

COMMUTATIVE DIRECTOIDS WITH SECTIONALLY ANTITONE BIJECTIONS*

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Abstract

We study commutative directoids with a greatest element, which can be equipped with antitone bijections in every principal filter. These can be axiomatized as algebras with two binary operations satisfying four identities. A minimal subvariety of this variety is described.

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Join-semilattices whose principal filters are Boolean lattices were used by J.C. Abbott [1] for a characterization of the logic connective implication in the classical propositional logic. These semilattices also have the property that on each principal filter of them an antitone involution is defined.

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Motivated by this observation, the notion of a *join semilattice with sectionally antitone involutions* in [2] and *join semilattice with sectionally antitone bijections* in [5] was defined. In this paper we introduce a further generalization of this concept defining the notion of a *commutative directoid with sectionally antitone bijections*. By means of these directoids we obtain “nice” algebraic structures, forming a variety with “nice” congruence properties.

The concept of directoid was introduced by J. Ježek and R. Quackenbush [8] and independently by V.M. Kopytov and Z.I. Dimitrov [9] in order to axiomatize algebraic structures defined on upward directed ordered sets. In a certain sense directoids generalize semilattices. For the reader’s convenience, we repeat definitions and basic properties of these concepts.

An ordered set $(A; \leq)$ is *upward directed* if $U(x, y) \neq \emptyset$ for every $x, y \in A$, where $U(x, y) = \{a \in A; x \leq a \text{ and } y \leq a\}$. Elements of $U(x, y)$ are referred to as common upper bounds of x, y . Of course, if $(A; \leq)$ has a greatest element then it is upward directed.

Let $(A; \leq)$ be an upward directed set and \sqcup denotes a binary operation on A . The pair $\mathcal{A} = (A; \sqcup)$ is called a \sqcup -*directoid* if

- (i) $x \sqcup y \in U(x, y)$ for all $x, y \in A$;
- (ii) if $x \leq y$ then $x \sqcup y = y$ and $y \sqcup x = y$.

The following axiomatization of directoids was established in [8]:

Proposition 1. *A groupoid $\mathcal{A} = (A; \sqcup)$ is a \sqcup -directoid if and only if it satisfies the following identities*

- (D1) $x \sqcup x = x$;
- (D2) $(x \sqcup y) \sqcup x = x \sqcup y$;
- (D3) $y \sqcup (x \sqcup y) = x \sqcup y$;
- (D4) $x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$.

A binary relation \leq defined on A by the rule

$$x \leq y \quad \text{if and only if} \quad x \sqcup y = y$$

is an order and $x \sqcup y \in U(x, y)$ for each $x, y \in A$.

A binary relation \leq will be called the *induced order* of $(A; \sqcup)$. A directoid $\mathcal{A} = (A; \sqcup)$ is called *commutative* if it satisfies the identity

$$(D5) \quad x \sqcup y = y \sqcup x.$$

It was shown in [8] that commutative directoids are axiomatized by the identities (D1), (D4) and (D5).

Let us denote a greatest element of an ordered set $(P; \leq)$ by 1. For each $a \in P$ the interval $[a, 1]$ will be called a *section*.

Definition 1. Let $(P; \leq)$ be a poset with a greatest element 1. $(P; \leq)$ is called a *poset with sectionally antitone bijections*, if for each element $a \in S$ there exists a bijection f_a of the interval $[a, 1] \subseteq P$ into itself such that

$$x \leq y \quad \Leftrightarrow \quad f_a(y) \leq f_a(x), \text{ for all } x, y \in [a, 1].$$

Of course, the inverse f_a^{-1} of f_a is also an antitone bijection on $[a, 1]$. If each f_a is an *involution*, i.e., $f_a^2(x) = x$, for all $x \in [a, 1]$, then $f_a^{-1} = f_a$ and $(P; \leq)$ is called a *poset with sectionally antitone involutions*.

Remark 1. Any poset $(P; \leq)$ with a greatest element possesses a commutative \sqcup -directoid $(P; \sqcup)$.

