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THE LATTICE OF VARIETIES OF FIBERED AUTOMATA

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Abstract

The class of all fibered automata is a variety of two-sorted algebras. This paper provides a full description of the lattice of varieties of fibred automata.

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1. Introduction

Fibered automata were introduced by J.D.H. Smith [5] as certain two sorted algebras. A motivation came from a construction that Ivo Rosenberg used in the proof of his theorem: each clone on a set may be definied as the set of all operations preserving a single relation [4]. This construction may be formulated as the specification of the variety of fibered automata. Also the Division Algorithm and the Continuous Fraction Algorithm can both be modelled by a fibered automaton.

General properties of fibered automata and of the variety of fibered automata were studied in [5]. The aim of this paper is to describe the lattice of varieties of fibered automata.

Basic definitions and simple examples are considered in Section 2. In Section 3, we construct the word algebra of fibered automata type. Next, in Section 4, we show that each term operation in fibered automata can be expressed in a standard form. In Section 5, we construct free fibered automata, and in Section 6, we investigate identities satisfied by fibered automata. Finally, in Section 7, we prove the following theorem.

[**Main Theorem 7.2**]. *Let* $\overline{\mathbb{N}}$ *be the ordinal sum of a singleton with the lattice* $(\mathbb{N}, |)$ *of natural numbers (including* 0*) under divisibilty. Then the lattice of varieties of fibered automata is isomorphic to the ordinal sum of the product* $\overline{\mathbb{N}} \times \underline{2}$ *with a singleton.*

2. Fibered automata

In this section we recall basic definitions and facts concerning fibered automata.

Definition 2.1 [5]**.** A *fibered automaton* is a two-sorted algebra $(S, E, \mu, \delta, \varepsilon)$ (or briefly (S, E)) with a unary *maternal operation*

$$\delta : S \to S,$$

a unary *paternal operation*

$$\varepsilon : S \to E,$$

and a binary operation

$$\mu : S \times E \to S; (s, e) \mapsto s \cdot e,$$

such that the operation μ and the *parent map*

$$(\delta, \varepsilon) : S \to S \times E; s \mapsto (s\delta, s\varepsilon)$$

are mutual inverses. The set S is called the *state space* of the automaton, and its elements are called *states*. The set E is called the *event set* of the automaton, and its elements are called *elementary events*. For a state s, the state $s\delta$ is called its *mother*, and the event $s\varepsilon$ its *father*.

Theorem 2.2 [5]**.** *The mutual inverse relation between the operation* μ *and the parent map is equivalent to satisfaction of the following identities*

$$s = s\delta \cdot s\varepsilon, \tag{2.3}$$

$$(s \cdot e)\delta = s, \tag{2.4}$$

$$(s \cdot e)\varepsilon = e \tag{2.5}$$

for all $s \in S$ and all $e \in E$.

Corollary 2.6. *The class $\mathcal{A}$ of fibered automata forms a variety of two-sorted algebras.*

The *transition diagram* of a fibered automaton (S, E) is a directed graph on the vertex set S whose edges are labelled by the elementary events. For each $s \in S$ and $e \in E$ there exists an edge $s \xrightarrow{e} s \cdot e$, and each vertex has indegree one.

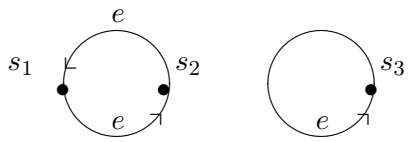
Example 2.7. Let $S = \{s_1, s_2, s_3\}$ and $E = \{e\}$. Define

$$s_1 \cdot e = s_2,\ s_2 \cdot e = s_1,\ s_3 \cdot e = s_3;$$

$$s_1\delta = s_2,\ s_2\delta = s_1,\ s_3\delta = s_3;$$

$$s_1\varepsilon = e,\ s_2\varepsilon = e,\ s_3\varepsilon = e.$$

The algebra $(S, E, \mu, \delta, \varepsilon)$ is a fibered automaton presented in the figure:



Example 2.8. Consider the set S of natural numbers given by their binary representations, and let $E = \{0, 1\}$. Let $c_0 \dots c_k$ denote the shortest binary representation of a natural number (so that $c_0 = 1$ unless $c_0 \dots c_n = 0$). Define

$$0\delta := 0 \text{ and } 1\delta := 0$$

and

$$(c_0 \dots c_k)\delta := c_0 \dots c_{k-1}, \text{ if } c_0 \dots c_k \notin \{0, 1\}.$$

