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ON THE LATTICE OF CONGRUENCES ON INVERSE SEMIRINGS

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Abstract

Let S be a semiring whose additive reduct $(S,+)$ is an inverse semigroup. The relations θ and k, induced by tr and ker (*resp.*), are congruences on the lattice $\mathcal{C}(S)$ of all congruences on S. For $\rho \in \mathcal{C}(S)$, we have introduced four congruences $\rho_{\min}$, $\rho_{\max}$, $\rho^{\min}$ and $\rho^{\max}$ on S and showed that $\rho\theta = [\rho_{\min}, \rho_{\max}]$ and $\rho\kappa = [\rho^{\min}, \rho^{\max}]$. Different properties of $\rho\theta$ and $\rho\kappa$ have been considered here. A congruence ρ on S is a Clifford congruence if and only if ρ_{max} is a distributive lattice congruence and ρ^{max} is a skew-ring congruence on S. If η (σ) is the least distributive lattice (resp. skew-ring) congruence on S then $\eta \cap \sigma$ is the least Clifford congruence on S.

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1. Introduction

The class of inverse semigroups is the most natural generalization of the class of groups. A semigroup S is called inverse if for each $a \in S$ there exists unique $x \in S$ such that

$$a = axa \text{ and } x = xax.$$

Whereas if for each $a \in S$ there exists $x \in S$ such that $a = axa$ then S is called regular semigroup. An element $e \in S$ is called an idempotent if $e = e^2$. A regular semigroup S is inverse if and only if $ef = fe$ for all idempotents e, f in S.

In this paper our objective is to study the lattice $\mathcal{C}(S)$ of all congruences on an inverse semiring S. It was recognized by Scheiblich [12] that every congruence ρ on an inverse semigroup is uniquely determined by its restriction to the idempotents, called the trace of ρ and the union of all its classes containing idempotents, called the kernel of ρ. The importance of trace was realized earlier by Reilly and Scheiblich [11]. They defined a congruence θ, induced by tr on the lattice of all congruences on an inverse semigroup and gave expressions for the least element $\rho_{\min}$ and greatest element $\rho_{\max}$ in $\rho\theta$. The congruence θ gives us a first decomposition of the lattice of all congruences that is useful in gaining some overview of the congruences on an inverse semigroup. For example, the θ-class of the equality relation consists of all idempotent separating congruences and the θ-class of the universal relation consists of all group congruences. Different such advantages of this way of looking at the congruences encouraged the researchers to continue their study in this way. Petrich [8] characterized the congruence θ in several ways in terms of congruences and the $\mathcal{H}$-equivalence. There he has drawn several interesting consequences concerning θ-classes and their least and greatest elements. Feigenbaum [1] first extended these results to an orthodox semigroup and later [2] to regular semigroups. Green [3] characterized the k-equivalence classes, where k is the relation on the lattice of all congruences on an inverse semigroup induced by kernel. Petrich and Reilly [10] determined the least element in a k-class and Pastijn and Petrich [7] generalizes these results to regular semigroups.

The unqualified success of these relations θ and k to study the lattice of all congruences on inverse semigroups including its diverse ramifications gave a certain hope that this may also turn out to be the case for the lattice of all congruences on inverse semirings. Sen, Ghosh and Mukhopadhyay [13] studied the congruences on inverse semirings whose additive reduct is commutative and Maity [6] improved this to the inverse semirings whose set of all additive idempotents is a bisemilattice.

The main aspect of this paper is to study the lattice of all congruences on inverse semiring by the congruences induced by trace and kernel, in particular characterization of $\rho_{\min}$, $\rho_{\max}$, $\rho^{\min}$ and $\rho^{\max}$. Such details are considered in Section 3 and Section 4.

In the last section we have considered the Clifford congruences on inverse semiring. A congruence ρ on an inverse semiring S is called a Clifford congruence if S/ρ is a distributive lattice of skew-rings. A congruence ρ on S is a Clifford congruence if and only if ρ_{max} is a distributive lattice congruence and ρ^{max} is a skew-ring congruence on S. If η is the least distributive lattice congruence and σ is the least skew-ring congruence on S then $\eta \cap \sigma$ is the least Clifford congruence on S.

