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INTERIOR AND CLOSURE OPERATORS ON BOUNDED COMMUTATIVE RESIDUATED ℓ-MONOIDS

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Abstract

Topological Boolean algebras are generalizations of topological spaces defined by means of topological closure and interior operators, respectively. The authors in [14] generalized topological Boolean algebras to closure and interior operators of MV-algebras which are an algebraic counterpart of the Łukasiewicz infinite valued logic. In the paper, these kinds of operators are extended (and investigated) to the wide class of bounded commutative $R\ell$-monoids that contains e.g. the classes of BL-algebras (i.e., algebras of the Hájek's basic fuzzy logic) and Heyting algebras as proper subclasses.

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1. Introduction

Topological Boolean algebras (= closure algebras, resp. interior algebras) (see for example [15]) are generalizations of topological spaces defined by means of topological closure, resp. interior, operators. In [14], closure, resp. interior, MV-algebras by means of additive closure and multiplicative interior, respectively, operators were introduced as a generalization of topological Boolean algebras.

It is well known that MV-algebras are an algebraic counterpart of the Łukasiewicz infinite valued propositional logic as well as Boolean algebras play this role for classical two valued logic. Every Boolean algebra is in fact an MV-algebra and conversely, every MV-algebra $\mathcal{A}$ contains the greatest Boolean subalgebra $B(\mathcal{A})$ formed by complemented (= additive, resp. multiplicative, idempotent) elements. According to [14], restriction of each additive closure operator of an MV-algebra $\mathcal{A}$ is a topological closure operator on the Boolean algebra $B(\mathcal{A})$. Moreover, in every complete MV-algebra, each topological closure operator on $B(\mathcal{A})$ can be extended on an additive closure operator on $\mathcal{A}$. (Analogously for multiplicative interior operators.)

The Łukasiewicz logic is one of the most important logics in the theory of fuzzy sets. Hájek's basic fuzzy logic generalizes all such logics. It is known that BL-algebras introduced also by Hájek are an algebraic counterpart of the basic fuzzy logic. Bounded commutative residuated lattice ordered monoids ($R\ell$-monoids) form a wide class of algebras, which contains not only the class of all BL-algebras, but also the class of all Heyting algebras. Therefore bounded commutative $R\ell$-monoids can be taken as an algebraic semantics of a more general logic than the basic logic.

In MV-algebras there are two binary operations "$\oplus$" and "$\odot$" which are mutually dual. Therefore by [14], for the MV-algebras the research of additive closure operators (ac-operators) is equivalent with that of multiplicative interior operators (mi-operators). Nevertheless, in the case of BL-algebras and then also in more general bounded commutative $R\ell$-monoids an operation with dual properties to the binary operation "$\odot$" does not generally exist.

So we introduce mi-operators on bounded commutative $R\ell$-monoids as the generalization of analogous operators on MV-algebras and we show their properties. Further, we introduce a new binary operation "$\oplus$", which need not to be dual to "$\odot$" in general, but it makes possible to introduce some analogy of an ac-operator from the theory of MV-algebras. We show mutual relationships between mi- and ac-operators, especially for the case of normal $R\ell$-monoids. Further, we describe mi- and ac-operators induced by operators on an $R\ell$-monoid $\mathcal{M}$ on the quotient $R\ell$-monoid $\mathcal{M}/D(\mathcal{M})$ of $\mathcal{M}$ by the filter $D(\mathcal{M})$ of dense elements in $\mathcal{M}$, on the $R\ell$-monoid $R(\mathcal{M})$ of regular elements in $\mathcal{M}$, on the $R\ell$-monoid $Fix(f)$ of fixed points of an mi-operator f on $\mathcal{M}$ and in the case of a BL-algebra $\mathcal{M}$ on the Heyting algebra $I(\mathcal{M})$ of idempotent elements in $\mathcal{M}$.

For notions and results related to BL- and MV-algebras see for example [3, 5, 6, 17].

2. Bounded commutative $R\ell$-monoids

Definition 2.1. An algebra $\mathcal{M} = (M; \odot, \vee, \wedge, \rightarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ is called a *bounded commutative $R\ell$-monoid* iff for each $x, y, z \in M$

(i) $(M; \odot, 1)$ is a commutative monoid,

(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice,

(iii) $x \odot y \leq z \Longleftrightarrow x \leq y \rightarrow z$,

(iv) $x \odot (x \rightarrow y) = x \wedge y$.

One can prove (see e.g. [5]) that bounded commutative $R\ell$-monoids are just commutative integral residuated lattices in the sense of [7] and [1], hence by [2] and [7], the operation "$\odot$" distributes over the lattice operations "$\vee$" and "$\wedge$". Further, the lattice $(M; \vee, \wedge)$ is distributive. Moreover, bounded commutative $R\ell$-monoids form a variety of type $\langle 2, 2, 2, 2, 0, 0\rangle$.

