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DISTRIBUTIVITY OF BOUNDED LATTICES WITH SECTIONALLY ANTITONE INVOLUTIONS

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Abstract

We present a simple condition under which a bounded lattice $\mathcal{L}$ with sectionally antitone involutions becomes an MV-algebra. In this case, $\mathcal{L}$ is distributive. However, we get a criterion characterizing distributivity of $\mathcal{L}$ in terms of antitone involutions only.

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The aim of our paper is to continue the treatements from [4]. We will use the same terminology and notation.

A mapping $f : A \to A$ is called an *involution* if $f(f(x)) = x$ for each $x \in A$. Let $(A; \leq)$ be an ordered set. A mapping $f : A \to A$ is *antitone* if $x \leq y$ implies $f(y) \leq f(x)$ for all $x, y \in A$.

Let $\mathcal{L} = (L; \vee, \wedge, \mathbf{0}, \mathbf{1})$ be a bounded lattice where $\mathbf{0}$ or $\mathbf{1}$ denotes the least or greatest element of $\mathcal{L}$, respectively. $\mathcal{L}$ is said to have *sectionally antitone involutions* if for every $x \in L$ there is an antitone involution on the interval $[x, \mathbf{1}]$; i.e., a mapping which assigns to each $a \in [x, \mathbf{1}]$ an element $a^x \in [x, \mathbf{1}]$ with $a^{xx} = a$ and $a \leq b$ entails $b^x \leq a^x$. The interval $[x, \mathbf{1}]$ is called a *section*.

Unfortunately, antitone involutions on corresponding sections are only partial operations on the whole set L. Moreover, every lattice $\mathcal{L}$ with sectionally antitone involutions has so many of these partial operations as many of elements it has. It prevent to define a type of these algebras in the sense of universal algebra. The way how to avoid these problems was settled in [4]: introduce a new binary operation " $\circ$ " on L as follows

$$x \circ y = (x \vee y)^y.$$

Since $x \vee y \in [y, \mathbf{1}]$, the definition is correct and " $\circ$ " is everywhere defined on L. Call " $\circ$ " the *assigned operation* of $\mathcal{L}$. The following was proved in [4] (see also [3]):

Proposition. *Let $\mathcal{L} = (L; \vee, \wedge, \mathbf{0}, \mathbf{1})$ be a bounded lattice.*

(a) *If $\mathcal{L}$ has sectionally antitone involutions then the assigned operation $\circ$ satisfies the folowing axioms:*

 (1) $\mathbf{1} \circ x = x$, $x \circ \mathbf{1} = \mathbf{1}$, $\mathbf{0} \circ x = \mathbf{1}$;

 (2) $(x \circ y) \circ y = (y \circ x) \circ x$;

 (3) $(((x \circ y) \circ y) \circ z) \circ (x \circ z) = \mathbf{1}$.

(b) *If $\circ$ is an operation on L satisfying* (1), (2) *and* (3) *then $\mathcal{L}$ is a lattice with sectionally antitone involutions where for each $a \in [x, \mathbf{1}]$ we have $a^x = a \circ x$. Moreover, $x \leq y$ if and only if $x \circ y = \mathbf{1}$ and $x \vee y = (x \circ y) \circ y$.*

Due to the foregoing Proposition, we can identify a lattice $\mathcal{L} = (L; \vee, \wedge, \mathbf{0}, \mathbf{1})$ with sectionally antitone involutions with an algebra $\mathcal{L} = (L; \vee, \wedge, \circ, \mathbf{0}, \mathbf{1})$ where " $\circ$ " satisfies simple identities (1), (2) and (3). From now on, whenever we will speak about such lattices, we will in fact consider the aforementioned algebra. Of course, this algebra is of type $(2, 2, 2, 0, 0)$ and, since its axioms are only identities, the class of sectionally antitone involutioned lattices (considered in this type) forms a variety.

Let us recall from [2] that by an MV-algebra is meant an algebra $\mathcal{A} = = (A; \oplus, \neg, \mathbf{0})$ of type $(2, 1, 0)$ satisfying the axioms

(MV1) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,

(MV2) $a \oplus b = b \oplus a$,

(MV3) $a \oplus \mathbf{0} = a$,

(MV4) $\neg\neg a = a$,

(MV5) $a \oplus \neg\mathbf{0} = \neg\mathbf{0}$,

(MV6) $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$.