2. Preliminaries

A *semiring* $(S, +, \cdot)$ is an algebra with two binary operations $+$ and $\cdot$ such that both the reducts $(S, +)$ and $(S, \cdot)$ are semigroups and in which the two distributive laws

$$x(y+z) = xy + xz \quad \text{and} \quad (y+z)x = yz + zx$$

are satisfied. Let S be a semiring. $a \in S$ is called an additive idempotent if $a + a = a$. We denote the set of all additive idempotents of a semiring S by E^+ or sometimes by $E^+(S)$. A subset $I \neq \emptyset$ of a semiring S is called a left [right] ideal of S if $a + b, sa[as] \in I$ for all $a, b \in I$ and $s \in S$. I is said to be an ideal of S if it is both a left and a right ideal of S. An ideal K of S is called a *k-ideal* if for $x \in S,\ x + k,\ k \in K$ implies that $x \in K$.

A semiring S is called an *inverse semiring* if for each $a \in S$ there exists a unique element $a' \in S$ such that $a = a + a' + a$ and $a' = a' + a + a'$. Following M. P. Grillet [4], we call a semiring S a *skew-ring* if its additive reduct $(S, +)$ is a group. A semiring S is called an *additive idempotent semiring* if the additive reduct is a semilattice. If moreover the multiplicative reduct $(S, \cdot)$ is a band then the semiring S is called a *b-lattice*.

Let S be an inverse semiring. Then the set of all congruences on S is a lattice which we will denote by $\mathcal{C}(S)$. For $\rho \in \mathcal{C}(S)$, we define the *trace* and *kernel* of ρ, respectively, by:

$$tr\rho = \rho \cap (E^+ \times E^+)$$

$$\text{and } \; ker\rho = \{x \in S \mid x\rho e, \; e \in E^+\}.$$

A semiring S is called a distributive lattice (b-lattice) D of skew-rings S_α if there exists a distributive lattice (b-lattice) congruence ρ on S such that $D = S/\rho$ and each ρ-class S_α; $\alpha \in D$ is a skew-ring.

Definition 2.1 [14] **.** A semiring S is called a completely regular semiring if for each $a \in S$ there exists $x \in S$ such that

(i) $a = a + x + a$ and $a + x = x + a$

(ii) $a(a + x) = a + x$.

Theorem 2.2 [14] **.** *A semiring S is completely regular if and only if it is a union of skew-rings.*

Theorem 2.3 [14] **.** *Let S be a completely regular semiring. Then $e^2 = e$ for all $e \in E^+$.*

Definition 2.4 [15] **.** An inverse semiring S is called a Clifford semiring if for all $a, b \in S$,

(2.1) $$a + a' = a' + a$$

(2.2) $$a(a + a') = a + a'$$

(2.3) $$a(b + b') = (b + b')a$$

(2.4) $$a + a(b + b') = a$$

(2.5) $$a + b = b \quad \text{implies that} \quad a + a = a.$$

Theorem 2.5 [15] **.** *A semiring S is a Clifford semiring if and only if it is a distributive lattice of skew-rings.*

Definition 2.6 [15]**.** An inversive semiring S is a generalized Clifford semiring if it satisfies the conditions (2.1), (2.2) and (2.5) for all $a, b \in S$.

Theorem 2.7 [15] **.** *A semiring S is a generalized Clifford semiring if and only if S is a b-lattice of skew-rings.*

Let S be a semiring. A congruence ρ on S is called a *Clifford (generalized) congruence* if S/ρ is a Clifford (generalized) semiring. For a class $\mathcal{F}$ of semirings, we define $\mathcal{F}$-congruences on S similarly.

Let S be an inversive semiring. We denote the Green's relations on $(S,+)$ by $\mathcal{L}^+, \mathcal{R}^+, \mathcal{H}^+$. Recall that, for $a, b \in S$,

$$a\mathcal{L}^+b \quad \text{if and only if} \quad a' + a = b' + b$$

$$a\mathcal{R}^+b \quad \text{if and only if} \quad a + a' = b + b'$$

$$a\mathcal{H}^+b \quad \text{if and only if} \quad a' + a = b' + b \quad \text{and} \quad a + a' = b + b'.$$

Again we refer [5] and [9] for the informations we need concerning inverse semigroups.

3. The congruence θ on $\mathcal{C}(S)$

In [11], Reilly and Scheiblich defined a relation θ on the lattice of congruences on inverse semigroups by: $\rho\theta\xi$ if and only if ρ, ξ induce the same partition of the idempotents of S. There they proved that θ is a complete congruence and each θ-class is a complete modular lattice.

