a finite sequence of metric spaces such that for any $1 \le i \le n$ the following conditions hold:

- $|Y_{k_i}| = k_i$,
- $d_i(a, b) = 1$ for distinct points $a, b \in Y_{k_i}$.

Fix a real number $\eta \in (0, 1)$. Define a finite sequence of scales $s_i(t) = \eta^{i-1} \cdot t$, $2 \le i \le n$. The functions from this sequence satisfy inequalities (3). Then we can consider the space

$$Y_1 wr_{s_2} Y_2 wr_{s_3} Y_3 wr_{s_4} \dots wr_{s_n} Y_n.$$

This space consists of $k_1 k_2 \dots k_n$ tuples of the form $(u_1, \dots, u_n) \in \prod_{i=1}^{n} Y_{k_i}$. The distance between distinct points of this space is defined by the following rule:

$$d((u_1, \dots, u_n), (v_1, \dots, v_n)) = \eta^l \text{ if } u_1 = v_1, \dots, u_{l-1} = v_{l-1}, u_l \ne v_l.$$

Denote this space by $B(k_1, \dots, k_n)$.

The well-known folklore result says that the isometry group of this space is isomorphic as a permutation group to the wreath product of the symmetric groups $S_{k_1}, S_{k_2}, \dots, S_{k_n}$.

Another way to describe the space $B(k_1, \dots, k_n)$ is as follows.

Recall, that a connected simple (non directed, without loops) graph is *a tree* if it has no cycles. It is easy to see that if a graph T is a tree then for any two vertices of T there exists a unique path connecting them. *A rooted tree* (T, v_0) is a tree with a fixed vertex v_0 named *the root* of the tree. For every nonnegative integer l *the level number* l (l*-th level*) is the set V_l of all vertices $v \in V(T)$ such that the length of the path between v and v_0 in T is equal to l. Respectively, the level number 0 contains only the root v_0. A homogeneous $(k_1, k_2, \dots, k_n)$–tree is a rooted tree such that each vertex of $(i-1)$-th level is connected with exactly k_i vertices of i-th level, $1 \le i \le n$.

Let (T, v_0) be a finite homogeneous $(k_1, k_2, \dots, k_n)$-rooted tree. Then all vertices $v \in V(T)$ have degree 1. We can introduce a natural ultrametric on V_n putting

$$\rho(v_i, v_j) = \eta^{(s+1)}, v_i \neq v_j$$

and $\rho(v_i, v_j) = 0, v_i = v_j$, where s is the length of the maximal common part of the paths connecting vertices v_i and v_j with the root.

- The space (V_n, ρ) is isometric to the space $B(k_1, \dots, k_n)$.

An endomorphism of a homogeneous rooted tree is called elliptic (due to J. Rhodes [7]) if it preserves numbers of levels.

- The semigroup of contractions of (V_n, ρ) is isomorphic to the semigroup of elliptic endomorphisms of rooted tree (T, v_0).

From theorem 2 it follows

- The semigroup of contractions of the space $B(k_1, \dots, k_n)$ is isomorphic as a transformation semigroup to the n-iterated wreath product of transformations semigroups $T_{k_1}, T_{k_2}, \dots, T_{k_n}$:

$$Ctr(B(k_1, \dots, k_n)) \simeq T_{k_1} \wr T_{k_2} \wr \dots \wr T_{k_n}.$$

This result immediately implies from [6].

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