We usually denote $\neg\mathbf{0}$ by $\mathbf{1}$ and we read (MV5) as

$$a \oplus \mathbf{1} = \mathbf{1}.$$

Let $\mathcal{L} = (L; \vee, \wedge, \circ, \mathbf{0}, \mathbf{1})$ be a lattice with sectionally antitone involutions. Introduce a new binary operation $\oplus$ on L as follows

$$(*) \qquad x \oplus y := (x \circ \mathbf{0}) \circ y$$

and a new unary operation $\neg$ by the rule

$$\neg x := x \circ \mathbf{0}$$

(hence $x \oplus y = \neg x \circ y$).

Lemma 1. *Let $\mathcal{L} = (L; \vee, \wedge, \circ, \mathbf{0}, \mathbf{1})$ be a lattice with sectionally antitone involutions and $\oplus$ be defined by $(*)$. Then $\oplus$ is commutative if and only if the assigned operation $\circ$ satisfies the identity*

$$\text{(WE)} \qquad x \circ (y \circ \mathbf{0}) = y \circ (x \circ \mathbf{0}).$$

Proof. Suppose (WE). Then, by (3) of Proposition, $(a \circ \mathbf{0}) \circ \mathbf{0} = a \vee \mathbf{0} = a$ and hence,

$$x \oplus y = (x \circ \mathbf{0}) \circ y = (x \circ \mathbf{0}) \circ ((y \circ \mathbf{0}) \circ \mathbf{0}) = (y \circ \mathbf{0}) \circ ((x \circ \mathbf{0}) \circ \mathbf{0}) = (y \circ \mathbf{0}) \circ x = y \oplus x.$$

Conversely, let $\oplus$ be commutative, then

$$x \circ (y \circ \mathbf{0}) = ((x \circ \mathbf{0}) \circ \mathbf{0}) \circ (y \circ \mathbf{0}) \neg x \oplus \neg y = \neg y \oplus \neg x$$

$$= ((y \circ \mathbf{0}) \circ \mathbf{0}) \circ (x \circ \mathbf{0}) = y \circ (x \circ \mathbf{0})$$

proving (WE). ■

More generally, consider a section $[p, \mathbf{1}]$ of $\mathcal{L}$ and introduce a binary operation $\oplus_p$ on $[p, \mathbf{1}]$ as follows

$$(**) \qquad x \oplus_p y = (x \circ p) \circ y \quad \text{for} \quad x, y \in [p, \mathbf{1}]$$

and $\neg_p x = x \circ p$. We can consider an algebra $([p, \mathbf{1}]; \oplus_p, \neg_p, p)$ for each $p \in L$ and the so-called *derived algebra* $\mathcal{A}(L) = (L; \oplus, \neg, \mathbf{0})$.

Theorem 1. *Let $\mathcal{L} = (L; \vee, \wedge, \circ, \mathbf{0}, \mathbf{1})$ be a lattice with sectionally antitone involutions. The derived algebra $\mathcal{A}(L) = (L; \oplus, \neg, \mathbf{0})$ is an MV-algebra if and only if $\oplus_p$ is commutative on every section $[p, \mathbf{1}]$.*

Proof. Let $\mathcal{L} = (L; \vee, \wedge, \circ, \mathbf{0}, \mathbf{1})$ be a lattice with sectionally antitone involutions. As it was shown by Lemma 2 in [4], the derived operations $\oplus$ and $\neg$ satisfy the axioms (MV3), (MV4), (MV5) and (MV6). Suppose now that $\oplus_p$ defined by $(**)$ is commutative for each $p \in L$. Similarly as in Lemma 1 it yields that the so-called Exchange Property (see [1])

$$(A) \qquad x \circ (y \circ z) = y \circ (x \circ z)$$

holds for $x, y \in [z, \mathbf{1}]$. Consider now arbitrary $x, y, z \in L$. Then $y \circ z = (y \vee z)^z$ thus $y \circ z \in [z, \mathbf{1}]$, i.e., $z \leq y \circ z$. Hence $y \circ z = z \vee (y \circ z)$. This yields

$$((x \circ z) \circ z) \circ (y \circ z) = (x \vee z) \circ (y \circ z) (x \vee z \vee (y \vee z)^z)^{(y \vee z)^z}$$

$$= (x \vee (y \vee z)^z)^{(y \vee z)^z} = x \circ (y \vee z)^z = x \circ (y \circ z).$$