Let S be an inverse semiring. Similarly we define a relation θ on $\mathcal{C}(S)$ by: for $\rho, \xi \in \mathcal{C}(S)$,

$$\rho\theta\xi \quad \text{if and only if} \quad tr\rho = tr\xi.$$

Let S be an inverse semiring and $\rho \in \mathcal{C}(S)$. We define a relation ρ_{max} on S by: for $a, b \in S$,

$$a\rho_{max}b \quad \text{if and only if} \quad (ra's + e + ras)\rho(rb's + e + rbs)$$
$$\text{for all} \quad e \in E^+ \quad \text{and for all} \quad r, s \in S^1.$$

Lemma 3.1. *Let S be an inverse semiring and ρ be a congruence on S. Then ρ_{max} is the greatest congruence on S such that $tr\rho = tr\rho_{max}$.*

Proof. Let $a, b \in S$ such that $a\rho_{max}b$. Then $(ra's+e+ras)\rho(rb's+e+rbs)$ for all $e \in E^+, r, s \in S^1$. Let $c \in S$. Then $rc's + e + rcs \in E^+$ for all $e \in E^+, r, s \in S^1$. So $(ra's+rc'a+e+rcs+ras)\rho(rb's+rc's+e+rcs+rbs)$

for all $e \in E^+, r, s \in S^1$. This implies that $(r(c+a)'s + e + r(c+a)s)$ $\rho(r(c+b)'s + e + r(c+b)s)$ so that $(c+a)\rho_{max}(c+b)$. Also for any $c \in S, a\rho_{max}b$ implies that $((rc)a's + e + (rc)as))\rho((rc)b's + e + (rc)bs)$ for all $e \in E^+, r, s \in S^1$. This implies that $(r(ca)'s+e+r(ca)s)\rho(r(cb)'s+e+r(cb)s)$ for all $e \in E^+, r, s \in S^1$. So $ca\rho_{max}cb$. Similarly $(a+c)\rho_{max}(b+c)$ and $ac\rho_{max}bc$. Therefore ρ_{max} is a congruence on S.

Let $\xi \in \mathcal{C}(S)$ be such that $tr\rho = tr\xi$. Then for $a, b \in S, a\xi b$ implies that $(ra's+e+ras)\xi(rb's+e+rbs)$. This implies that $(ra's+e+ras)\rho(rb's+e+rbs)$ so that $a\rho_{max}b$. Hence $\xi \subseteq \rho_{max}$, in particular $\rho \subseteq \rho_{max}$. This implies that $tr\rho \subseteq tr\rho_{max}$. Now for any $e, f \in E^+, e\rho_{max}f$ implies that $(e+g+e)$ $\rho(f+g+f)$ for all $g \in E^+$ that is $(e+g)\rho(f+g)$ for all $g \in E^+$. This implies that $e = (e+e)\rho(f+e) = (e+f)\rho(f+f) = f$. Thus $tr\rho_{max} \subseteq tr\rho$. Hence $tr\rho = tr\rho_{max}$. Thus ρ_{max} is the greatest congruence on S with the same trace. ■

As in the inverse semigroup, for $\rho \in \mathcal{C}(S)$, we define another relation ρ_{min} on S by: for $a, b \in S$,

$$a\rho_{min}b \text{ if and only if } a+e = b+e,\ e\rho(a'+a)\rho(b'+b) \quad \text{for some } e \in E^+.$$

Then it can be checked that ρ_{min} is the least congruence on S with the same trace.

From Lemma 3.1, we can prove the following theorem similarly to Theorem III.2.5 [9].

Theorem 3.2. *Let S be an inversive semiring. Then*

(i) *θ is a complete congruence on $\mathcal{C}(S)$,*

(ii) *for any $\rho \in \mathcal{C}(S), \rho\theta = [\rho_{min}, \rho_{max}]$,*

(iii) *$\rho\theta$ is a complete modular sublattice of $\mathcal{C}(S)$.*

Theorem 3.3. *Let S be an inverse completely regular semiring and $\rho, \xi \in \mathcal{C}(S)$. Then the following conditions are equivalent:*

(i) *$\rho\theta\xi$,*

(ii) *$(\rho \cap \xi)|_{e\rho}$ and $(\rho \cap \xi)|_{e\xi}$ are skew-ring congruences for all $e \in E^+$.*

Proof. (i) $\Rightarrow$ (ii): Let $e \in E^+$. Then $e^2 = e$, by Theorem 2.3. Hence $e\rho$ is a subsemiring of S. Let $f, g \in e\rho \cap E^+$. Then $f\rho g$ and so $f\xi g$. Therefore $f(\rho \cap \xi)g$. Hence $(\rho \cap \xi)|_{e\rho}$ is a skew-ring congruence. Similarly $(\rho \cap \xi)|_{e\xi}$ is a skew-ring congruence.