In the sequel, an *$R\ell$-monoid* will mean a bounded commutative $R\ell$-monoid. For example, every BL-algebra and every Heyting algebra are special cases of $R\ell$-monoids. Hence the class of BL-algebras is a proper subclass of the class of all $R\ell$-monoids (which is by [11] a subvariety of the variety of $R\ell$-monoids defined by the identity $(x \rightarrow y) \vee (y \rightarrow x) = 1$).

On an arbitrary $R\ell$-monoid $\mathcal{M}$ we define the unary operation *negation* $^- : M \rightarrow M$ by

$$x^- := x \rightarrow 0$$

for each element $x \in M$.

Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be an MV-algebra. Set $x \rightarrow y := \neg x \oplus y$ and $x^- := x \rightarrow 0 = \neg x$. Then $\mathcal{A}$ becomes an $R\ell$-monoid. We can characterize MV-algebras by means of the negation, because by [9, 10], the class of MV-algebras is a subvariety of the variety of $R\ell$-monoids determined by the identity $x^{--} = x$.

In the following lemma we show some properties of $R\ell$-monoids, which we will use in the sequel.

Lemma 2.1. ([12, 13, 16]). *In any $R\ell$-monoid $\mathcal{M}$ we have for any $x, y, z \in M$*

(i) $x \odot y \leq x \wedge y \leq x, y,$

(ii) $x \leq y \Longrightarrow x \odot z \leq y \odot z,$

(iii) $x \leq y \Longrightarrow z \to x \leq z \to y,$

(iv) $x \leq y \Longrightarrow y \to z \leq x \to z,$

(v) $x \leq y \Longleftrightarrow x \to y = 1,$

(vi) $x \to x = 1,\ 1 \to x = x,\ x \to 1 = 1,$

(vii) $y \leq x \to y,$

(viii) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z),$

(ix) $(x \vee y) \to z = (x \to z) \wedge (y \to z),$

(x) $x \to (y \wedge z) = (x \to y) \wedge (x \to z),$

(xi) $(x \vee y) \odot (x \wedge y) = x \odot y,$

(xii) $(x \to y) \odot (y \to z) \leq x \to z,$

(xiii) $1^{--} = 1,\ 0^{--} = 0,$

(xiv) $x \leq x^{--},\ x^{-} = x^{---},$

(xv) $x \leq y \Longrightarrow y^{-} \leq x^{-},$

(xvi) $(x \vee y)^{-} = x^{-} \wedge y^{-},$

(xvii) $(x \wedge y)^{--} = x^{--} \wedge y^{--},$

(xviii) $(x \odot y)^{-} \geq x^{--} \odot y^{--},$

(xix) $(x \to y)^{--} = x^{--} \to y^{--}.$

3. Multiplicative interior and additive closure operators on $R\ell$-monoids

Definition 3.1. Let $\mathcal{M} = (M; \odot, \vee, \wedge, \to, 0, 1)$ be an $R\ell$-monoid and $f : M \to M$ a mapping. Then f is called a *multiplicative interior operator* (or *mi-operator*) on $\mathcal{M}$ iff for each $x, y \in M$

1. $f(x \odot y) = f(x) \odot f(y)$,
2. $f(x) \leq x$,
3. $f(f(x)) = f(x)$,
4. $f(1) = 1$.

Lemma 3.1. *Each mi-operator on an $R\ell$-monoid $\mathcal{M}$ is an interior operator on the lattice $(M; \vee, \wedge)$.*

Proof. Let us consider an *mi*-operator f on $\mathcal{M}$ and $x, y \in M$ such that $x \leq y$. Then

$$f(x) = f(x \wedge y) = f(y \odot (y \to x)) = f(y) \odot f(y \to x),$$

so with regard to Lemma 2.1 (i) we get that $f(y) \odot f(y \to x) \leq f(y)$, therefore $f(x) \leq f(y)$. Thus f is isotone and by 2 and 3 from Definition 3.1, f is an interior operator on $(M; \vee, \wedge)$. ■

Lemma 3.2. *For an mi-operator f on an $R\ell$-monoid $\mathcal{M}$ and for each $x, y \in M$,*

$$f(x \to y) \leq f(x) \to f(y).$$

Proof. Let $x, y \in M$. Then $(x \to y) \odot x = x \wedge y \leq y$ and by Lemma 3.1 $f((x \to y) \odot x) = f(x \to y) \odot f(x) \leq f(y)$, so $f(x \to y) \leq f(x) \to f(y)$. ■

Remark 3.3. The identity $f(x \to y) = f(x) \to f(y)$ is generally not satisfied on the class of $R\ell$-monoids. For example, for the canonical MV-algebra $\mathcal{A} = ([0, 1]; \oplus, \neg, 0)$ where $[0, 1]$ is the real unit interval, $x \oplus y = \min(1, x + y)$ and $\neg x = 1 - x$ for any $x, y \in [0, 1]$, and for the *mi*-operator f on $\mathcal{A}$ such that

$$f(x) = \begin{cases} 1 & \text{for} \quad x = 1, \\ 0 & \text{for} \quad x \neq 1, \end{cases}$$

we get $0,5 \to 0,2 \neq 1$, hence $f(0,5 \to 0,2) = 0$, but at the same time $f(0,5) \to f(0,2) = 0 \to 0 = 1$.

