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SEMILATTICES WITH SECTIONAL MAPPINGS

Ivan Chajda

Department of Algebra and Geometry,
Palacký University of Olomouc
Tomkova 40, 779 00 Olomouc, Czech Republic

e-mail: chajda@risc.upol.cz

AND

Günther Eigenthaler

Institute of Discrete Mathematics and Geometry,
Vienna University of Technology
Wiedner Hauptstraße 8–10/104, 1040 Wien, Austria

e-mail: g.eigenthaler@tuwien.ac.at

Abstract

We consider join-semilattices with 1 where for every element p a mapping on the interval $[p, 1]$ is defined; these mappings are called sectional mappings and such structures are called semilattices with sectional mappings. We assign to every semilattice with sectional mappings a binary operation which enables us to classify the cases where the sectional mappings are involutions and / or antitone mappings. The paper generalizes results of [3] and [4], and there are also some connections to [1].

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In the whole paper we deal with join-semilattices with a greatest element 1, denoted by $\mathcal{S} = (S; \vee, 1)$. Its induced order will be denoted by $\leq$. For $p \in S$, the interval $[p, 1]$ of $\mathcal{S}$ is called a *section*. A mapping of $[p, 1]$ into itself is called a *sectional mapping*. To discern such mappings for distinct sections, we will use the notation $x \mapsto x^p$ for $x \in [p, 1]$. If $\mathcal{S}$ is endowed with a sectional mapping on every section, $\mathcal{S}$ will be called a *semilattice with sectional mappings*.

For a semilattice $\mathcal{S} = (S; \vee, 1)$ with sectional mappings we define the so-called *assigned operation* $\circ$ as follows:

$$x \circ y := (x \vee y)^y.$$

Since $x \vee y \in [y, 1]$, $\circ$ is well-defined on S and the new structure will be denoted by $\mathcal{S}^* = (S; \vee, \circ, 1)$.

It is easily seen that the following properties hold in $\mathcal{S}^*$:

(1) $x \circ y \in [y, 1]$, i.e., $(x \circ y) \vee y = x \circ y$ for all $x, y \in \mathcal{S}$;

(2) $(x \vee y) \circ y = x \circ y$ for all $x, y \in \mathcal{S}$;

(3) $x \circ y = x^y$ for all $y \in \mathcal{S}$ and $x \in [y, 1]$.

Conversely, let $\mathcal{S}^* = (S; \vee, \circ, 1)$ be a join-semilattice with a greatest element 1 and a binary operation $\circ$ satisfying (1) and (2). If we define $x^p := x \circ p$ for any $p \in \mathcal{S}$ and $x \in [p, 1]$, then $\mathcal{S} = (S; \vee, 1)$ is a semilattice with sectional mappings.

Furthermore, using (1)–(3) one can easily check that the mappings $\mathcal{S} \mapsto \mathcal{S}^*$ and $\mathcal{S}^* \mapsto \mathcal{S}$ are inverse bijections between the class of all semilattices with sectional mappings and the variety of all structures $(S; \vee, \circ, 1)$ where $(S; \vee, 1)$ is a join-semilattice with 1 and $\circ$ satisfies (1) and (2).

A sectional mapping $x \mapsto x^p$ on $[p, 1]$ is called a *switching mapping* if the following holds:

$$x^p = p \quad \text{if and only if} \quad x = 1,$$

$$x^p = 1 \quad \text{if and only if} \quad x = p.$$

We say that $\mathcal{S}$ is a semilattice with *sectional switching mappings* if $\mathcal{S}$ is a semilattice with sectional mappings which all are switching mappings.

Lemma 1. *Let* $\mathcal{S}^* = (S; \vee, \circ, 1)$ *be the algebra assigned to a semilattice with sectional switching mappings. Then* $\mathcal{S}^*$ *satisfies the following:*

(4) $x \leq y$ *if and only if* $x \circ y = 1$;

(5) $x \vee y = 1$ *if and only if* $x \circ y = y$;

(6) $x \circ x = 1,\ x \circ 1 = 1$ *and* $1 \circ x = x$.

Conversely, suppose that $\mathcal{S}^*$ *satisfies* (4)–(6), *then* $\mathcal{S} = (S; \vee, 1)$ *is a semilattice with sectional switching mappings.*

Proof. If $x \leq y$ then $x \vee y = y$ and hence $x \circ y = (x \vee y)^y = y^y = 1$.

Conversely, $x \circ y = 1$ yields $(x \vee y)^y = 1$, from which we infer $x \vee y = y$ thus $x \leq y$.

If $x \vee y = 1$ then $x \circ y = (x \vee y)^y = 1^y = y$. Conversely, $x \circ y = y$ yields $(x \vee y)^y = y$, which implies $x \vee y = 1$.