Indeed, take any choice function $\varphi : 2^P \rightarrow P$, such that for any nonempty subset $A \subseteq P$, $\varphi(A) \in A$. Then define a binary operation \sqcup on P as follows

$$x \sqcup y = \begin{cases} \varphi(U(x, y)), & \text{if } x \text{ and } y \text{ are incomparable} \\ y & \text{if } x \leq y \\ x & \text{if } y \leq x. \end{cases}$$

It is evident that this operation satisfies the axioms (D1)–(D5) and that

$$x \leq y \quad \Leftrightarrow \quad x \sqcup y = y.$$

Given a commutative \sqcup -directoid S with sectionally antitone bijections, we can introduce two new binary operations on S as follows:

$$(P) \quad x \circ y = f_y(x \sqcup y), \quad x * y = f_y^{-1}(x \sqcup y).$$

Since $x \sqcup y \in [y, 1]$, \circ and $*$ are everywhere defined operations on the set S . Conversely, one can check immediately that for any $a \in S$ and $x \in [a, 1]$

$$(A) \quad f_a(x) = x \circ a, \quad f_a^{-1}(x) = x * a.$$

Clearly, if all the mappings f_a are involutions, then $x \circ y = x * y$, for all $x, y \in S$.

Lemma 1. *Let \mathcal{S} be a commutative \sqcup -directoid with sectionally antitone bijections and $\circ, *$ be operations defined by (P). Then*

$$a \leq b \Leftrightarrow a \circ b = 1 \Leftrightarrow a * b = 1.$$

Proof. Suppose $a, b \in S$ and $a \leq b$. Since f_b is an antitone bijection on $[b, 1]$, we have $f_b(b) = 1$. Then

$$(Q) \quad a \circ b = f_b(a \sqcup b) = f_b(b) = 1, \quad a * b = f_b^{-1}(a \sqcup b) = f_b^{-1}(1) = 1.$$

To prove the converse implication we note that for all $x \in [b, 1]$ it is $b \leq x$ and thus $f_b(b) \geq f_b(x)$. Let $a \circ b = 1$. Then $f_b(a \sqcup b) = 1 = f_b(b)$, but f_b is a bijection, hence $a \sqcup b = b$, i.e., $a \leq b$. Analogously we obtain $a * b = 1 \Rightarrow a \leq b$. \blacksquare

Theorem 1. *Let \mathcal{S} be a commutative \sqcup -directoid with sectionally antitone bijections and $\circ, *$ be operations defined by (P). Then*

- (1) $x \circ x = x * x = 1, x \circ 1 = x * 1 = 1, 1 \circ x = 1 * x = x;$
- (2) $(x \circ y) * y = (x * y) \circ y = (y \circ x) * x = (y * x) \circ x;$
- (3) $x \circ (((x \circ y) * y) \circ z) * z = 1;$
- (4) $((((x \circ z) * z) \circ y) * y) \circ z \circ (x \circ z) = (((((x \circ z) * z) \circ y) * y) * z) \circ (x * z) = 1.$

Moreover, each of the terms of (2) is equal to $x \sqcup y$.

Proof.

- (1) By Lemma 1, $x \circ x = x * x = 1$ and $x \circ 1 = x * 1 = 1$. We also conclude $1 \circ x = f_x(1) = x$ and $1 * x = f_x^{-1}(1) = x$. Thus (1) is satisfied.
- (2) $(x \circ y) * y = f_y^{-1}(f_y(x \sqcup y) \sqcup y) = f_y^{-1}(f_y(x \sqcup y)) = x \sqcup y$ since $f_y(x \sqcup y) \geq y$. Analogously, we can check $(x * y) \circ y = x \sqcup y$, $(y \circ x) * x = x \sqcup y$, and $(y * x) \circ x = x \sqcup y$. Due to the fact that \mathcal{S} is commutative, we can establish the last two equalities.