Let us consider a mapping $f : M \to M$ on an $R\ell$-monoid $\mathcal{M}$. We define a mapping $f^- : M \to M$ such that

$$f^-(x) := (f(x^-))^-,$$

for any $x \in M$.

Proposition 3.4. If f is an *mi*-operator on an $R\ell$-monoid $\mathcal{M}$ then the mapping f^- is isotone.

Proof. Let us consider elements $x, y \in M$ such that $x \leq y$. Then $y^- \leq x^-$ (see Lemma 2.1 (xv)), so $f(y^-) \leq f(x^-)$. Therefore $(f(x^-))^- \leq (f(y^-))^-$, or equivalently $f^-(x) \leq f^-(y)$. ■

Proposition 3.5. *If f is an mi-operator on an $R\ell$-monoid $\mathcal{M}$ then we have for each element $x \in M$*

$2'$. $x \leq f^-(x)$,

$3'$. $f^-(f^-(x)) = f^-(x)$,

$4'$. $f^-(0) = 0$.

Proof.

$2'$. If $x \in M$ then $f^-(x) = (f(x^-))^- \geq x^{--} \geq x$.

$3'$. For any $x \in M$ we have $f^-(f^-(x)) = f^-(f(x^-)^-) = (f(f(x^-))^{--})^-$ and $f(x^-) \leq (f(x^-))^{--}$. Hence $f(f(x^-)) = f(x^-) \leq f((f(x^-))^{--})$, therefore $(f(x^-))^- \geq (f((f(x^-))^{--}))^-$, and so $f^-(x) \geq f^-(f^-(x))$. By $2'$ we have also $f^-(x) \leq f^-(f^-(x))$, that means $f^-(f^-(x)) = f^-(x)$.

$4'$. $f^-(0) = (f(0^-))^- = (f(1))^- = 1^- = 0$. ■

Remark 3.6.

a) By Propositions 3.4 and 3.5, f^- is a closure operator on the lattice $(M; \vee, \wedge)$.

b) By the part $2'$ of the proof of Proposition 3.5 the stronger inequality $x^{--} \leq f^-(x)$ is satisfied in $\mathcal{M}$.

Let us consider an $R\ell$-monoid $\mathcal{M} = (M; \odot, \vee, \wedge, \rightarrow, 0, 1)$. Given $x, y \in M$, we put

$$x \oplus y := (x^- \odot y^-)^-.$$

Then $\oplus$ has the following properties.

Lemma 3.7. *If $\mathcal{M}$ is an $R\ell$-monoid and $x, y, z \in M$, then*

(a) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(b) $x \oplus y \geq x^{--} \vee y^{--} \geq x \vee y$,

(c) $x \oplus 0 = x^{--}$,

(d) $(x \oplus y)^{--} = x^{--} \oplus y^{--} = x \oplus y$,

(e) $x \odot x^- = 0, \quad x \oplus x^- = 1$.

Proof. Properties (a), (c) and (d) are proved in [13].

(b) $(x \oplus y) = (x^- \odot y^-)^- \geq x^{--}, y^{--}$, so $x \oplus y \geq x^{--} \vee y^{--} \geq x \vee y$.

(e) Clearly $x \odot x^- = 0$. Hence $x \oplus x^- = (x^- \odot x^{--})^- = 0^- = 1$.

■

Corollary 3.8. *An $R\ell$-monoid $\mathcal{M}$ is an MV-algebra iff $x \oplus 0 = x$ for each $x \in M$.*

Proof. By [9], an (bounded commutative) $R\ell$-monoid $\mathcal{M}$ is an MV-algebra if and only if $x^{--} = x$ for each $x \in M$. Now, the corollary follows from Lemma 3.7 (c). ■

Definition 3.2. An $R\ell$-monoid $\mathcal{M}$ is *normal* if $\mathcal{M}$ satisfies identity

$$(x \odot y)^{--} = x^{--} \odot y^{--}.$$

Remark 3.9. It follows from [12], Proposition 5 that every BL-algebra and every Heyting algebra is also a normal $R\ell$-monoid, hence the variety of normal $R\ell$-monoids is considerably large.