(ii) $\Rightarrow$ (i): Follows easily. ■

Corollary 3.4. *In a Clifford or generalized Clifford semiring S the follwing conditions are equivalent for $\rho, \xi \in \mathcal{C}(S)$:*

(i) $\rho\theta\xi$,

(ii) $(\rho \cap \xi)|_{e\rho}$ *and* $(\rho \cap \xi)|_{e\xi}$ *are skew-ring congruences for all* $e \in E^+$.

Lemma 3.5. *Let S be an inverse semiring and $\rho \in \mathcal{C}(S)$. Then*

(i) $ker\rho_{min} = \{a \in S : a + e = e \text{ for some } e \in E^+, e\rho(a' + a)\}$,

(ii) $ker\rho_{max} = \{a \in S : (ras + e)\rho(e + ras) \text{ for all } e \in E^+, r, s \in S^1\}$.

Proof. (i) It is similar to Proposition 5.6 [8].

(ii) For $a \in S$,

$$
\begin{aligned}
& a \in ker\rho_{max} \\
\Leftrightarrow\ & a\rho_{max}(a' + a) \\
\Leftrightarrow\ & (ra's + e + ras)\rho(r(a' + a)'s + e + r(a' + a)s) \text{for all } r, s \in S^1, e \in E^+ \\
\Leftrightarrow\ & (ra's + e + ras)\rho(r(a' + a)s + e + r(a' + a)s) \text{for all } r, s \in S^1, e \in E^+ \\
\Leftrightarrow\ & (ra's + e + ras)\rho(r(a' + a)s + e) \text{for all } r, s \in S^1, e \in E^+ \\
\Leftrightarrow\ & (ras + ra's + e + ras)\rho(ras + r(a' + a)s + e) \text{for all} r, s \in S^1, e \in E^+ \\
\Leftrightarrow\ & (e + ras + ra's + ras)\rho(ras + e) \text{for all } r, s \in S^1, e \in E^+ \\
\Leftrightarrow\ & (e + ras)\rho(ras + e) \text{for all } r, s \in S^1, e \in E^+.
\end{aligned}
$$
■

A congruence ρ on an inverse semiring S is called idempotent separating if for $e, f \in E^+$,

$$e\rho f \text{ implies that } e = f.$$

It is clear that ε, the equality relation on S is the minimum idempotent separating congruence on S. Theorem 3.2 implies that ε_{max} is the maximum idempotent separating congruence on S. We denote it by μ or sometimes by μ_S. Therefore, for $a, b \in S$,

$$a\mu b \Leftrightarrow ra's + e + ras = rb's + e + rbs \text{ for all } e \in E^+, r, s \in S^1.$$

Theorem 3.6. *Let S be a generalized Clifford semiring. Then $\rho_{max} = \rho \vee \mu$ for every congruence ρ on S.*

Proof. Let ρ be a congruence on S and $a, b \in S$ such that $a\mu b$. Then $ra's + e + ras = rb's + e + rbs$ for all $r, s \in S^1, e \in E^+$. So $(ra's + e + ras)\rho(rb's + e + rbs)$ for all $r, s \in S^1, e \in E^+$. Hence $a\rho_{max}b$. Also $\rho \subseteq \rho_{max}$. This implies that $\rho \vee \mu \subseteq \rho_{max}$. Now $tr\rho_{max} = tr\rho \subseteq tr(\rho \vee \mu)$. Since S is a generalized Clifford, $ker\rho_{max} = S = ker\varepsilon_{max} = ker\mu \subseteq ker(\rho \vee \mu)$, by Lemma 3.5. Therefore $\rho_{max} \subseteq \rho \vee \mu$. Hence $\rho_{max} = \rho \vee \mu$. ■

4. The congruence κ on $\mathcal{C}(S)$

As in the lattice of congruences on inverse semigroups, we define another relation κ on $\mathcal{C}(S)$ by:

$$\rho\kappa\xi \quad \text{if and only if} \quad ker\rho = ker\xi.$$

A congruence ρ on an inverse semiring S is said to saturate a non-empty subset H of S if H is a union of some ρ-classes. In [13], Sen, Ghosh and Mukhopadhyay determined the greatest congruence τ^H on S, which saturates a given nonempty subset H as follows:

$$a\tau^H b \quad \text{if and only if} \quad x + ras + y \in H \Leftrightarrow x + rbs + y \in H$$
$$\text{for all } x, y \in S^o, r, s \in S^1.$$