Define

$$(c_0 \dots c_k)\varepsilon := c_k,$$

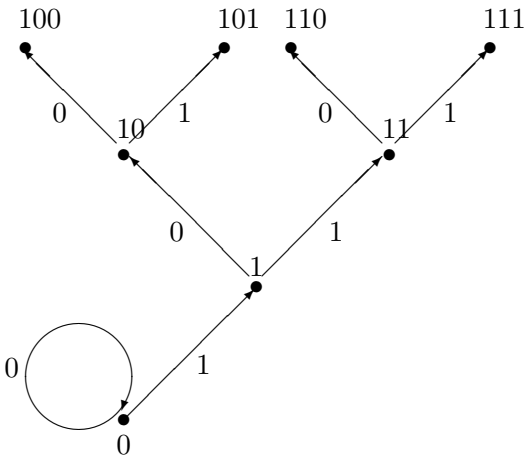
and for $e \in E$, define the operation μ by:

$$0 \cdot e := e;$$

$$c_0 \dots c_k \cdot e := c_0 \dots c_k e, \text{ if } c_0 \dots c_k \neq 0.$$

The algebra $(S, E, \mu, \delta, \varepsilon)$ is a fibered automaton with the following transition diagram:

...



Example 2.9. Let $m > 0$ be a natural number. Let C_m be a fibered automaton with the event set $E = \{e\}$ and the state space $S = \{s_0, \dots, s_{m-1}\}$ satisfying

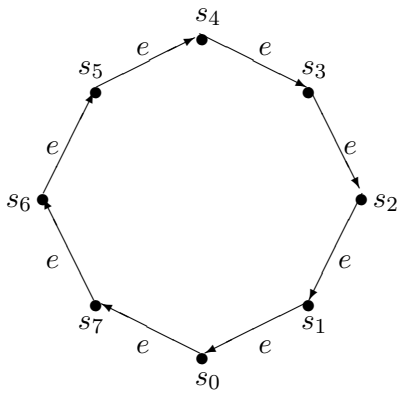
$$s_i\delta = s_{(i+1) \bmod m},$$

$$s_i \cdot e = s_{(i-1) \bmod m},$$

and

$$s_i \varepsilon = e.$$

In the following figure we present C_8



Further examples of fibered automata can be found in [5].

3. The word algebra

We will construct the word algebra of fibered automata type over two generator sets X and A. We define two sequences X_k and A_k, for $k = 0, 1, 2, \ldots$, of sets of words inductively as follows:

$$X_0 = X,$$

$$A_0 = A,$$

$$X_1 := \{w \mid w = x \cdot a \text{ or } w = x\delta, \text{ where } x \in X,\ a \in A\},$$

$$A_1 := \{t \mid t = x\varepsilon, \text{ where } x \in X\}.$$

Suppose that X_j and A_j are defined for $j < k$. Let

$$X_k := \{w\delta \mid w \in X_{k-1}\} \cup \left\{ w \cdot t \mid t \in \bigcup_{i<k} A_i, w \in X_{k-1} \right\}$$

$$\cup \left\{ w \cdot t \mid w \in \bigcup_{i<k-1} X_i, t \in A_{k-1} \right\},$$

$$A_k := \{w\varepsilon \mid w \in X_{k-1}\}.$$

Define

$$XW := \bigcup_{k\in\mathbb{N}} X_k,$$

$$AW := \bigcup_{k\in\mathbb{N}} A_k.$$

The operations μ, δ, ε are defined on (XW, AW) by

$$\mu\colon XW \times AW \to XW; (w,t) \mapsto w \cdot t,$$

$$\delta\colon XW \to XW; w \mapsto w\delta,$$

and

$$\varepsilon\colon XW \to AW; w \mapsto w\varepsilon.$$

Then the algebra $(XW, AW, \mu, \delta, \varepsilon)$ is the *word algebra* of fibered automata type over (X, A).

4. Term operations of fibered automata

This section shows that in fibered automata each term operation has a special form.

Terms of the sort XW will be called *XW-terms*, and terms of the sort AW will be called *AW-terms*. Denote by x or x_i the elements of X and by

a or a_j the elements of A. For positive integers n and r, we define $x\delta^0 := x$, $x\delta^n := (x\delta^{n-1})\delta$, and $xa_1 \dots a_r := (\dots((x \cdot a_1) \cdot a_2)\dots) \cdot a_r$. In particular $xa^r := (xa^{r-1}) \cdot a$.

Lemma 4.1. *For all $n \geq 1$ and $k, l \geq 0$, the following identities hold in the variety $\mathcal{A}$:*

1. $x\delta^n \cdot x\delta^{n-1}\varepsilon = x\delta^{n-1}$;
2. $(x\delta^k \cdot x\delta^l\varepsilon)\delta = x\delta^k$;
3. $(x\delta^k \cdot x\delta^l\varepsilon)\varepsilon = x\delta^l\varepsilon$;
4. $(xa_1 \dots a_r)\delta^n = \begin{cases} x\delta^{n-r}, & \text{if } n > r; \\ xa_1 \dots a_{r-n}, & \text{if } n < r; \\ x, & \text{if } n = r, \end{cases}$
5. $(xa_1 \dots a_r)\delta^n\varepsilon = \begin{cases} x\delta^{n-r}\varepsilon, & \text{if } n > r; \\ a_{r-n}, & \text{if } n < r; \\ x\varepsilon, & \text{if } n = r. \end{cases}$

We omit the easy proofs.