Lemma 3.10. *In every normal $R\ell$-monoid $\mathcal{M}$ it holds for each $x, y \in M$*

(f) $(x \oplus y)^- = x^- \odot y^-$,

(g) $(x \odot y)^- = x^- \oplus y^-$.

Proof. Let us consider arbitrary elements $x, y \in M$. Then

(f) $(x \oplus y)^- = (x^- \odot y^-)^{--} = x^{---} \odot y^{---} = x^- \odot y^-$.

(g) We will use Lemma 3.7(d). We have got

$$x^- \oplus y^- = (x^- \oplus y^-)^{--} = ((x^{--} \odot y^{--})^-)^{--} =$$
$$= (x^{--} \odot y^{--})^- = (x \odot y)^{---} = (x \odot y)^-.$$

■

Proposition 3.11. *If $\mathcal{M}$ is a normal $R\ell$-monoid and f is an mi-operator on $\mathcal{M}$ then the mapping f^- satisfies on $\mathcal{M}$ also identity*

1'. $f^-(x \oplus y) = f^-(x) \oplus f^-(y)$.

Proof. Let $x, y \in M$. Then

$$f^-(x) \oplus f^-(y) = ((f^-(x))^- \odot (f^-(y))^-)^- = ((f(x^-))^{--} \odot (f(y^-))^{--})^- =$$
$$= (f(x^-) \odot f(y^-))^{---} = (f(x^-) \odot f(y^-))^- = (f(x^- \odot y^-))^- =$$
$$= (f(x^{---} \odot y^{---}))^- = (f((x^- \odot y^-)^{--})^- = f^-((x^- \odot y^-)^-) = f^-(x \oplus y).$$

■

In the theory of MV-algebras there exist additive closure operators as the duals to multiplicative interior operators (see [14]). Now, we will introduce additive closure operators on $R\ell$-monoids and we will describe their properties as well as their relationship to mi-operators.

Definition 3.3. If $\mathcal{M}$ is an $R\ell$-monoid and $g : M \to M$ a mapping then g is called an *additive closure operator* (*ac-operator*) on $\mathcal{M}$ iff for each $x, y \in M$

1′. $g(x \oplus y) = g(x) \oplus g(y)$.

2′. $x \leq g(x)$,

3′. $g(g(x)) = g(x)$,

4′. $g(0) = 0$.

Proposition 3.12. *If $\mathcal{M}$ is a normal $R\ell$-monoid and is f an mi-operator on $\mathcal{M}$ then the mapping f^- is an ac-operator on $\mathcal{M}$.*

Proof. It follows from Propositions 3.5 and 3.11. ■

Lemma 3.13. *Any ac-operator g satisfies on an $R\ell$-monoid $\mathcal{M}$ the identity*

$$g(x^{--}) = (g(x))^{--}.$$

Proof. For any element $x \in M$ and any ac-operator g on $\mathcal{M}$ we have by Lemma 3.7 (c)

$$g(x^{--}) = g(x \oplus 0) = g(x) \oplus g(0) = g(x) \oplus 0 = (g(x))^{--}.$$

■

Proposition 3.14. *Let us consider an additive closure operator g on a normal $R\ell$-monoid $\mathcal{M}$. Then we have for each $x, y \in M$*

1″. $g^-(x \odot y) = g^-(x) \odot g^-(y)$,

2″. $g^-(x) \leq x^{--}$,

3″. $g^-(g^-(x)) = g^-(x)$,

4″. $g^-(1) = 1$.

Moreover, if g is isotone then g^- is also an isotone operator.

Proof. Let us choose arbitrary elements $x, y \in M$.

$1''$. By Lemma 3.10,

$$g^-(x \odot y) = (g((x \odot y)^-))^- = (g(x^- \oplus y^-))^- =$$
$$= (g(x^-) \oplus g(y^-))^- = (g(x^-))^- \odot (g(y^-))^- = g^-(x) \odot g^-(y),$$

$2''$. $g^-(x) = (g(x^-))^- \leq (x^-)^- = x^{--}$,

$3''$. By Lemma 3.13,

$$g^-(g^-(x)) = (g((g(x^-))^{--}))^- = (g(g(x^-)))^{---} = (g(x^-))^- = g^-(x),$$

$4''$. $g^-(1) = (g(1^-))^- = (g(0))^- = 0^- = 1$.

Let g be isotone and let $x \leq y$. Then $x^- \geq y^-$ and further $g(x^-) \geq g(y^-)$. Hence $(g(x^-))^- \leq (g(y^-))^-$ which is equivalent to $g^-(x) \leq g^-(y)$. ■

Remark 3.15. So, Propositions 3.4, 3.5 and 3.11 tell us that for each *mi*-operator f on a normal $R\ell$-monoid $\mathcal{M}$ the induced mapping f^- is an isotone *ac*-operator on $\mathcal{M}$. Therefore the mapping $(f^-)^-$ is isotone and has all the properties from Proposition 3.14.