$$(3) \quad x \circ (((x \circ y) * y) \circ z) * z = x \circ ((x \sqcup y) \sqcup z) = f_{(x \sqcup y) \sqcup z}(x \sqcup ((x \sqcup y) \sqcup z)) = f_{(x \sqcup y) \sqcup z}((x \sqcup y) \sqcup z) = 1 \text{ by (D4).}$$

$$(4) \quad (((x \circ z) * z) \circ y) * y \circ z = ((x \sqcup z) \sqcup y) \circ z = f_z(((x \sqcup z) \sqcup y) \sqcup z). \text{ Since } f_z \text{ is antitone and } z \leq (x \sqcup z) \sqcup y, x \sqcup z \leq (x \sqcup z) \sqcup y \text{ we have:}$$

$$f_z(((x \sqcup z) \sqcup y) \sqcup z) = f_z((x \sqcup z) \sqcup y) \leq f_z(x \sqcup z) = x \circ z,$$

thus, by (Q):

$$((((x \circ z) * z) \circ y) * y) \circ z \circ (x \circ z) = 1.$$

Analogously we obtain:

$$((((x \circ z) * z) \circ y) * y) * z \circ (x * z) = 1. \quad \blacksquare$$

Theorem 2. *Let $\mathcal{A} = (A; \circ, *, 1)$ be an algebra of type $(2, 2, 0)$ satisfying the identities (1), (2) and (3). Define a binary relation \leq on A as follows:*

$$(R) \quad a \leq b \text{ if and only if } a \circ b = 1.$$

*Then \leq is a partial order on A with a greatest element 1 and $(A; \leq)$ is a commutative \sqcup -directoid $(A; \sqcup)$ where $(x \circ y) * y = x \sqcup y$ and the following assertions are equivalent:*

- (i) *The algebra \mathcal{A} satisfies the identity (4);*
- (ii) *for any $x, y, a \in A, x \in [a, 1]$ the following implication holds:*

$$(5) \quad x \leq y \Rightarrow y \circ a \leq x \circ a \text{ and } y * a \leq x * a.$$

*Moreover, for any $a \in A$ the mappings $f_a(x) = x \circ a, f_a^{-1}(x) = x * a$ are mutually inverse antitone bijections on $[a, 1]$.*

Proof. First we prove that the relation \leq defined by (R) is a partial order.

Due to (1), \leq is reflexive. Suppose $x \leq y$ and $y \leq x$. Then $x \circ y = 1$ and $y \circ x = 1$ hence, by (1) and (2), $x = 1 * x = (y \circ x) * x = (x \circ y) * y = 1 * y = y$, thus \leq is antisymmetrical.

Suppose $x \leq y$ and $y \leq z$. Then $x \circ y = 1$ and $y \circ z = 1$. Hence $x \circ z = x \circ (1 * z) = x \circ ((y \circ z) * z) = x \circ (((1 * y) \circ z) * z) = x \circ (((x \circ y) * y) \circ z) * z = 1$ by (3). Thus \leq is transitive, i.e., it is a partial order. Also (1) tells us that $a \circ 1 = 1$ for all a , so 1 is the greatest element.

Define $a \sqcup b = (a \circ b) * b$. Put $x = y = a$, $z = b$ in (3). Then, with (1), we can easily derive

$$a \circ ((a \circ b) * b) = 1.$$

Putting $x = y = b$, $z = a$ in (3) we obtain analogously (using (1) and (2)) that

$$b \circ ((a \circ b) * b) = 1.$$

Thus $a \leq a \sqcup b$ and $b \leq a \sqcup b$.

Suppose now $a \leq b$. Then $a \sqcup b = (a \circ b) * b = 1 * b = b$. Clearly, $a \sqcup b = (a \circ b) * b = (b \circ a) * a = b \sqcup a$, thus $(A; \sqcup)$ is a commutative \sqcup -directoid.