Lemma 4.2. *In the variety of fibered automata each XW-term can be expressed in the standard form. It can be written as*

$$x\delta^n$$

or

$$x\delta^n t_1 \dots t_m = (\dots(x\delta^n \cdot t_1) \cdot \dots) \cdot t_m,$$

for some AW-terms $t_1, \dots, t_m$, where $n \in \mathbb{N}$ and $m \in \mathbb{Z}^+$.

Proof. The proof is by induction on k.

If $v \in X_0$ or X_1, then v has the required form. Assume that the lemma holds for all words in X_j with $j < k$, and let $v \in X_k$. Then v may have one of the three forms given in the definition of X_k with $w \in X_{k-1}$ or $w \in \bigcup_{i<k-1} X_i$ having the required form.

If $v = w \cdot t$ for some $t \in A_l$ and $l < k$, then the term w is

$$x\delta^n$$

or

$$x\delta^n t_1 \dots t_m$$

for some AW-terms $t_1, \dots, t_m$. Then clearly, the word v has the required form in both cases. Similarly, one shows that $v = w \cdot t$ where $w \in X_i$ for some $i < k-1$ and for $t \in A_{k-1}$ has the required form.

Otherwise, $v = w\delta$. If

$$w = x\delta^n,$$

then

$$v = x\delta^{n+1}.$$

If

$$w = x\delta^l t_1 \dots t_m$$

for some AW-words $t_1, \dots, t_m$, then by Lemma 4.1 (4)

$$v = x\delta^l t_1 \dots t_{m-1}.$$

It follows that all XW-terms have the required form. ■

Lemma 4.3. *In the variety of fibered automata each AW-term can be expressed in the standard form. It belongs to the set A or can be written as:*

$$x\delta^n\varepsilon$$

for some $n \in \mathbb{N}$.

Proof. The proof is by induction on k. If $t \in A_1$, then t has the required form. Assume that the Lemma holds for all words in A_j for $j < k$, and let $t \in A_k$. Then $t = w\varepsilon$ where $w \in X_{k-1}$. By Lemma 4.2 the term w is expressed as $x\delta^n$ or $\delta^n t_1 \dots t_m$ for some AW-terms $t_1, \dots, t_m$ (given in the required form). In the first case

$$t = x\delta^n\varepsilon.$$

In the second case Lemma 4.1 (5) implies that

$$t = (x\delta^n t_1 \dots t_m)\varepsilon = t_m,$$

again giving the required form. ■

For $X = \{x_1, \dots, x_k\}$ and $A = \{a_1, \dots, a_l\}$, we have the following sets of standard terms

$$A \cup \{x_i\delta^n\varepsilon \mid 1 \le i \le k,\, n \in \mathbb{N}\} (\subseteq AW)$$

and

$$\{x_i\delta^n \mid 1 \le i \le k,\, n \in \mathbb{N}\}$$

$$\cup \{x_i\delta^n t_1 \dots t_m \mid 1 \le i \le k,\, n \in \mathbb{N},\, t_i \in AW\} (\subseteq XW).$$

In particular, for $X = \{x\}$ and empty set A, we have the following sets of standard terms

$$\{x\delta^n\varepsilon \mid n \in \mathbb{N}\} (\subseteq AW)$$

and

$$\{x\delta^n \mid n \in \mathbb{N}\} \cup \{(x\delta^n)(x\delta^{n_1}\varepsilon)\dots(x\delta^{n_k}\varepsilon) \mid n, n_i \in \mathbb{N}\} (\subseteq XW).$$

5. Free fibered automata

We define a two-sorted algebra $(XF, AF, \mu, \delta, \varepsilon)$ as follows:

$$\begin{aligned} AF :=& A \cup \{x\delta^k\varepsilon \mid x \in X,\, k \in \mathbb{N}\}, \\ XF :=& \{x\delta^k \mid x \in X,\, k \in \mathbb{N}\} \\ & \cup \{x\delta^k t_1 \dots t_m \mid x \in 0,\, k \in \mathbb{N},\, m \in \mathbb{Z}^+, \\ & t_1, \dots, t_m \in AF,\, t_1 \neq x\delta^{k-1}\varepsilon\}. \end{aligned}$$

Define a binary operation $\mu : XF \times AF \to XF$ by

$$(x\delta^k, t)\mu := \begin{cases} x\delta^{k-1}, \text{ if } t = x\delta^{k-1}\varepsilon; \\ x\delta^k \cdot t, \text{ otherwise} \end{cases}$$

and

$$(x\delta^k t_1 \dots t_m, t)\mu := (x\delta^k t_1 \dots t_m) \cdot t.$$

Define unary operations δ and ε by

$$\delta \colon XF \to XF; w \mapsto \begin{cases} w\delta, & \text{if } w = x\delta^n; \\ v, & \text{if } w = v \cdot t \end{cases}$$

and

$$\varepsilon \colon XF \to AF; w \mapsto \begin{cases} w\varepsilon, & \text{if } w = x\delta^n; \\ t, & \text{if } w = v \cdot t \end{cases}$$

where $n \in \mathbb{N}$, $v \in XF$ and $t \in AF$.