4. Operators on algebras derived from $R\ell$-monoids

Let us have an $R\ell$-monoid $\mathcal{M}$ and its *mi*-operator f. In this chapter, the algebra $(\mathcal{M}, f) = (M; \odot, \vee, \wedge, \rightarrow, 0, 1, f)$ will be called an *interior $R\ell$-monoid* (analogously as for the MV-algebras in [14]).

Definition 4.1. If $\mathcal{M}$ is an $R\ell$-monoid then a nonempty subset F of M is called a *filter* in $\mathcal{M}$ iff

(F1) $x, y \in F \Longrightarrow x \odot y \in F$,

(F2) $x \in F,\ y \in M,\ x \leq y \Longrightarrow y \in F$.

It is known that filters of $R\ell$-monoids coincide with kernels of their congruences. If F is a filter of an $R\ell$-monoid $\mathcal{M}$ then F is the kernel of a unique congruence $\Theta(F)$ such that

$$\langle x, y \rangle \in \Theta(F) \Longleftrightarrow (x \rightarrow y) \wedge (y \rightarrow x) \in F$$

for each $x, y \in M$. Therefore, for each $R\ell$-monoid M we can consider the quotient $R\ell$-monoid $\mathcal{M}/F$ by its filter F.

We will denote by $D(\mathcal{M}) = \{x \in M : x^{--} = 1\}$ the set of all *dense* elements of an $R\ell$-monoid $\mathcal{M}$. By [12], Theorem 8 or [5], Proposition 3.3, $D(\mathcal{M})$ is a proper filter in $\mathcal{M}$ and that the quotient $R\ell$-monoid $\mathcal{M}/D(\mathcal{M})$ is an MV-algebra. Moreover, for each $x, y \in M$,

$$x/D(\mathcal{M}) = y/D(\mathcal{M}) \Longleftrightarrow x^{--} = y^{--}.$$

Theorem 4.1. *Let us consider an interior $R\ell$-monoid $(\mathcal{M}, f)$. Further, we will consider a mapping $\tilde{f} : \mathcal{M}/D(\mathcal{M}) \to \mathcal{M}/D(\mathcal{M})$ such that for each element $x \in M$,*

$$\tilde{f}(x/D(\mathcal{M})) := f(x)/D(\mathcal{M}).$$

Then $\tilde{f}$ is an mi-operator on the MV-algebra $\mathcal{M}/D(\mathcal{M})$.

Proof. First, we will show that the mapping $\tilde{f}$ is correctly defined. Let $x, y \in M$ be such elements that $\langle x, y\rangle \in \Theta(D(\mathcal{M}))$. Since $\mathcal{M}/D(\mathcal{M})$ is an MV-algebra, we get

$$\tilde{f}(x/D(\mathcal{M})) = \tilde{f}((x/D(\mathcal{M}))^{--}) = \tilde{f}(x^{--}/D(\mathcal{M})) =$$

$$= f(x^{--})/D(\mathcal{M}) = f(y^{--})/D(\mathcal{M}) = \cdots = \tilde{f}(y/D(\mathcal{M})).$$

Now, we will check that $\tilde{f}$ satisfies conditions from the definition of a multiplicative interior operator on an MV-algebra.

1′. $\tilde{f}(x/D(\mathcal{M})) \odot \tilde{f}(y/D(\mathcal{M})) = f(x)/D(\mathcal{M}) \odot f(y)/D(\mathcal{M}) =$
$= (f(x) \odot f(y))/D(\mathcal{M}) = f(x \odot y)/D(\mathcal{M}) =$
$= \tilde{f}((x \odot y)/D(\mathcal{M})) = \tilde{f}((x/D(\mathcal{M})) \odot (y/D(\mathcal{M})))$.

2′. $\tilde{f}(x/D(\mathcal{M})) = f(x)/D(\mathcal{M}) \leq x/D(\mathcal{M})$.

3′. $\tilde{f}(\tilde{f}(x/D(\mathcal{M}))) = \tilde{f}(f(x)/D(\mathcal{M})) = f(f(x))/D(\mathcal{M}) =$
$= f(x)/D(\mathcal{M}) = \tilde{f}(x/D(\mathcal{M}))$.

4′. $\tilde{f}(1/D(\mathcal{M})) = f(1)/D(\mathcal{M}) = 1/D(\mathcal{M})$.