Also $z \leq x \circ z$ and, by means of (A), we conclude

$$x \circ (y \circ z) = ((x \circ z) \circ z) \circ (y \circ z) = (x \circ z) \oplus_z (y \circ z)$$

$$= (y \circ z) \oplus_z (x \circ z) = ((y \circ z) \circ z) \circ (x \circ z) = y \circ (x \circ z)$$

thus the Exchange Property holds for arbitrary $x, y, z \in L$. By Lemma 1 we immediately see that $\oplus$ (i.e., $\oplus_0$) is commutative. Further,

$$(x \oplus y) \oplus z = z \oplus (x \oplus y) = (z \circ \mathbf{0}) \circ ((x \circ \mathbf{0}) \circ y)$$

$$= (z \circ \mathbf{0}) \circ ((x \circ \mathbf{0}) \circ ((y \circ \mathbf{0}) \circ \mathbf{0})) = (x \circ \mathbf{0}) \circ ((z \circ \mathbf{0}) \circ ((y \circ \mathbf{0}) \circ \mathbf{0}))$$

$$= (x \circ \mathbf{0}) \circ ((y \circ \mathbf{0}) \circ ((z \circ \mathbf{0}) \circ \mathbf{0})) = (x \circ \mathbf{0}) \circ ((y \circ \mathbf{0}) \circ z) = x \oplus (y \oplus z)$$

thus $\oplus$ satisfies both (MV1) and (MV2) and hence the derived algebra $\mathcal{A}(L)$ is an MV-algebra.

Conversely, let $\mathcal{A}(L)$ be an MV-algebra (where $x \oplus y = (x \circ \mathbf{0}) \circ y$ and $\neg x = x \circ \mathbf{0}$). By Theorem 10 in [5] the assigned operation $\circ$ satisfies the Exchange Property and hence for any $p \in L$ and $x, y \in [p, \mathbf{1}]$ we obtain

$$x \oplus_p y = (x \circ p) \circ y = (x \circ p) \circ (y \vee p) = (x \circ p) \circ ((y \circ p) \circ p)$$

$$= (y \circ p) \circ ((x \circ p) \circ p) = (y \circ p) \circ (x \vee p) = (y \circ p) \circ x = y \oplus_p x$$

whence $\oplus_p$ is commutative for each $p \in L$. ■

If $\mathcal{L} = (L; \vee, \wedge, \circ, \mathbf{0}, \mathbf{1})$ is a lattice with sectionally antitone involutions and $\mathcal{A}(L)$ is the derived algebra then, whenever $\mathcal{A}(L)$ is an MV-algebra, $\mathcal{L}$ is distributive (see e.g., [5]). However, distributivity of $\mathcal{L}$ does not imply that $\mathcal{A}(L)$ is an MV-algebra:

Example. Consider the lattice as shown in Figure 1

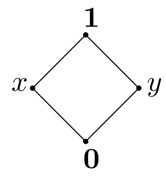

Figure 1

where the operation $\circ$ is given by the table

$\circ$	$\mathbf{0}$	x	y	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
x	x	$\mathbf{1}$	y	$\mathbf{1}$
y	y	x	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	x	y	$\mathbf{1}$

Then $\mathcal{L}$ is a distributive lattice with sectionally antitone involutions but

$$x \oplus y = (x \circ \mathbf{0}) \circ y = x \circ y = y \neq x = y \circ x = (y \circ \mathbf{0}) \circ x = y \oplus x$$

thus the derived algebra $\mathcal{A}(L)$ does not satisfy (MV2).

In what follows, we will characterize distributivity of $\mathcal{L}$ in terms of the assigned operation $\circ$.