(i) \Rightarrow (ii): Suppose now that also (4) is satisfied and $x \in [a, 1]$, $x \leq y$. Then $a \leq x$ and thus $(x \circ a) * a = x \sqcup a = x$. Therefore from (4) for $z = a$ it follows: $((x \circ y) * y) \circ a \circ (x \circ a) = 1$, which yields: $(y \circ a) \circ (x \circ a) = ((1 * y) \circ a) \circ (x \circ a) = (((x \circ y) * y) \circ a) \circ (x \circ a) = 1$. Thus $y \circ a \leq x \circ a$. Analogously we obtain: $y * a \leq x * a$, i.e., the condition (5) holds.

Now we consider $a \in A$, and f_a, f_a^{-1} defined by (A). Assume $x \in [a, 1]$. Then

$$f_a^{-1}(f_a(x)) = (x \circ a) * a = x \sqcup a = x,$$

$$f_a(f_a^{-1}(x)) = (x * a) \circ a = x \sqcup a = x,$$

using (3). Since $x \in [a, 1]$, by [5], also $f_a(x) \in [a, 1]$. Hence f_a and f_a^{-1} are bijections on $[a, 1]$ (each being the inverse of the other).

For $x \in [a, 1]$ and $x \leq y$ we have by (5):

$$f_a(y) = y \circ a \leq x \circ a = f_a(x),$$

$$f_a^{-1}(y) = y * a \leq x * a = f_a^{-1}(x),$$

therefore f_a and f_a^{-1} are antitone bijections.

(ii) \Rightarrow (i): By the assumptions $(A; \sqcup)$ is a commutative \sqcup -directoid with sectionally antitone bijections. Take $x, y \in A$. Since

$$f_y(x \sqcup y) = (x \sqcup y) \circ y = ((x \circ y) * y) \circ y = f_y(f_y^{-1}(x \circ y)) = x \circ y,$$

$$f_y^{-1}(x \sqcup y) = (x \sqcup y) * y = ((x * y) \circ y) * y = f_y^{-1}(f_y(x * y)) = x * y,$$

\circ and $*$ can be also defined by (P). Applying Theorem 1, we obtain that the algebra $(A; \circ, *, 1)$ satisfies (4). \blacksquare

Remark 2.

- (i) According to Theorem 1, to any commutative \sqcup -directoid $(P; \leq)$ with sectionally antitone bijections it corresponds an algebra $(P; \circ, *, 1)$ satisfying the axioms (1)–(4).
- (ii) In view of Theorem 2, to any algebra $\mathcal{A} = (A; \circ, *, 1)$ satisfying the identities (1)–(4) it corresponds a commutative directoid $(A; \sqcup)$ with sectionally antitone bijections, where \leq is the induced order of $(A; \sqcup)$.

The following example shows, that different algebras satisfying the identities (1)–(4) can be assigned to the same poset $(P; \leq)$ with sectionally antitone bijections.

A poset $(P; \leq)$ is called a *complete bipartite* poset if there exists two nonempty subset A, B of P such that $A \cap B = \emptyset$, $A \cup B = P$ and such that for any $x, y \in P$

$$x \leq y \quad \text{if and only if} \quad \text{either } x = y \text{ or } x \in A \text{ and } y \in B.$$

Example 1. Let us consider the finite poset $P \cup \{1\}$, where $(P; \leq)$ is a complete bipartite poset and 1 is an element with the property $x \leq 1$, for all $x \in P$. Denote the sets A and B as follows: $A = \{a_1, a_2, \dots, a_n\}$, $B = \{b_1, b_2, \dots, b_m\}$. Clearly $P \cup \{1\} = \{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m, 1\}$ and $a_i < b_j < 1$ for all $1 \leq i \leq n$, $1 \leq j \leq m$, see Figure 1. We show that $(P \cup \{1\}; \leq)$ is a poset with antitone bijections. Take an arbitrary permutation π of the set $\{b_1, b_2, \dots, b_m\}$ and let us define an arbitrary bijection for each section $[p, 1]$, (where $p \in P \cup \{1\}$) as follows:

$$\begin{aligned} f_1(1) &= 1, \\ f_{b_j}(1) &= b_j, \quad f_{b_j}(b_j) = 1, \quad 1 \leq j \leq m, \end{aligned}$$

$$f_{a_i}(1) = a_i, \quad f_{a_i}(a_i) = 1, \quad 1 \leq i \leq n$$

and $f_{a_i}(b_j) = \pi(b_j)$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