■

Corollary 4.2. *Let f be an mi-operator on an $R\ell$-monoid $\mathcal{M}$ and let $\tilde{f}$ be the induced mi-operator on the MV-algebra $\mathcal{M}/D(\mathcal{M})$. Then the operator $(\tilde{f})^-$ is an ac-operator on the MV-algebra $\mathcal{M}/D(\mathcal{M})$ and the operator $(\tilde{f}^-)^-$ is an mi-operator on $\mathcal{M}/D(\mathcal{M})$ again. Moreover, it holds $\tilde{f} = (\tilde{f}^-)^-$.*

For an arbitrary $R\ell$-monoid $\mathcal{M}$ denote by $B(\mathcal{M})$ the set of all elements from M for which x^- is a complement of x in the lattice $(M; \vee, \wedge)$. Note that if $x \in B(\mathcal{M})$ then its complement x' in $(M; \vee, \wedge)$ is equal to x^-. By [8], Proposition 2.8.9, $B(\mathcal{M})$ is in fact a subalgebra of $\mathcal{M}$. It is known that if $\mathcal{M}$ is an MV-algebra then $B(\mathcal{M})$ is a Boolean algebra.

Corollary 4.3. *Let $(\mathcal{M}, f)$ be an interior $R\ell$-monoid. Then $B(\mathcal{M}/D(\mathcal{M}))$ as a subalgebra of $\mathcal{M}/D(\mathcal{M})$ endowed with the restriction of the operator $\tilde{f}^-$ on $B(\mathcal{M}/D(\mathcal{M}))$ is a topological Boolean algebra.*

Proof. It follows from [14], Theorem 5 where it is proved that the restriction of an arbitrary *ac*-operator on an arbitrary MV-algebra $\mathcal{A}$ on $B(\mathcal{A})$ is a closure operator on the Boolean algebra $B(\mathcal{A})$. ∎

Definition 4.2. Let F be a filter in an interior $R\ell$-monoid $(\mathcal{M}, f)$. Then F is called an *i-filter* (or *interior filter*) iff

(F3) $x \in F \Longrightarrow f(x) \in F$.

Theorem 4.4. *Let $(\mathcal{M}, f)$ be an interior $R\ell$-monoid, let F be its i-filter and let $\tilde{f} : \mathcal{M}/F \to \mathcal{M}/F$ be the mapping such that for each $x \in M$,*

$$\tilde{f}(x/F) := f(x)/F.$$

Then the $R\ell$-monoid $\mathcal{M}/F$ endowed with $\tilde{f}$ is an interior $R\ell$-monoid.

Proof. Let us consider $x, y \in M$ such that $x/F = y/F$. So we have $\langle x, y\rangle \in \Theta(F)$ or equivalently $(x \to y) \wedge (y \to x) \in F$. Hence we get $x \to y, y \to x \in F$ with regard to (F2) and further $f(x \to y), f(y \to x) \in F$ with regard to (F3). Acording to Lemma 3.2,

$$f(x \to y) \leq f(x) \to f(y), \quad f(y \to x) \leq f(y) \to f(x),$$

therefore also $f(x) \to f(y), f(y) \to f(x) \in F$. By Lemma 2.1 (vii),

$$f(y) \to f(x) \leq (f(x) \to f(y)) \to (f(y) \to f(x)),$$

thus $(f(x) \to f(y)) \to (f(y) \to f(x)) \in F$. Finally

$$(f(x) \to f(y)) \wedge (f(y) \to f(x)) =$$

$$(f(x) \to f(y)) \odot ((f(x) \to f(y)) \to (f(y) \to f(x))) \in F.$$

Therefore $\langle f(x), f(y)\rangle \in \Theta(F)$ and the unary operation $\tilde{f}$ is correctly defined on $\mathcal{M}/F$. Analogously as in the proof of Theorem 4.1, one can verify that $\tilde{f}$ satisfies conditions 1–4 from the defition of *mi*-operator on the $R\ell$-monoid, and so the proof is done. ∎

Corollary 4.5. *There is a one-to-one correspondence between the i-filters and the congruences of the interior $R\ell$-monoids.*

If $\mathcal{M}$ is an $R\ell$-monoid and $x \in M$ then the element x is said to be *regular* in $\mathcal{M}$ iff $x^{--} = x$. Let us denote by $R(\mathcal{M})$ the set of all regular elements in $\mathcal{M}$.