Further, $1 \circ x = x$ follows from (5), and $x \circ x = 1$, $x \circ 1 = 1$ follow from (4).

The second statement of the Lemma follows straightforward from (4)–(6). ■

In what follows, we call also $\mathcal{S}^* = (S; \vee, \circ, 1)$ a semilattice with sectional (switching) mappings in case that it corresponds to such a structure by the bijection defined above.

For an algebra $\mathcal{A} = (A; F)$, denote by Con $\mathcal{A}$ the lattice of all congruences on $\mathcal{A}$ (with respect to set inclusion), i.e., the lattice of all equivalence relations Θ on A which are subalgebras of the algebra $\mathcal{A} \times \mathcal{A}$. In particular, for a semilattice $\mathcal{S}^* = (S; \vee, \circ, 1)$ with sectional mappings, Con $\mathcal{S}^*$ denotes the congruence lattice of $\mathcal{S}^*$.

Note that the meet $\wedge$ in the lattice Con $\mathcal{A}$ is given by the set-theoretical intersection, i.e., $\Theta_1 \wedge \Theta_2 = \Theta_1 \cap \Theta_2$ for all $\Theta_1, \Theta_2 \in$ Con $\mathcal{A}$. For $x \in A$ and $\Theta \in$ Con $\mathcal{A}$, $[x]\Theta$ denotes the congruence class of x w.r.t. Θ.

An algebra $\mathcal{A} = (A; F)$ with a constant 1 is called *distributive at* 1 if

$$[1](\Theta \cap (\Phi \vee \Psi)) = [1]((\Theta \cap \Phi) \vee (\Theta \cap \Psi))$$

for any $\Theta, \Phi, \Psi \in$ Con $\mathcal{A}$ (where $\vee$ is the join in the lattice Con $\mathcal{A}$);

$\mathcal{A}$ is *permutable at* 1 if

$$[1](\Theta \bullet \Phi) = [1](\Phi \bullet \Theta)$$

for any $\Theta, \Phi \in \mathrm{Con}\ \mathcal{A}$ (where $\bullet$ denotes the relational product). The following result is a straightforward modification of Theorem 8.3.2 in [2].

Lemma 2. *An algebra of a variety $\mathcal{V}$ with constant* 1 *is both distributive at* 1 *and permutable at* 1 *if and only if there exists a binary term $t(x, y)$ such that*

$$t(x, x) = 1, \quad t(x, 1) = 1 \quad \textit{and} \quad t(1, x) = x.$$

By Lemma 1, every semilattice with sectional switching mappings is a member of a variety with a binary term $t(x, y) = x \circ y$ satisfying the identities of Lemma 2. Hence, we infer immediately:

Corollary 1. *Every semilattice $\mathcal{S}^* = (S; \vee, \circ, 1)$ with sectional switching mappings is distributive at* 1 *and permutable at* 1.

Call $\mathcal{S}^* = (S; \vee, \circ, 1)$ a *semilattice with sectional involutions* if every sectional mapping is an involution, i.e., $x^{pp} = x$ for each $p \in S$ and each $x \in [p, 1]$.

Lemma 3. *Let $\mathcal{S}^* = (S; \vee, \circ, 1)$ be a semilattice with sectional mappings. Then $\mathcal{S}^*$ is a semilattice with sectional involutions if and only if it satisfies the identity $(x \circ y) \circ y = x \vee y$.*

Proof. Suppose that every sectional mapping is an involution. We have $(x \vee y)^y \in [y, 1]$, i.e., $y \leq (x \vee y)^y$ and hence

$$(x \circ y) \circ y = ((x \vee y)^y \vee y)^y = (x \vee y)^{yy} = x \vee y.$$

Conversely, suppose $x \in [p, 1]$ for $p \in S$. Then $p \leq x$ and

$$x^p = x \circ p \quad \text{(by (3)), i.e.,} \quad x^{pp} = (x \circ p) \circ p = x \vee p = x$$

thus every sectional mapping is an involution. ∎

Remark 1. Every semilattice $\mathcal{S} = (S; \vee, 1)$ can be considered as a semilattice with sectional switching involutions. Namely, for any $p \in S$ we can define $x \mapsto x^p$ as follows:

$$1^p = p, \quad p^p = 1 \quad \text{and} \quad x^p = x \quad \text{for} \quad x \neq p, \quad x \neq 1.$$

Of course, these mappings are both involutions and switching mappings. Hence, we can state:

Theorem 1. *On every semilattice $\mathcal{S} = (S; \vee, 1)$ we can define a binary operation $\circ$ satisfying*

(i) *$x \leq y$ if and only if $x \circ y = 1$,*

(ii) *$x \vee y = 1$ if and only if $x \circ y = y$,*

(iii) *$(x \circ y) \circ y = x \vee y$,*

(iv) *$x \circ x = 1, \quad 1 \circ x = x, \quad x \circ 1 = 1$.*

We are going to show that every congruence on such a semilattice is uniquely determined by its class containing 1.