Further, a commutative directoid $(P \cup \{1\}; \sqcup)$ can be defined on the set $P \cup \{1\}$ as follows:

Fix an arbitrary element b_k ($1 \leq k \leq m$). Then define \sqcup in the following way:

$$x \sqcup y = y \quad \text{iff} \quad x \leq y,$$

for any b_i, b_j , $b_i \neq b_j$ let $b_i \sqcup b_j = 1$, for any a_i, a_j , $a_i \neq a_j$ let $a_i \sqcup a_j = b_k$.

According to Remark 1, $(P \cup \{1\}, \sqcup)$ is a commutative directoid with sectionally antitone bijections.

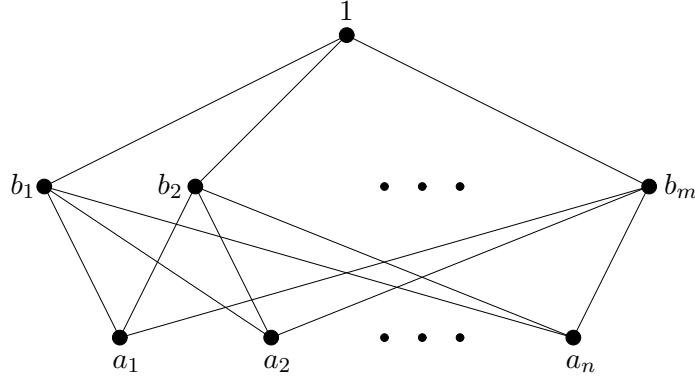


Figure 1

In view of Theorem 1, to any permutation π on the set $\{b_1, \dots, b_m\}$ and to any $b_k \in \{b_1, b_2, \dots, b_m\}$ there corresponds an algebra \mathcal{A} satisfying the identities (1)–(4). In fact we have $a_i \circ a_i = b_i \circ b_i = a_i \circ b_j = b_j \circ 1 = 1 \circ 1 = 1$ and $a_i * a_i = b_i * b_i = a_i * b_j = b_j * 1 = 1 * 1 = 1$ for all $i \neq j$ such that $1 \leq i \leq n$ and $1 \leq j \leq m$. Further, we have $b_i \circ b_j = b_i * b_j = b_j$, for all $i \neq j$ such that $1 \leq i, j \leq m$, $a_i \circ a_j = \pi(b_k)$, $a_i * a_j = \pi^{-1}(b_k)$, for all $1 \leq i, j \leq n$ and $b_j \circ a_i = \pi(b_j)$, $b_j * a_i = \pi^{-1}(b_j)$, for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Let us recall that an algebra $(L; \sqcup, \sqcap)$ of type $(2, 2)$ is called a λ -lattice if its both reducts $(L; \sqcup)$ and $(L; \sqcap)$ are commutative directoids and hence the operations are connected by the absorption laws, see e.g. [11].

Corollary 1. *Let $\mathcal{A} = (A; \circ, *, 1)$ be an algebra of type $(2, 2, 0)$ satisfying the identities (1), (2), (3) and (4) and let \leq be the induced order. Then $(A; \leq)$ is a commutative \sqcup -directoid with 1, where $x \sqcup y = (x \circ y) * y$ and for each $a \in A$ the section $[a, 1]$ becomes a λ -lattice where*

$$x \sqcap_a y = (((x \circ a) \circ (y \circ a)) * (y \circ a)) * a.$$

Proof. In view of Theorem 2 (ii), the section $[a, 1]$ is a commutative \sqcup -directoid in which $x \sqcup y = (x \circ y) * y$. Since f_a, f_a^{-1} are antitone bijections it follows that letting $x \sqcap_a y = f_a^{-1}(f_a(x) \sqcup f_a(y))$ we get a commutative \sqcap -directoid with the same order. Hence $([a, 1]; \sqcup, \sqcap_a)$ is a λ -lattice.