For any $\rho \in \mathcal{C}(S)$, we define two relations on S by:

$$\rho^{max} = \tau^{ker\rho}$$

$$\text{and } \rho^{min} = (\rho \cap \mathcal{L}^+)^*.$$

Theorem 4.1. *Let S be an inverse semiring. Then*

(i) κ *is a* $\cap$*-complete congruence on* $\mathcal{C}(S)$,

(ii) $\rho\kappa = [\rho^{min}, \rho^{max}]$ *for any* $\rho \in \mathcal{C}(S)$,

(iii) $\rho\kappa$ *is a complete sublattice of* $\mathcal{C}(S)$ *for all* $\rho \in \mathcal{C}(S)$.

Proof. (i) This is similar to Theorem III.4.8 [9].

(ii) Let $\xi \in \rho\kappa$. Then $ker\rho = ker\xi$. Since $\tau^{ker\xi}$ is the largest congruence saturating $ker\xi$, so $\xi \subseteq \tau^{ker\xi} = \tau^{ker\rho} = \rho^{max}$. Also $ker\rho^{max} = ker\tau^{ker\rho} = ker\rho$. Hence ρ^{max} is the largest element in $\rho\kappa$.

Let $\xi \in \rho\kappa$. Then $ker\rho = ker\xi$. Now $a(\rho \cap \mathcal{L}^+)b$ implies that $a\rho b$ and $a' + a = b' + b$. Then $(a + b')\rho(b + b')$ implies that $a + b' \in ker\rho = ker\xi$, so that $(a+b')\xi((a+b')'+(a+b')) = (b+a')+(a+b')$. Hence $a = a+(a'+a) = a + (b' + b) = ((a + b') + b)\xi(b + (a' + a) + (b' + b)) = b$. Thus $\rho \cap \mathcal{L}^+ \subseteq \xi$, which gives $ker(\rho \cap \mathcal{L}^+)^* \subseteq ker\xi$. Again $a \in ker\xi = ker\rho$ implies that $a\rho(a' + a)$. Then $a(\rho \cap \mathcal{L}^+)(a' + a)$ and so $ker\xi \subseteq ker(\rho \cap \mathcal{L}^+)^*$. Thus $ker\xi = ker(\rho \cap \mathcal{L}^+)^*$. Again $\rho \cap \mathcal{L}^+ \subseteq \xi$ implies that $(\rho \cap \mathcal{L}^+)^* \subseteq \xi$ that is $\rho^{min} \subseteq \xi$. Thus ρ^{min} is the least element of $\rho\kappa$.

(iii) Since any interval in a complete lattice is a complete sublattice, so $\rho\kappa$ is a complete sublattice of $\mathcal{C}(S)$. ∎

Now we give an alternative presentation for ρ^{min}. For this we state the followong Lemma without proof. In fact it can be proved easily as in semigroup.

Lemma 4.2. *Let θ be an equivalence relation on a semiring S. Then the congruence θ^* on S generated by θ is gien by:*

$$(a, b) \in \theta^* \quad \text{if and only if} \quad \exists x, y \in S^o, r, s \in S^1 \quad \text{and} \quad (c, d) \in \theta$$

$$\text{such that} \quad a = x + rcs + y \quad \text{and} \quad b = x + rds + y.$$

Now the following theorem is straightforward.

Theorem 4.3. *Let S be an inverse semiring and ρ be a congruence on S. Then*

$$(a,b) \in \rho^{min} \text{ if and only if } \exists x, y \in S^o, r, s \in S^1 \text{ and } (c,d) \in \rho$$
$$\text{such that } a = x + rcs + y, b = x + rds + y \text{ and } c' + c = d' + d.$$

From Theorem 3.2 and Theorem 4.1 the following theorem follows immediately.

Theorem 4.4. *Let S be an inverse semiring. Then for any congruence ρ on S,*

$$\rho = \rho_{min} \vee \rho^{min} = \rho_{max} \cap \rho^{max}.$$

The following theorem shows that each idempotent separating congruence on an inverse semiring is the least element of its kernel class.

Theorem 4.5. *Let S an inverse semiring and ρ be an idempotent separating congruence on S. Then $\rho = \rho^{min}$. In particular $\mu = \mu^{min}$.*

Proof. Let $a, b \in S$ such that $a\rho b$. Then $(a' + a)\rho(b' + b)$ implies that $a' + a = b' + b$. Now $a\mathcal{L}^+(a' + a) = (b' + b)\mathcal{L}^+ b$ implies that $a(\rho \cap \mathcal{L}^+)b$. Hence $\rho \subseteq \rho^{min}$ and so $\rho = \rho^{min}$. ■

Theorem 4.6. *Let S be an inverse completely regular semiring and $\rho, \xi \in \mathcal{C}(S)$. Then the following conditions are equivalent:*

(i) $\rho\kappa\xi$.