Lemma 4. *Let $\mathcal{S}^* = (S, \vee, \circ, 1)$ be a semilattice with sectional switching involutions and let $\Theta \in$ Con $\mathcal{S}^*$. Then*

$$(x, y) \in \Theta \quad \textit{if and only if} \quad x \circ y, \quad y \circ x \in [1]\Theta.$$

Proof. Suppose $(x, y) \in \Theta$. Then also

$$(x \circ y, 1) = (x \circ y, y \circ y) \in \Theta \quad \text{and}$$

$$(y \circ x, 1) = (y \circ x, y \circ y) \in \Theta, \quad \text{i.e.,} \quad x \circ y, \ y \circ x \in [1]\Theta.$$

Conversely, suppose $x \circ y, \ y \circ x \in [1]\Theta$. Then $(x \circ y, 1) \in \Theta$ thus also $(x \vee y, y) = ((x \circ y) \circ y, 1 \circ y) \in \Theta$ and $(x \vee y, x) = ((y \circ x) \circ x, 1 \circ x) \in \Theta$. Due to symmetry and transitivity of Θ, we get $(x, y) \in \Theta$. ∎

An algebra $\mathcal{A} = (A; F)$ with a constant 1 is called *weakly regular at* 1 if every congruence on $\mathcal{A}$ is uniquely determined by its 1-class, i.e., if $\Theta, \Phi \in \text{Con}\ \mathcal{A}$ and $[1]\Theta = [1]\Phi$ then $\Theta = \Phi$.

Theorem 2. *Let $\mathcal{S}^* = (S; \vee, \circ, 1)$ be a semilattice with sectional switching involutions. Then $\mathcal{S}^*$ is weakly regular at* 1 *and, moreover,* Con $\mathcal{S}^*$ *is distributive.*

Proof. The first assertion follows from Lemma 4. Moreover $[1](\Theta\cap(\Phi\vee\Psi)) = [1]((\Theta\cap\Phi)\vee(\Theta\cap\Psi))$ by Corollary 1 and hence, by the previous assertion, also

$$\Theta \cap (\Phi \vee \Psi) = (\Theta \cap \Phi) \vee (\Theta \cap \Psi)$$

thus Con $\mathcal{S}^*$ is distributive. ■

Call $\mathcal{S}^* = (S; \vee, \circ, 1)$ a *semilattice with sectional antitone mappings* if every sectional mapping is antitone, i.e., if $x \leq y$ for $x, y \in [p, 1]$ implies $y^p \leq x^p$ for any $p \in S$.

Lemma 5. *Let $\mathcal{S}^* = (S; \vee, \circ, 1)$ be a semilattice with sectional mappings. Then $\mathcal{S}^*$ is a semilattice with sectional antitone mappings if and only if it satisfies*

$$x \circ z \geq (x \vee y) \circ z.$$

Proof. Let every sectional mapping be antitone. Since $z \leq x \vee z \leq x \vee y \vee z$, we have

$$x \circ z = (x \vee z)^z \geq (x \vee y \vee z)^z = (x \vee y) \circ z.$$

Conversely, if $\mathcal{S}^*$ satisfies the given identity and $x, y \in [z, 1]$ with $x \leq y$, then

$$y^z = y \circ z = (x \vee y) \circ z \leq x \circ z = (x \vee z)^z = x^z$$

thus every sectional mapping is antitone. ■

Remark 2. It is evident that if $\mathcal{S}^* = (S; \vee, \circ, 1)$ is a semilattice with sectional antitone involutions then every sectional mapping is also a switching mapping. Hence, these algebras have distributive congruence lattices and every congruence on $\mathcal{S}^*$ is determined by its 1-class (Theorem 2). Moreover, we can characterize such semilattices by three simple identities, as the following theorem shows.

Theorem 3. *Let $\mathcal{S}^* = (S; \vee, \circ, 1)$ be a semilattice with sectional mappings. Then $\mathcal{S}^*$ is a semilattice with sectional antitone involutions if and only it satisfies the following identities:*

(A1) $1 \circ x = x$,

(A2) $(x \circ y) \circ y = x \vee y$,

(A3) $((x \vee y) \circ z) \circ (x \circ z) = 1$.

Proof. Let $\mathcal{S}^* = (S; \vee, \circ, 1)$ satisfy (A1), (A2) and (A3). If $a \circ b = 1$ for $a, b \in S$ then, by (A2) and (A1),

$$a \vee b = (a \circ b) \circ b = 1 \circ b = b \quad \text{thus} \quad a \leq b.$$

Hence, (A3) yields $(x \vee y) \circ z \leq x \circ z$ and, by Lemma 5, the sectional mappings are antitone. By (A2) and Lemma 3, they are involutions.