Proposition 5.1. *The algebra $(XF, AF, \mu, \delta, \varepsilon)$ is the free fibered automaton over (X, A).*

Proof. Clearly, the algebra (XF, AF) satisfies the identities (2.3), (2.4) and (2.5). Hence it is a fibered automaton. And obviously, it is generated by (X, A).

Let (S, E) be a fibered automaton, and let $f = (f_X, f_A)$ be a pair of functions:

$$f_X \colon X \to S; x \mapsto xf_X,$$

$$f_A \colon A \to E; a \mapsto af_A.$$

Define the mapping $\overline{f} = (\overline{f_X}, \overline{f_A})$, where $\overline{f_X} : XF \to S$ and $\overline{f_A} : AF \to E$, by

$$x\overline{f_X} = xf_X \text{ for } x \in X,$$

$$a\overline{f_A} = af_A \text{ for } a \in A,$$

and for more complicated terms by

$$x\delta^k\varepsilon\overline{f_A} = xf_X\delta^k\varepsilon,$$

$$x\delta^k\overline{f_X} = xf_X\delta^k,$$

$$x\delta^k t_1 \dots t_m\overline{f_X} = ((xf_X)\delta^k)(t_1\overline{f_A}) \dots (t_m\overline{f_A}).$$

The proof that $\overline{f}$ is a homomorphism uniquely extending f is routine, and we omit obvious though long calculations. ∎

6. Identities

In this section, we describe all types of equations that may be satisfied by fibered automata. As in all fibered automata, each word of fibered automata type equals some word in the standard form, it is sufficient to consider only pairs of such terms and investigate relationships between them.

We set $X = \{x_1, x_2, \dots\}$ and $A = \{a_1, a_2, \dots\}$, but will denote the elements of the first set also by x, y etc. and the elements of the second set by a, b etc. First note that for an equation $t = w$ to be satisfied by fibered automata it is necessary that both terms t and w are XW-terms or both are AW-terms. The first type will be called an *XW-equation* or an *XW-identity*, and the second an *AW-equation* or an *AW-identity*.

First recall that standard AW-terms are just variables $a, b, \dots$ of the event sort and terms $x\delta^n\varepsilon$ with x a variable of the state sort.

Proposition 6.1. *Let $k, l \in \mathbb{N}$ with $l > k$. The following AW-identities are equivalent in each fibered automaton (S, E) with $|S| > 0$.*

$$x\delta^k\varepsilon = y\delta^l\varepsilon, \tag{6.2}$$

$$x\delta^k\varepsilon = y\delta^k\varepsilon, \tag{6.3}$$

$$x\delta^k\varepsilon = x\delta^l\varepsilon, \tag{6.4}$$

$$x\delta^k\varepsilon = a, \tag{6.5}$$

$$a = b, \tag{6.6}$$

$$x\varepsilon = y\varepsilon. \tag{6.7}$$

Proof. (6.7) ⇒ (6.3) This is obvious.
(6.3) ⇒ (6.7) Assume that (6.3) holds. By Lemma 4.1 (4), any x, y can be presented in the form $x = xa^k\delta^k$ and $y = ya^k\delta^k$, whence

$$x\varepsilon = xa^k\delta^k\varepsilon = ya^k\delta^k\varepsilon = y\varepsilon.$$

(6.4) ⇒ (6.7) Assume that (6.4) holds. By Lemma 4.1 (4), any y can be presented in the form $y = ya^k\delta^k$, whence

$$y\varepsilon = ya^k\delta^k\varepsilon = ya^k\delta^l\varepsilon = y\delta^{l-k}\varepsilon.$$

Then by (2.5)

$$x\varepsilon = (y\delta \cdot x\varepsilon)\varepsilon = (y\delta \cdot x\varepsilon)\delta^{l-k}\varepsilon = y\delta^{l-k}\varepsilon = y\varepsilon.$$

The implications (6.7) ⇒ (6.2) and (6.2) ⇒ (6.4) are obvious, and so are (6.7) ⇒ (6.6) ⇒ (6.5) ⇒ (6.4). As (6.4) implies (6.7), this completes the proof. ■