(ii) $(\rho \cap \xi)|_{e\rho}$ *and* $(\rho \cap \xi)|_{e\xi}$ *are additive idempotent semiring congruences for all* $e \in E^+$.

Proof. (i) $\Rightarrow$ (ii): Let $\rho \in \mathcal{C}(S)$ and $e \in E^+$. Since S is completely regular, by Theorem 2.3 it follows that $e^2 = e$ and so $e\rho$ is a subsemiring of S. Let $a, b \in e\rho$. Then $(a + a)\rho a$ and $(a + b)\rho e\rho(b + a)$. Again $ker\rho = ker\xi$ implies that $a, b \in ker\xi$ and so $(a + a)\xi a$ and $(a + b)\xi(b + a)$. Thus $(\rho \cap \xi)|_{e\rho}$ is an additive idempotent semiring congruence.

(ii) $\Rightarrow$ (i): Let $a \in ker\rho$. Then there is $e \in E^+$ such that $a \in e\rho$. Hence $a(\rho \cap \xi)f$ for some $f \in E^+$. Then $a\xi f$, that is $a \in ker\xi$. So $ker\rho \subseteq ker\xi$. Similarly $ker\xi \subseteq ker\rho$. Therefore $ker\rho = ker\xi$, which implies that $\rho\kappa\xi$. ■

Theorem 4.7. *Let S be a semiring which is a distributive lattice of skew-rings and $\rho, \xi \in \mathcal{C}(S)$. Then the following conditions are equivalent:*

(i) $\rho\kappa\xi$.

(ii) $(\rho \cap \xi)|_{e\rho}$ *and* $(\rho \cap \xi)|_{e\xi}$ *are distributive lattice congruences for all* $e \in E^+$.

Proof. (i) $\Rightarrow$ (ii): From above Theorem, it follows that $(\rho \cap \xi)|_{e\rho}$ is an additive idempotent semiring congruence. So $a(\rho \cap \xi)(a + a')$ for all $a \in e\rho$. Since S is a distributive lattice of skew-rings, so by Theorem 2.5, it satisfies (1.1.1), (1.1.2), (1.1.3) and (1.1.4). Then for $a, b \in e\rho$ we get $a^2(\rho \cap \xi)a(a + a') = (a + a')(\rho \cap \xi)a$ and $ab(\rho \cap \xi)(ab + ab') = a(b + b') = (b + b')a = (ba + b'a)(\rho \cap \xi)ba$. Also $(a + ab)(\rho \cap \xi)(a + ab + ab' + a') = a + a(b + b') + a' = (a + a')(\rho \cap \xi)a$. Hence $(\rho \cap \xi)|_{e\rho}$ is a distributive lattice congruence.

(ii) $\Rightarrow$ (i): Follows from Theorem 4.6. ∎

Corollary 4.8. *Let S be a semiring which is a b-lattice of skew-rings and $\rho, \xi \in \mathcal{C}(S)$. Then the following conditions are equivalent:*

(i) $\rho\kappa\xi$.

(ii) $(\rho \cap \xi)|_{e\rho}$ *and* $(\rho \cap \xi)|_{e\xi}$ *are b-lattice congruences for all* $e \in E^+$.

5. Clifford congruences

Let S be an inverse semiring. A congruence ρ on an inverse semiring S is idempoetnt pure if for all $a \in S, e \in E^+, a\rho e$ implies that $a \in E^+$. So ρ is idempotent pure if and only if $ker\rho = E^+$. Hence ε is the least idempotent pure congruence and ε^{max} is the greatest idempotent pure congruence on S, which will be denoted by τ or sometimes by τ_S. Therefore, for $a, b \in S$,

$$a\tau b \quad \text{if and only if} \quad x + ras + y \in E^+ \Leftrightarrow x + rbs + y \in E^+$$
$$\text{for all} \quad x, y \in S^o, r, s \in S^1.$$

We will denote the least distributive lattice congruence on S by η and the least b-lattice congruence on S by ν.