We have

$$\begin{aligned} x \sqcap_a y &= f_a^{-1}(f_a(x) \sqcup f_a(y)) = ((x \circ a) \sqcup (y \circ a)) * a = \\ &= (((x \circ a) \circ (y \circ a)) * (y \circ a)) * a. \end{aligned}$$

■

When $\mathcal{A} = (A; \sqcup, \sqcap)$ is a λ -lattice then the induced operation \sqcap_a in a section $[a, 1]$ need not coincide with the operation \sqcap , see e.g. the following:

Example 2. Let $\mathcal{L} = (L; \sqcup, \sqcap)$, $L = \{0, w, z, y, x, 1\}$ be a λ -lattice depicted in Figure 2. Pick $w = x \sqcap y$ and $z \sqcup w = x$.

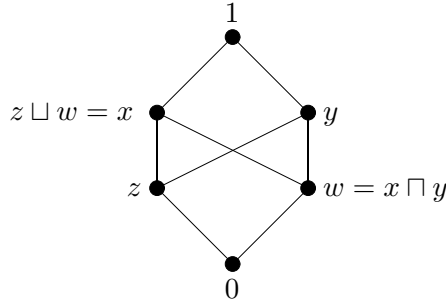


Figure 2

Further, let $f_z(1) = z$, $f_z(z) = 1$, $f_z(x) = x$, $f_z(y) = y$ and $f_z^{-1} = f_z$. Then $x = x \circ z$, $y = y \circ z$, $z = (x \sqcup y) \circ z = (x \circ z) \sqcap_z (y \circ z)$, i.e., $x \sqcap_z y = z$ but $w = x \sqcap y = (x \circ z) \sqcap (y \circ z)$, thus \sqcap_z is not the same as \sqcap .

An *implication algebra* [1] is an algebra $(A; \circ)$ satisfying the identities

$$(I1) \quad (x \circ y) \circ x = x;$$

$$(I2) \quad (x \circ y) \circ y = (y \circ x) \circ x;$$

$$(I3) \quad x \circ (y \circ z) = y \circ (x \circ z).$$

It is well-known that any implication algebra $(A; \circ)$ contains an element $1 \in A$ such that $x \circ x = 1$, for all $x \in A$ (see e.g. [1]). The algebra $(A; \circ, \circ, 1)$ of type $(2,2,0)$ where the operation \circ is doubled will be called a *double implication algebra*.

It is easy to see that any double implication algebra satisfies the identities (1)–(4).

Indeed, for the algebra $\mathcal{A} = (A; \circ, \circ, 1)$ the identity (2) is the same as (I2) and we have $x \circ x = 1$, $1 \circ x = x$ and $x \circ 1 = (1 \circ x) \circ (1 \circ 1) = 1 \circ ((1 \circ x) \circ 1) = 1 \circ 1 = 1$, i.e., (1) is also satisfied by \mathcal{A} . Clearly also (3) holds. According to [1], (R) defines a partial order \leq with the property (5) on any implication algebra $(A; \circ)$. Hence, by Theorem 2, we obtain that $(A; \circ, \circ, 1)$ satisfies the identity (4), as well.

Example 3. Observe that for a two-element chain $(\{0, 1\}; \leq)$ the algebra $\mathcal{S}_2 = (\{0, 1\}; \circ, \circ, 1)$ (where, of course, $f_0(0) = 1$, $f_0(1) = 0$) is a double implication algebra. The operation \circ is given by the table

\circ	0	1
0	1	1
1	0	1.

It is proved in [10] that the implication algebras form a minimal quasivariety which is generated by the two-element implication algebra $(\{0, 1\}; \circ)$. Hence it is not hard to see that the variety generated by the algebra \mathcal{S}_2 is also a minimal quasivariety and it coincides with the variety of all double implication algebras.