Proposition 6.8. *The identities* (6.2)–(6.5) *and* (6.7) *are equivalent in each fibered automaton.*

Proof. If a fibered automaton has an empty state space, then each of these identities is satisfied. Note that an automaton $(\varnothing, E)$ with $|E| \geq 2$ satisfies the identity (6.7), but does not satisfy (6.6). On the other hand, if an automaton satisfies (6.6), then it satisfies (6.7) too. ■

The variety $\mathcal{P}$ of fibered automata defined by any of the equivalent identities of Proposition 6.8 is called the variety of *permutational* fibered automata (see [5]). The subvariety defined by $a = b$ is denoted by $\mathcal{P}_0$. The event set of each fibered automaton $(S, E) \in \mathcal{P}_0$ is empty or it consists of precisely one element. The state space of each permutational automaton is empty or the event set is a singleton. Note that the identities of Proposition 6.1 form all types of AW-equations.

Standard XW-terms are just variables $x, y, \ldots$ of the state sort, terms $x\delta^n$, and terms $x\delta^k t_1 \ldots t_m$ with standard AW-subterms $t_1, \ldots, t_m$.

Proposition 6.9. *The following XW-identities are equivalent in each fibered automaton for any $k, l, j \in \mathbb{N}$, $m, n \in \mathbb{Z}^+$ with $m \geq n$, and standard AW-terms t_i, t'_i.*

$$x\delta^k = y\delta^l, \tag{6.10}$$

$$x\delta^k t_1 \ldots t_m = y\delta^l t'_1 \ldots t'_n, \tag{6.11}$$

$$x\delta^k t_1 \ldots t_m = y\delta^j, \tag{6.12}$$

$$x = y. \tag{6.13}$$

Proof. If a fibered automaton has an empty state space then each XW-identity is satisfied and all are equivalent.

Let (S, E) be a fibered automaton with non-empty state space.

(6.10) $\Rightarrow$ (6.13) Assume that (6.10) holds. Hence and by Lemma 4.1 (4) we have:

$$x = xa^k\delta^k = ya^l\delta^l = y.$$

(6.13) $\Rightarrow$ (6.10), (6.11), (6.12). Note that (6.13) implies $|S| = 1$. Hence any substitution of elements of S and E for variables of any side of first three equations provides elements of S, and the three identities hold.

(6.11) $\Rightarrow$ (6.13) Assume that a fibered automaton satisfies (6.11). Hence and by Lemma 4.1 (4), we have

$$x\delta^k = (x\delta^k t_1 \ldots t_m)\delta^m = (y\delta^l t'_1 \ldots t'_n)\delta^m = y\delta^{m-n}.$$

And by (6.13) $\Leftrightarrow$ (6.10) we have $x = y$.

(6.12) $\Rightarrow$ (6.13) Assume that a fibered automaton satisfies (6.12). Hence and by Lemma 4.1 (4), we have

$$x\delta^k = (x\delta^k t_1 \dots t_m)\delta^m = y\delta^j\delta^m = y\delta^{j+m}.$$

And by (6.13) $\Leftrightarrow$ (6.10) we have $x = y$. ∎

Definition 6.14. *The* trivial fibered automata *are* $(\varnothing, \varnothing)$, $(\varnothing, \{e\})$, *and* $(\{s\}, \{e\})$.

Corollary 6.15. *If a fibered automaton* (S, E) *has a non-empty state space and satisfies any of the equivalent identities* (6.10)–(6.13)*, then it is trivial.*

Proof. By Propositoin 6.9 (S, E) satisfies $x = y$, hence the set S has only one element. By (6.13), (S, E) satisfies $x\varepsilon = y\varepsilon$. Hence and by Proposition 6.1 also $a = b$ holds in (S, E). Hence the set E has only one element. ∎

Proposition 6.16. *For any* $m, n, s \in \mathbb{Z}^+$ *and* $k \in \mathbb{N}$ *such that* $m \leq k + s$*, and any standard form* AW*-terms* t_i *and* t'_i*, the following* XW*-identities are equivalent in each fibered automaton* (S, E)*.*

$$x\delta^s = x, \tag{6.17}$$

$$x\delta^k = x\delta^{s+k}, \tag{6.18}$$

$$x\delta^k t_1 \dots t_m = x\delta^{k+s-m}, \tag{6.19}$$

$$x\delta^k t_1 \dots t_m = x\delta^{k+s-m+n} t'_1 \dots t'_n. \tag{6.20}$$

Proof. Since all XW-identities hold in any fibered automaton with an empty state space, let us consider any fibered automaton (S, E) such that $S \neq \varnothing$.

(6.18) $\Rightarrow$ (6.17) Assume that a fibered automaton (S, E) satisfies (6.18).