Theorem 5.1. *Let S be an inverse semiring and ρ be a congruence on S. Then the following conditions are equivalent.*

(i) *ρ is a generalized Clifford congruence.*

(ii) *ρ_{max} is a b-lattice congruence and ρ^{max} is a skew-ring congruence on S.*

(iii) *$\rho_{max} = \rho \vee \nu$ and $\rho^{max} = \rho \vee \sigma$.*

(iv) *$tr\rho = tr(\rho \vee \nu)$ and $ker\rho = ker(\rho \vee \sigma)$.*

Proof. (i) $\Rightarrow$ (ii): Since ρ is a generalized Clifford congruence on S, so $(S/\rho, +)$ is a Clifford semigroup, that is $(a+e)\rho(e+a)$ for all $a \in S, e \in E^+$. Let $a, b \in S$. Then for all $r, s \in S^1, e \in E^+$,

$$\begin{aligned} r(a+b)'s + e + r(a+b)s =&(rb's + (ra's + e + ras) + rbs) \\ &\rho(ra's + e + ras + rb's + rbs) \\ &\rho(ra's + rb's + rbs + e + ras) \\ &\rho(ra's + rb's + e + rbs + ras) \\ &= r(b+a)'s + e + r(b+a)s \end{aligned}$$

shows that $(a+b)\rho_{max}(b+a)$. Similarly $(a+a)\rho_{max}a$ for all $a \in S$. Now $r(a^2)'s + e + ra^2s = (raa's + e + raas)\rho(ra(a'+a)s + e)\rho(r(a'+a)s + e)\rho(ra's + e + ras)$ for all $r, s \in S^1, e \in E^+$ implies that $a^2\rho_{max}a$ for all $a \in S$. Therefore ρ_{max} is a b-lattice congruence on S.

It can be proved that ρ^{max} is a skew-ring congruence, similarly to Theorem 4.6 [6].

(ii) $\Rightarrow$ (iii): By our hypothesis, $\nu \subseteq \rho_{max}$. Also $\rho \subseteq \rho_{max}$. Hence $\nu \vee \rho \subseteq \rho_{max}$. Again $ker\rho_{max} \subseteq S = ker\nu = ker(\nu \vee \rho)$ and $tr\rho_{max} = tr\rho \subseteq tr(\nu \vee \rho)$ implies that $\rho_{max} \subseteq \nu \vee \rho$. Thus $\rho_{max} = \nu \vee \rho$.

For ρ^{max}, again we refer Theorem 4.6 [6].

(iii) $\Rightarrow$ (iv): Trivial.

(iv) $\Rightarrow$ (i): Let $a \in S$. Then $(a+a')\nu a\nu(a'+a)$ and so $(a+a')\nu(a'+a)$. Then $tr\rho = tr(\rho \vee \nu)$ implies that $(a+a')\rho(a'+a)$. Also $a^2\nu a$ implies that $(a^2 + (a^2)')\nu(a+a')$, that is $a(a+a')\nu(a+a')$, which implies that $a(a+a')\rho(a+a')$. Again let $a, b \in S$ such that $a\rho + b\rho = b\rho$.

Then $(a+b)\rho b$ implies that $(a+b)+b' \in ker\rho = ker(\rho \vee \sigma)$. Since σ and so $\rho \vee \sigma$ are skew-ring congruences and $b+b' \in E^+$, so $a \in ker(\rho \vee \sigma) = ker\rho$. Hence $a\rho + a\rho = a\rho$. Thus ρ is a generalized Clifford congruence on S. ■

Corollary 5.2. *Let S be a semiring. Then the following conditions are equivalent:*

(i) *S is a generalized Clifford semiring.*

(ii) *For every $\rho \in \mathcal{C}(S), \rho_{max} = \rho \vee \nu$ and $\rho^{max} = \rho \vee \sigma$.*

(iii) *$\mu = \nu$ and $\tau = \sigma$.*

Note that Theorem 3.6 is a direct consequence of this corollary.

Theorem 5.3. *Let S be an inverse semiring. Then $\sigma \cap \nu$ is the least generalized Clifford congruence on S.*

Proof. Let $\lambda = \sigma \cap \nu$. Then it can be proved that $(a+a')\nu(a'+a)$ and $a(a+a')\nu(a+a')$ for all $a \in S$, similarly to the Theorem 5.1. Also σ being skew-ring congruence $e\sigma f$ for all $e, f \in E^+$. So $(a+a')\lambda(a'+a)$ and $a(a+a')\lambda(a+a')$. Let $a, b \in S$ be such that $a\lambda + b\lambda = b\lambda$. Then $a\sigma + b\sigma = b\sigma$, which implies that $a\sigma + a\sigma = a\sigma$. Also $(a+a)\nu a$. Hence $(a+a)\lambda a$. Therefore λ is a generalized Clifford congruence. Let ρ be a generalized Clifford congruence. Then $\nu \subseteq \rho_{max}$ and $\sigma \subseteq \rho^{max}$. So $\sigma \cap \nu \subseteq \rho_{max} \cap \rho^{max} = \rho$, by Theorem 4.4. This implies that $\sigma \cap \nu$ is the least generalized Clifford congruence on S. ■