Proposition 2. *The variety \mathcal{V} of the algebras $(A; \circ, *, 1)$ of type $(2, 2, 0)$ satisfying (1)–(4) contains a single minimal quasivariety, namely the variety of double implication algebras.*

Proof. Let \mathcal{W} be a nontrivial subquasivariety of \mathcal{V} and $\mathcal{A} = (A; \circ, *, 1)$ a nontrivial algebra in \mathcal{W} . In view of Theorem 1, the corresponding poset $(A; \leq)$ is a commutative \sqcup -directoid with sectionally antitone bijections. As $|A| \geq 2$, there exists an element $a \in A$ with $a \neq 1$. In view of Theorem 1 we have $a \circ a = a * a = 1 \circ 1 = 1 * 1 = 1$, $a \circ 1 = a * 1 = 1$ and $1 \circ a = 1 * a = a$. Hence $(\{a, 1\}; \circ, *, 1)$ is a subalgebra of $(A; \circ, *, 1)$. Since $(\{a, 1\}; \leq)$ is a two-element chain, the algebra $\mathcal{S}_2 = (\{a, 1\}; \circ, *, 1)$ is a double implication algebra with two elements (see Example 3). Denote the variety generated by it as \mathcal{V}_2 . We already shown that \mathcal{V}_2 is the variety of all double implication algebras. Since \mathcal{V}_2 is a minimal quasivariety as well, we have $\mathcal{V}_2 = Q(\mathcal{S}_2)$, where $Q(\mathcal{S}_2)$ denotes the quasivariety generated by the algebra \mathcal{S}_2 . As $\mathcal{S}_2 \in \mathcal{W}$ we get $\mathcal{V}_2 = Q(\mathcal{S}_2) \subseteq \mathcal{W}$, and this proves that \mathcal{V}_2 is the unique minimal quasivariety contained in variety \mathcal{V} . ■

Let $\mathcal{A} = (A; \circ, *, 1)$ be an algebra of type $(2, 2, 0)$. A nonempty subset $K \subseteq A$ is called a *congruence kernel* of \mathcal{A} if $K = [1]_{\Theta} = \{x \in A; (x, 1) \in \Theta\}$ for some congruence Θ of \mathcal{A} . Recall that \mathcal{A} is called *3-permutable* if $\Theta_1 \circ \Theta_2 \circ \Theta_1 = \Theta_2 \circ \Theta_1 \circ \Theta_2$ holds for every $\Theta_1, \Theta_2 \in \text{Con}\mathcal{A}$. According to J. Hagemann and A. Mitschke [7], a variety \mathcal{V} of algebras is 3-permutable if and only if there exist ternary terms p_0, p_1, p_2 and p_3 in \mathcal{V} such that the following identities hold in \mathcal{V} :

$$(B) \quad \begin{cases} p_0(x, y, z) = x, p_3(x, y, z) = z, \\ p_i(x, x, y) = p_{i+1}(x, y, y) \text{ for } i \in \{0, 1, 2\}. \end{cases}$$

An algebra \mathcal{A} with a constant 1 is called *weakly regular* if every $\Theta \in \text{Con}\mathcal{A}$ is determined by its kernel, i.e., if $[1]_{\Theta} = [1]_{\Phi}$ implies $\Theta = \Phi$ for every $\Phi, \Theta \in \text{Con}\mathcal{A}$. A variety \mathcal{V} is weakly regular if every algebra $\mathcal{A} \in \mathcal{V}$ has this property. The following characterization of weakly regular varieties was established by B. Csákány in [6].

Proposition 3 [6]. A variety \mathcal{V} with 1 is weakly regular if and only if there exists $n \in \mathbb{N}$ and binary terms $q_1(x, y), q_2(x, y), \dots, q_n(x, y)$ such that

$$(C) \quad q_1(x, y) = q_2(x, y) = \cdots = q_n(x, y) = 1 \Leftrightarrow x = y$$

is satisfied for every algebra $\mathcal{A} \in \mathcal{V}$.