By Lemma 4.1 (4), we have $x = xa^k\delta^k$, whence

$$x = xa^k\delta^k = xa^k\delta^{s+k} = xa^k\delta^k\delta^s = x\delta^s.$$

(6.17) $\Rightarrow$ (6.18) Assume that (S, E) satisfies (6.17). Then

$$x\delta^k = x\delta^k\delta^s = x\delta^{k+s}.$$

(6.18) $\Rightarrow$ (6.19) Assume that (S, E) satisfies (6.18). First note that $x\delta^k\varepsilon = x\delta^l\varepsilon$. Then by Proposition 6.1, $|E| = 1$, and we have $a = t_1 = \cdots = t_m$. Hence and by (2.3)

$$x\delta a = x\delta \cdot x\varepsilon = x,$$

and then for all natural j, l with $j > 1$, and $0 < l \leq j$ we have

$$x\delta^j a^l = (x\delta^{j-1}\delta)aa^{l-1} = (x\delta^{j-1}\delta)(x\delta^{j-1}\varepsilon)a^{l-1} = x\delta^{j-1}a^{l-1}$$

$$= \ldots = x\delta^{j-l}.$$

Hence

$$x\delta^k t_1 \ldots t_m = x\delta^{k+s}t_1 \ldots t_m = x\delta^{k+s}a^m = x\delta^{k+s-m}.$$

(6.19) $\Rightarrow$ (6.18) Assume that (S, E) satisfies (6.19). Hence and by Lemma 4.1 (4), we have:

$$x\delta^k = (x\delta^k t_1 \ldots t_m)\delta^m = x\delta^{k+s-m}\delta^m = x\delta^{k+s}.$$

(6.18) $\Rightarrow$ (6.20) Assume that (S, E) satisfies (6.18). As in the proof of (6.19) $\Leftrightarrow$ (6.18) we have $a = t_1 = \cdots = t_m = t'_1 = \ldots = t'_n$, and for all natural $j > 1$ and $0 < l \leq j$ we have

$$x\delta^j a^l = x\delta^{j-l}.$$

Hence

$$x\delta^k t_1 \dots t_m = x\delta^{k+s} t_1 \dots t_m = x\delta^{k+s} a^m = (x\delta^{k+s} a^m)\delta^n a^n$$

$$= x\delta^{k+s-m}\delta^n a^n = x\delta^{k+s-m+n} a^n = x\delta^{k+s-m+n} t'_1 \dots t'_n.$$

(6.20) $\Rightarrow$ (6.18) Assume that (S, E) satisfies (6.20). Hence and by Lemma 4.1 (4), we have:

$$x\delta^k = (x\delta^k t_1 \dots t_m)\delta^m = (x\delta^{k+s-m+n} t'_1 \dots t'_n)\delta^m$$

and if $m \geq n$

$$(x\delta^{k+s-m+n} t'_1 \dots t'_n)\delta^m = (x\delta^{k+s-m+n} t'_1 \dots t'_n)\delta^n \delta^{m-n} = x\delta^{k+s}.$$

Otherwise, since $a = t_1 = \dots = t_m = t'_1 = t'_n$ and

$$x\delta^j a^l = x\delta^{j-l},$$

for all natural $j > 1$ and $0 < l \leq j$, we have

$$(x\delta^{k+s-m+n} t'_1 \dots t'_n)\delta^m = x\delta^{k+s-m+n} t'_1 \dots t'_{n-m}$$

$$= x\delta^{k+s-m+n} a^{n-m} = x\delta^{k+s}.$$

■

Lemma 6.21. *If a fibered automaton (S, E) satisfies any non-trivial XW-identity, then it is a permutational fibered automaton. Moreover, if $|S| > 0$, then $(S, E) \in \mathcal{P}_0$.*

Proof. By Propositions 6.9 and 6.16, it is sufficient to consider only the identities (6.13) or (6.17).

Let (S, E) satisfy (6.13). Then $x\varepsilon = y\varepsilon$.

Let (S, E) satisfy (6.17) for some $s \in \mathbb{N}$. Then $x\delta^s\varepsilon = x\varepsilon$. Hence and by Proposition 6.8, we have $x\varepsilon = y\varepsilon$.

By Proposition 6.1, if $|S| > 0$, then (S, E) satisfies $a = b$, and hence $(S, E) \in \mathcal{P}_0$. ■

Let $n \in \mathbb{Z}^+$, and let $\mathcal{A}_n$ be the class of fibered automata defined by the identity $x\delta^n = x$, and let $\mathcal{A}_n^0$ be the class of fibered automata defined by the identities $x\delta^n = x$ and $a = b$.