Conversely, if the sectional mappings are antitone involutions then they are switching mappings and, by Lemma 1, $\mathcal{S}^*$ satisfies (A1), by Lemma 3, it satisfies (A2) and, by Lemma 5 and Lemma 1 (4), it satisfies also (A3). ■

Corollary 2. *If* $\mathcal{S}^* = (S; \vee, \circ, 1)$ *satisfies* (A1), (A2) *and* (A3) *then every section is a lattice* $([p, 1], \wedge_p, \vee)$ *with respect to the induced order, and for* $x, y \in [p, 1]$ *we have*

$$x \wedge_p y = ((x \circ p) \vee (y \circ p)) \circ p.$$

Proof. Since the sectional mappings are antitone involutions by Theorem 3, we obtain that

$$(x^p \vee y^p)^p = ((x \circ p) \vee (y \circ p)) \circ p$$

is a greatest lower bound of x, y in $[p, 1]$. ■

Up to now we did not consider the case that sectional mappings of different sections are dependent. If, however, $(S, \vee, \wedge, 1)$ is an upper bounded modular lattice, x^q is a complement of x in the section $[q, 1]$, $q \leq p$ and $x \in [p, 1]$ then

$$\text{(cc)} \qquad x^p = x^q \vee p$$

is a complement of x in $[p, 1]$. We can generalize this as follows: A semilattice $\mathcal{S}^* = (S; \vee, \circ, 1)$ with sectional mappings satisfies the *compatibility condition* if for any $q \leq p \leq x$ the condition (cc) holds. This condition can be expressed equivalently as an identity in the operations $\vee$ and $\circ$:

$$x \circ (y \vee z) = ((x \vee y) \circ z) \vee y \vee z \quad \text{for all} \quad x, y, z \in \mathcal{S}.$$

Indeed, applying the compatibility condition to the case

$$z \leq y \vee z \leq x \vee y \vee z$$

we obtain

$$(x \vee y \vee z)^{y \vee z} = (x \vee y \vee z)^{z} \vee y \vee z,$$

$$\text{i.e., } x \circ (y \vee z) = ((x \vee y) \circ z) \vee y \vee z.$$

Conversely, (cc) follows from this identity by taking $y = p$ and $z = q$ with $q \leq p \leq x$.

By an *ortholattice* we mean a bounded lattice with an antitone involution $x \mapsto x'$ such that x' is a complement of x. In this case x' is called an *orthocomplement* of x. An *orthomodular lattice* is an ortholattice which satisfies the orthomodular law:

$$x \leq y \quad \text{implies} \quad x \vee (y \wedge x') = y$$

or, equivalently,

$$x \leq y \quad \text{implies} \quad y \wedge (x \vee y') = x.$$

For a semilattice with sectional mappings satisfying (cc) we are able to characterize – in the following theorem – the case where all sections are orthomodular lattices. Let us note that the case where all sections are Boolean algebras was characterized by J.C. Abbott [1].

Theorem 4. *Let $\mathcal{S}^* = (S; \vee, \circ, 1)$ be a semilattice with sectional mappings satisfying the compatibility condition. Then every section is an orthomodular lattice if and only if $\mathcal{S}^*$ satisfies* (A1), (A2) *and* (A3).

Proof. Suppose $\mathcal{S}^*$ satisfies (A1), (A2) and (A3). Then, by Corollary 2, every section is a lattice $([p, 1], \wedge_p, \vee)$. Let $x \in [p, 1]$, then $p \leq x \leq x$ and, by the compatibility condition, we obtain

$$1 = x^x = x^p \vee x.$$

Since, by Theorem 3, the sectional mappings are antitone involutions, we have $x^p \wedge_p x = (x \vee x^p)^p = 1^p = p$ thus x^p is a complement and hence an orthocomplement of x in the section $[p, 1]$. Suppose $x, y \in [p, 1]$, $x \leq y$. Then $p \leq x \leq y$ and, again by the compatibility condition, we have

$$y^x = y^p \vee x$$

thus $y \wedge_p (x \vee y^p) = y \wedge_p y^x = x$ which is just the orthomodular law in $[p, 1]$, i.e., $([p, 1], \wedge_p, \vee)$ is an orthomodular lattice.

Conversely, if every section of $\mathcal{S}^*$ is an orthomodular lattice and x^p is an orthocomplement of x in $[p, 1]$ then the mapping $x \mapsto x^p$ is an antitone involution and thus, by Theorem 3, $\mathcal{S}^*$ satisfies (A1), (A2) and (A3). ∎

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