For other congruence conditions, the reader is asked to consult e.g. [3].

Theorem 3. *The variety \mathcal{V} of the algebras $(A; \circ, *, 1)$ of type $(2, 2, 0)$ satisfying the identities (1)–(4) is weakly regular, 3-permutable, arithmetical at 1 and congruence distributive.*

Proof. Consider the terms $q_1(x, y) = x \circ y$ and $q_2(x, y) = y \circ x$. Then $q_1(x, x) = q_2(x, x) = x \circ x = 1$. If $q_1(x, y) = 1$ and $q_2(x, y) = 1$, then by (R) we have $x \leq y$ and $y \leq x$ thus $x = y$. In view of Proposition 3, we conclude that \mathcal{V} is weakly regular (at 1).

Now, let $p_0(x, y, z) = x$, $p_3(x, y, z) = z$ and $p_1(x, y, z) = (z * y) \circ x$, $p_2(x, y, z) = (x * y) \circ z$. It is easy to see that these terms satisfy the identities (B). Consequently, \mathcal{V} is congruence 3-permutable.

It is proved in [3] that a variety is arithmetical at 1 if and only if there exists a binary term $b(x, y)$ of it such that $b(x, x) = b(1, x) = 1$ and $b(x, 1) = x$. Obviously, we can take $b(x, y) = y \circ x$.

Since \mathcal{V} is arithmetical at 1 it is congruence distributive at 1, i.e., $[1]_{\Theta \cap (\Phi \vee \Psi)} = [1]_{(\Theta \cap \Phi) \vee (\Theta \cap \Psi)}$, for all $\Theta, \Phi, \Psi \in \text{Con}\mathcal{A}$ for $\mathcal{A} \in \mathcal{V}$.

As \mathcal{V} is weakly regular, this equality implies $\Theta \cap (\Phi \vee \Psi) = (\Theta \cap \Phi) \vee (\Theta \cap \Psi)$ (for all $\Theta, \Phi, \Psi \in \text{Con}\mathcal{A}$ and $\mathcal{A} \in \mathcal{V}$), thus the variety \mathcal{V} is congruence distributive. ■

REFERENCES

- [1] J.C. Abbott, *Semi-Boolean algebras*, Matem. Vestnik **4** (1967), 177–198.
- [2] I. Chajda, *Lattices and semilattices having an antitone bijection in any upper interval*, Comment. Math. Univ. Carolinae **44** (2003), 577–585.
- [3] I. Chajda, G. Eigenthaler and H. Länger, *Congruence Classes in Universal Algebra*, Heldermann Verlag, Lemgo (Germany), 2003, ISBN 3–88538–226–1.
- [4] I. Chajda and M. Kolařík, *Directoids with sectionally antitone involutions and skew MV-algebras*, Math. Bohemica **132** (2007), 407–422.
- [5] I. Chajda and R. Radeleczki, *Semilattices with sectionally antitone bijections*, Novi Sad J. Math. **35** (2005), 93–101.

- [6] B. Csákány, *Characterization of regular varieties*, Acta Sci. Math. Szeged **31** (1970), 187–189.
- [7] J. Hagemann and A. Mitschke, *On n -permutable congruences*, Algebra Universalis **3** (1973), 8–12.
- [8] J. Ježek and R. Quackenbush, *Directoids: algebraic models of up-directed sets*, Algebra Universalis **27** (1990), 49–69.
- [9] V.M. Kopytov and Z.I. Dimitrov, *On directed groups*, Siberian Math. J. **30** (1989), 895–902. (Russian original: Sibirsk. Mat. Zh. **30** (6) (1988), 78–86.)
- [10] S. Radeleczki, *The congruence lattice of implication algebras*, Math. Pannonica **3** (1992), 115–123.
- [11] V. Snášel, *λ -lattices*, Math. Bohemica **122** (1997), 267–272.

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