Let an $R\ell$-monoid $\mathcal{M}$ be normal. Then by [12], Theorem 7, it holds that $\mathcal{R}(\mathcal{M}) = (R(\mathcal{M}); \odot, \vee_{R(\mathcal{M})}, \wedge, \to, 0, 1)$, where $x \vee_{R(\mathcal{M})} y := (x \vee y)^{--}$ for each $x, y \in R(\mathcal{M})$ and the remaining operations are the restrictions of the original ones from M on $R(\mathcal{M})$, is an MV-algebra. Moreover, the MV-algebra $\mathcal{R}(\mathcal{M})$ is isomorphic with $\mathcal{M}/D(\mathcal{M})$. The mappings

$\varphi : R(\mathcal{M}) \to M/D(\mathcal{M})$ where $\varphi : x \mapsto x/D(\mathcal{M})$

and

$\psi : M/D(\mathcal{M}) \to R(\mathcal{M})$ where $\psi : y/D(\mathcal{M}) \mapsto y^{--}$

are the mutually inverse isomorphisms. From this we obtain the following theorem.

Theorem 4.6. *Let us consider an mi-operator f on an $R\ell$-monoid $\mathcal{M}$. Then the mapping $\hat{f} : R(\mathcal{M}) \to R(\mathcal{M})$ such that $\hat{f} : x \mapsto (f(x))^{--}$ is an mi-operator on the MV-algebra $\mathcal{R}(\mathcal{M})$.*

Let $\mathcal{M}$ be an $R\ell$-monoid. Let us denote by $I(\mathcal{M}) = \{a \in M : a \odot a = a\}$ the set of all *idempotent elements* in M.

Lemma 4.7. *If $\mathcal{M}$ is an $R\ell$-monoid, $a \in I(\mathcal{M})$ and $x \in \mathcal{M}$ then $a \wedge x = a \odot x$.*

Proof. See [8], Lemma 2.8.3. ■

Lemma 4.8. *If $\mathcal{M}$ is a normal $R\ell$-monoid and $a \in I(\mathcal{M})$ then $a^{--} \in I(\mathcal{M})$ and $a^- \oplus a^- = a^-$.*

Proof. For an arbitrary element $a \in I(\mathcal{M})$ we have

$$a^{--} \odot a^{--} = (a \odot a)^{--} = a^{--}$$

due to normality of $\mathcal{M}$. Moreover

$$a^- \oplus a^- = (a^{--} \odot a^{--})^- = (a \odot a)^{---} = a^-.$$

■

Theorem 4.9. *If $\mathcal{M}$ is an $R\ell$-monoid then $I(\mathcal{M})$ is a subalgebra of the reduct $(M; \odot, \vee, \wedge, 0, 1)$ of the $R\ell$-monoid $\mathcal{M}$.*

Proof. Closedness of $I(\mathcal{M})$ to the operation "$\wedge$" follows directly from Lemma 4.7. It is enough to check that $I(\mathcal{M})$ is closed to the operation "$\vee$". Let $a, b \in I(\mathcal{M})$. Since "$\odot$" is distributive over joins and meets, we have

$$(a \vee b) \odot (a \vee b) = (a \odot a) \vee (a \odot b) \vee (b \odot b) = a \vee b \vee (a \odot b) = a \vee b.$$

■

Since $I(\mathcal{M})$ is a subalgebra of the reduct $(M; \odot, \vee, \wedge, 0, 1)$, we denote by $B(I(\mathcal{M}))$ the set of all complemented elements in the lattice $(I(\mathcal{M}); \vee, \wedge)$. It is known that $B(\mathcal{M}) \subseteq I(\mathcal{M})$ and that if $\mathcal{M}$ is an MV-algebra then $I(\mathcal{M}) = B(\mathcal{M})$.

Theorem 4.10. *For each $R\ell$-monoid $\mathcal{M}$ we have $B(I(\mathcal{M})) = B(\mathcal{M})$.*

Proof. Clearly $0, 1 \in I(\mathcal{M})$, so 0 is the least and 1 the greatest element in the lattice $(I(\mathcal{M}); \vee, \wedge)$. Let $x \in B(I(\mathcal{M}))$. Then there exists an element $y \in I(\mathcal{M})$ such that $x \vee y = 1$ and $x \wedge y = 0$ in the lattice $(I(\mathcal{M}); \vee, \wedge)$. Since $I(\mathcal{M}) \subseteq M$ and since operations "$\vee$", "$\wedge$" in $(I(\mathcal{M}); \vee, \wedge)$ are restrictions of the "same" operations in $(M; \vee, \wedge)$, y is also a complement of x in the lattice $(M; \vee, \wedge)$. So $x \in B(\mathcal{M})$.

The converse inclusion follows from Lemma 2.8.8 of [8], where it is proved that $B(\mathcal{M}) \subseteq I(\mathcal{M})$. ■

Theorem 4.11. *If M is a BL-algebra then $I(\mathcal{M})$ is a subalgebra of the algebra $\mathcal{M}$ and, moreover, $I(\mathcal{M})$ is a Heyting algebra.*

Proof. Let $\mathcal{M}$ be a BL-algebra. To prove that $I(\mathcal{M})$ is a subalgebra of $\mathcal{M}$ it is by Theorem 4.9 enough to check that $I(\mathcal{M})$ is closed to "$\rightarrow$".