Theorem 5.4. *Let S be an inverse semiring and ρ be a congruence on S. Then the following conditions are equivalent:*

(i) *ρ is a Clifford congruence on S.*

(ii) *ρ_{max} is a distributive lattice congruence on S and ρ^{max} is a skew-ring congruence on S.*

(iii) *$\rho_{max} = \rho \vee \eta$ and $\rho^{max} = \rho \vee \sigma$.*

(iv) *$tr\rho = tr(\rho \vee \eta)$ and $ker\rho = ker(\rho \vee \sigma)$.*

Proof. (i) $\Rightarrow$ (ii): Every Clifford semiring is a generalized Clifford semiring. So the Theorem 5.1 implies that $a\rho_{max}(a+a), (a+b)\rho_{max}(b+a), a\rho_{max}a^2$ for all $a, b \in S$. Let $a, b \in S$. Then for $r, s \in S^1, e \in E^+$, $r(ab)'s+e+r(ab)s = (rab's+e+rabs)\rho(ra(b'+b)s+e)\rho(r(b'+b)as+e)\rho(rb'as+e+rbas) = r(ba)'+e+r(ba)s$, since ρ is a Clifford congruence. Thus $ab\rho_{max}ba$. Now $r(a+ab)'s+e+r(a+ab)s = (rab's+ra's+e+ras+rabs)\rho(ra's+e+ras+rab's+rabs)\rho(ra's+e+r(a+a(b'+b))s)\rho(ra's+e+ras)$ implies that $(a+ab)\rho_{max}a$. Also $a(a+b) = (a^2+ab)\rho_{max}(a+ab)\rho_{max}a$. Hence ρ_{max} is a distributive lattice congruence.

Similarly to Theorem 5.1, it can be proved that ρ^{max} is a skew-ring congruence.

(ii) $\Rightarrow$ (iii): Similar to Theorem 5.1.

(iii) $\Rightarrow$ (iv): Trivial.

(iv) $\Rightarrow$(i): Let $\rho \in \mathcal{C}(S)$ has the given trace and kenel. Then ρ is a generalized Clifford congruence, which can be proved similarly to the Theorem 5.1. Now for all $a, b \in S$, $ab\eta ba$, implies that

$$\begin{aligned}
&(ab+(ab)')\eta(ba+(ba)')\\
\Rightarrow\ &(ab+ab')\rho\vee\eta(ba+b'a)\\
\Rightarrow\ &a(b+b')\rho\vee\eta(b+b')a\\
\Rightarrow\ &a(b+b')\rho(b+b')a, \text{ since } tr\rho = tr\rho\vee\eta.
\end{aligned}$$

Again for all $a, b \in S$, $(a+ab)\eta a$ implies that

$$\begin{aligned}
&(a+ab)\rho\vee\eta a\\
\Rightarrow\ &(a+ab+(a+ab)')\rho\vee\eta(a+a')\\
\Rightarrow\ &(a+ab+ab'+a')\rho\vee\eta(a+a')\\
\Rightarrow\ &(a+a(b+b')+a')\rho(a+a'), \text{ since } tr\rho = tr\rho\vee\eta\\
\Rightarrow\ &(a+a'+a)\rho(a+a(b+b')+a'+a) = a+a'+a+a(b+b')\\
\Rightarrow\ &a\rho(a+a(b+b')).
\end{aligned}$$

Thus ρ is a Clifford congruence. ■

Corollary 5.5. *On a semiring S the following conditions are equivalent:*

(i) *S is a Clifford semiring.*

(ii) *For every $\rho \in \mathcal{C}(S), \rho_{max} = \rho \vee \eta$ and $\rho^{max} = \rho \vee \sigma$.*

(iii) *$\mu = \eta$ and $\tau = \sigma$.*

The following theorem follows from Theorem 5.4 similarly to the Theorem 5.3.

Theorem 5.6. *Let S be an inverse semiring. Then $\sigma \cap \eta$ is the least Clifford congruence on S.*

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