Proposition 6.22. *If $m \neq n \in \mathbb{N}$, then $\mathcal{A}_n \neq \mathcal{A}_m$ and $\mathcal{A}_n^0 \neq \mathcal{A}_m^0$.*

Proof. Assume that $m > n$. Let C_m be the fibered automaton defined in Example 2.9. Then for each $i = 0, \ldots, m-1$ we have $s_i\delta^m = s_i$ and $s_0\delta^n = s_n$. The automaton C_m satisfies $a = b$ and $x\delta^m = x$, and does not satisfy $x\delta^n = x$. Hence $\mathcal{A}_n \neq \mathcal{A}_m$ and $\mathcal{A}_n^0 \neq \mathcal{A}_m^0$. ■

Proposition 6.23. *For each $n \in \mathbb{Z}^+$, $\mathcal{A}_n^0 \subset \mathcal{A}_n$ and $\mathcal{A}_n^0 \nsubseteq \mathcal{A}_m$ for $m < n$.*

Proof. It is obvious that $\mathcal{A}_n^0 \subseteq \mathcal{A}_n$. A fibered automaton with empty state space and at least two element event set lies in $\mathcal{A}_n$, but not in $\mathcal{A}_n^0$. Hence $\mathcal{A}_n^0 \neq \mathcal{A}_n$.

Clearly $C_n \in \mathcal{A}_n^0$ and $C_n \notin \mathcal{A}_m$, and hence $\mathcal{A}_n^0 \nsubseteq \mathcal{A}_m$. ■

Theorem 6.24. [5] *If a fibered automaton $(S, E, \mu, \delta, \varepsilon)$ is permutational, then the maternal operation $\delta\colon X \to X$ bijects.*

Lemma 6.25. *A fibered automaton (S, E) satisfies the identities $x\delta^n = x$ and $x\delta^m = x$ for $m \neq n$ if and only if it satisfies the identity $x\delta^{GCD(n,m)} = x$.*

Proof. ($\Leftarrow$) Let (S, E) satisfy $x\delta^{GCD(n,m)} = x$. Then

$$x\delta^n = x\delta^{GCD(n,m)(n/GCD(n,m))} = x\delta^{GCD(n,m)} \ldots \delta^{GCD(n,m)} = x.$$

Similarly, one shows that the second identity also holds.

($\Rightarrow$) Let (S, E) satisfy $x\delta^n = x$ and $x\delta^m = x$. (S, δ) is a dynamical system. The operation δ generates the free group $F_G(\delta) \cong \mathbb{Z}$ acting on S. The equation $x\delta^n = x$ induces the relation $\delta^n = 1$ on $F_G(\delta)$. There are integers $r, s \in \mathbb{Z}$ such that $GCD(n, m) = rm + sn$. As $\delta^m = 1$ and $\delta^n = 1$, it follows that $\delta^{GCD(,n,m)} = \delta^{rm+sn} = 1$. This means that $x\delta^{GCD(,n,m)} = x$. ■

Corollary 6.26. *The inclusions $\mathcal{A}_n \subseteq \mathcal{A}_m$ and $\mathcal{A}_n^0 \subseteq \mathcal{A}_m^0$ hold if and only if $n|m$*

Proof. ($\Leftarrow$) This is obvious.

($\Rightarrow$) Suppose that n does not divide m. Then $C_n \in \mathcal{A}_n^0 \subseteq \mathcal{A}_n$ but $C_n \notin \mathcal{A}_m^0 \subseteq \mathcal{A}_m$, a contradiction. Hence $n|m$. ■

Let $\mathcal{A}_0$ be the variety of fibered automata definied by $x = y$.

Lemma 6.27. *Each non-trivial, proper subvariety $\mathcal{D}$ of the variety $\mathcal{P}$ of permutational fibered automata coincides with $\mathcal{P}_0$ or $\mathcal{A}_0$, or with one of $\mathcal{A}_n$, or $\mathcal{A}_n^0$.*

Proof. If a subvariety $\mathcal{D}$ does not satisfy any of the non-trivial XW-identities and does not satisfy $a = b$, then $\mathcal{D} = \mathcal{P}$. If $\mathcal{D}$ does not satisfy any of the non-trivial XW-identities but satisfies $a = b$, then $\mathcal{D} = \mathcal{P}_0$. Assume that $\mathcal{D}$ satisfies satisfies some XW-identity. By Propositions 6.9 and 6.16, all such identities have the form $x\delta^s = x$ where $s \in S$, and S is a subset of $\mathbb{Z}^+$ or $x = y$. If $\mathcal{D}$ satisfies $x = y$ and $a = b$, then it is trivial. If $\mathcal{D}$ satisfies $x = y$ and does not satisfy $a = b$, then it coincides with $\mathcal{A}_0$. If D does not satisfy $x = y$, then let s_0 be the smallest number in the set S. By Lemma 6.25, the variety $\mathcal{D}$ satisfies $x\delta^{GCD(s,s_0)} = x$ for each $s \in S$. Hence $GCD(s, s_0) \in S$. As s_0 is the smallest number in S, $GCD(s, s_0) = s_0$. By Lemma 6.25, if the variety $\mathcal{D}$ satisfies $a = b$, then it is defined by $x\delta^{s_0} = x$ and $a = b$. Otherwise it is defined by $x\delta^{s_0} = x$. ■

7. The lattice of varieties of fibered automata

In this section we construct the lattice of varieties of fibered automata.