By [11], Theorem 1, an (bounded) $R\ell$-monoid is a BL-algebra if and only if it is a subdirect product of (bounded) linearly ordered $R\ell$-monoids (= BL-chains).

Let BL-algebra $\mathcal{M}$ be a subdirect product of BL-chains $\mathcal{M}_\alpha$ ($\alpha \in \Gamma$). Let $a = (a_\alpha; \alpha \in \Gamma) \in M$. It is clear that $a \in I(\mathcal{M})$ iff $a_\alpha \in I(\mathcal{M}_\alpha)$ for each $\alpha \in \Gamma$. At the same time, if $a_\alpha, b_\alpha \in I(\mathcal{M}_\alpha)$ then $a_\alpha \rightarrow b_\alpha = 1_\alpha$ for $a_\alpha \leq b_\alpha$. Since $(a_\alpha \rightarrow b_\alpha) \odot a_\alpha = (a_\alpha \rightarrow b_\alpha) \wedge a_\alpha$, we get $a_\alpha \rightarrow b_\alpha = b_\alpha$ for $a_\alpha > b_\alpha$ because, by Definition 2.1, $a_\alpha \rightarrow b_\alpha$ is the greatest element $c \in M$ such that $c \wedge a_\alpha = b_\alpha$. So if $a_\alpha, b_\alpha \in I(\mathcal{M}_\alpha)$ then $a_\alpha \rightarrow b_\alpha \in I(\mathcal{M}_\alpha)$. Thus, $I(\mathcal{M})$ is closed to "$\rightarrow$".

By [11], Proposition 4, $I(\mathcal{M})$ is moreover a Heyting algebra. ■

Theorem 4.12. *If $\mathcal{M}$ is a BL-algebra and if f is an mi-operator on $\mathcal{M}$ then $f(a) \in I(\mathcal{M})$ for each element $a \in I(\mathcal{M})$. The restriction of f on $I(\mathcal{M})$ is an mi-operator on the Heyting algebra $I(\mathcal{M})$.*

Proof. Since $I(\mathcal{M})$ is closed to "$\odot$", it is enough to show that $f(I(\mathcal{M})) \subseteq I(\mathcal{M})$. Let $a \in I(\mathcal{M})$. Then $f(a) = f(a \odot a) = f(a) \odot f(a)$, so $f(a) \in I(\mathcal{M})$. ■

Let $\mathcal{M}$ be an $R\ell$-monoid and f an *mi*-operator on $\mathcal{M}$. We will denote by $Fix(f) = \{x \in M : f(x) = x\}$ the set of all fixed points of the operator f. Since for an arbitrary element $x \in M$ the equality $f(f(x)) = f(x)$ holds, $Fix(f) = Im(f) = f(M)$.

Theorem 4.13. *Let $\mathcal{M}$ be an $R\ell$-monoid and f an mi-operator on $\mathcal{M}$. Then $Fix(f) = (Fix(f); \odot, \vee, \wedge_f, \rightarrow_f, 0, 1)$, where $x \wedge_f y := f(x \wedge y)$ and $x \rightarrow_f y := f(x \rightarrow y)$ for each $x, y \in Fix(f)$ and the other operations are restrictions of the original ones on $\mathcal{M}$ on $Fix(f)$, is an $R\ell$-monoid. Moreover, $f \,|\, Fix(f)$ is the identity on $Fix(f)$.*

Proof. Clearly $0, 1 \in Fix(f)$.

1. Let $x, y \in Fix(f)$. Then $f(x \odot y) = f(x) \odot f(y) = x \odot y$, so $x \odot y \in Fix(f)$. So $(Fix(f); \odot, 1)$ is a commutative monoid.

2. Further, f is an interior operator on the lattice $(M; \vee, \wedge)$, therefore $(Fix(f); \vee, \wedge_f, 0, 1)$ is a bounded lattice.

3. Let $x, y \in Fix(f)$. It holds that $f(x \to y) \leq x \to y$, therefore $f(x \to y) \odot x \leq y$. Let $u \in Fix(f)$ be such an element that $u \odot x \leq y$. Then $u \leq x \to y$ and so $u = f(u) \leq f(x \to y)$. Now, $x \to_f y = f(x \to y)$ is the greatest element among all $z \in Fix(f)$ such that $z \odot x \leq y$.

4. Let $x, y \in Fix(f)$. Then $x \odot (x \to_f y) = x \odot f(x \to y) =$

   $= f(x) \odot f(x \to y) = f(x \odot (x \to y)) = f(x \wedge y) = x \wedge_f y$.

Hence $Fix(f)$ is an $R\ell$-monoid and $f \,|\, Fix(f) = id_{Fix(f)}$. ∎

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