Lemma 7.1. *The following posets are isomorphic:*

$$(\{\mathcal{A}_n^0 \mid n \in \mathbb{Z}^+\} \cup \{\mathcal{P}_0\}, \subseteq) \cong (\mathbb{N}, |)$$

and

$$(\{\mathcal{A}_n \mid n \in \mathbb{Z}^+\} \cup \{\mathcal{P}\}, \subseteq) \cong (\mathbb{N}, |).$$

Proof. Consider the maps

$$i\colon \{\mathcal{A}_n \mid n \in \mathbb{Z}^+\} \cup \{\mathcal{P}\} \to \mathbb{N}; \begin{cases} \mathcal{A}_n \mapsto n, \\ \mathcal{P} \mapsto 0 \end{cases}$$

and

$$j\colon \{\mathcal{A}_n^0 \mid n \in \mathbb{Z}^+\} \cup \{\mathcal{P}_0\} \to \mathbb{N}; \begin{cases} \mathcal{A}_n^0 \mapsto n, \\ \mathcal{P}_0 \mapsto 0. \end{cases}$$

The maps

$$i|_{\{\mathcal{A}_n \mid n \in \mathbb{Z}^+\}}\colon \{\mathcal{A}_n \mid n \in \mathbb{Z}^+\} \to \mathbb{Z}^+$$

and

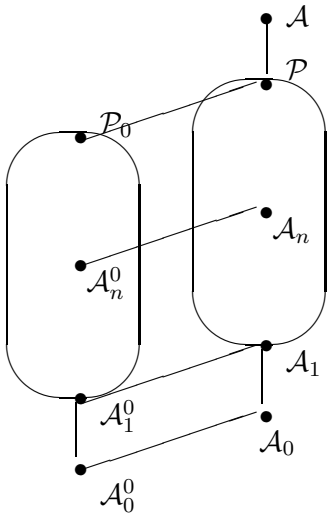
$$j|_{\{\mathcal{A}_n^0 \mid n \in \mathbb{Z}^+\}}\colon \{\mathcal{A}_n^0 \mid n \in \mathbb{Z}^+\} \to \mathbb{Z}^+$$

are bijections and, by Corollary 6.26, order preserving. Hence they are order-isomorphisms. Lemma 6.21 shows that $\mathcal{P}$ is the greatest element in the poset $(\{\mathcal{A}_n \mid n \in \mathbb{Z}^+\} \cup \{\mathcal{P}\}, \subseteq)$, and that $\mathcal{P}_0$ is the greatest element in $(\{\mathcal{A}_n^0 \mid n \in \mathbb{Z}^+\} \cup \{\mathcal{P}_0\}, \subseteq)$, similarly as 0 in the poset $(\mathbb{N}, |)$. ■

Let $\mathcal{A}_0^0$ consist of trivial fibered automata. Then $\mathcal{A}_0^0 \leq \mathcal{A}_0$, $\mathcal{A}_0^0 \leq \mathcal{A}_n^0$, and $\mathcal{A}_0 \leq \mathcal{A}_n$ for each positive integer n.

Let $\underline{2}$ be a two element lattice. Note that both posets of Lemma 7.1 are isomorphic to the lattice $\mathbb{N}$ of natural numbers with the divisibility relation as the ordering relation.

Theorem 7.2. *Let $\overline{\mathbb{N}}$ be the ordinal sum of a singleton with the lattice $(\mathbb{N}, |)$ of natural numbers (including 0) under divisibilty. Then the lattice of varieties of fibered automata is isomorphic to the ordinal sum of the product $\overline{\mathbb{N}} \times \underline{2}$ with a singleton.*



Proof. Lemma 6.27 we show that the varieties $\mathcal{A}_n$, $\mathcal{A}_n^0$ and $\mathcal{P}_0$ are all non-trivial, proper subvarieties of $\mathcal{P}$. By Lemma 7.1, the lattices $(\{\mathcal{A}_n \mid n \in \mathbb{Z}^+\} \cup \{\mathcal{P}\}, \subseteq)$ and $(\{\mathcal{A}_n^0 \mid n \in \mathbb{Z}^+\} \cup \{\mathcal{P}_0\}, \subseteq)$ are isomorphic to the lattice $(\mathbb{N}, |)$. By Proposition 6.23, for each $n \in \mathbb{N}$, $\mathcal{A}_n^0 \subset \mathcal{A}_n$, and $\mathcal{P}_0 \subset \mathcal{P}$. ■

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