Lemma 2. *Let $\mathcal{L} = (L; \vee, \wedge, \circ, \mathbf{0}, \mathbf{1})$ be a lattice with sectionally antitone involutions. The following conditions are equivalent:*

(a) *$\mathcal{L}$ is distributive;*

(b) *$\mathcal{L}$ does not contain M_3 or N_5 with the greatest element equal to $\mathbf{1}$;*

(c) *$\mathcal{L}$ does not contain M_3 or N_5 with the least element equal to $\mathbf{0}$.*

Proof. (b)$\Rightarrow$(a) and (c)$\Rightarrow$(a) are trivial. Suppose that $\mathcal{L}$ contains either M_3 or N_5 with elements denoted as shown in Figure 2.

Figure 2

Consider the section $[x, \mathbf{1}]$ and the involution $w \mapsto w^x$ in $[x, \mathbf{1}]$. Since every antitone involution is an antiautomorphism, it yields that $\mathcal{L}$ contains also M_3 or N_5 with the elements y^x, a^x, b^x, c^x and $\mathbf{1} = x^x$, i.e., (a)⇒(b) is valid. However, if $\mathcal{L}$ contains M_3 or N_5 with the greatest element equal to $\mathbf{1}$, we can apply the involution $w \mapsto w^0$ in the whole $[\mathbf{0}, \mathbf{1}]$ to obtain M_3 or N_5 with the least element $\mathbf{1}^0 = \mathbf{0}$. Thus (b)⇒(c). ■

Theorem 2. *Let $\mathcal{L} = (L; \vee, \wedge, \circ, \mathbf{0}, \mathbf{1})$ be a lattice with sectionally antitone involutions. The following conditions are equivalent:*

(I) *$\mathcal{L}$ is distributive;*

(II) *$y \circ x = x$ and $z \circ x = x$ and $(y \circ w) \circ (x \circ w) = (z \circ w) \circ (x \circ w)$ imply $y = z$;*

(III) *$y \circ x = z \circ x$ and $(y \circ \mathbf{0}) \circ (x \circ \mathbf{0}) = (z \circ \mathbf{0}) \circ (x \circ \mathbf{0})$ imply $y = z$.*

Proof. $\mathcal{L}$ is distributive if and only if it satisfies the so-called cancellation law

(CL) $$x \vee y = x \vee z \quad \text{and} \quad x \wedge y = x \wedge z \quad \text{imply} \quad y = z.$$

(i) Due to (b) of Lemma 2, we need only consider the case $x \vee y = \mathbf{1} = x \vee z$. By the Proposition it means $(y \circ x) \circ x = \mathbf{1} = (z \circ x) \circ x$. However, $y \circ x$, $z \circ x \in [x, \mathbf{1}]$ and the involution $w \mapsto w^x$ in $[x, \mathbf{1}]$ is a bijection, thus the previous condition can be reduced to

(+) $$y \circ x = x \quad \text{and} \quad z \circ x = x.$$

Now, consider $w = x \wedge y \wedge z$. Then $x, y, z \in [w, \mathbf{1}]$ and by the Proposition and De Morgan laws

$$x \wedge y = (x^w \vee y^w)^w = ((y \circ w) \circ (x \circ w)) \circ w;$$

now since the element in brackets is above w, we conclude

$$(y \circ w) \circ (x \circ w) = (z \circ w) \circ (x \circ w).$$

Analogously for

$$x \wedge z = ((z \circ w) \circ (x \circ w)) \circ w.$$

But $(y \circ w) \circ (x \circ w)$, $(z \circ w) \circ (x \circ w) \in [x \circ w, \mathbf{1}]$ and the corresponding involution in $[x \circ w, \mathbf{1}]$ is a bijection, i.e., $x \wedge y = x \wedge z$ if and only if

$$(++) \qquad (y \circ w) \circ (x \circ w) = (z \circ w) \circ (x \circ w).$$

Altogether, (+), (++) and (CL) get (I) $\Leftrightarrow$ (II).

(ii) Due to (c) of Lemma 2, it is enough to consider only the case $x \wedge y = \mathbf{0} = x \wedge z$. Hence, $x \vee y = (y \circ x) \circ x$, $x \vee z = (z \circ x) \circ x$ and $x \vee y = x \vee z$ will get $y \circ x = z \circ x$ and, due to the Proposition and De Morgan laws, we can transform $x \wedge y = (x^0 \vee y^0)^0$, $x \wedge z = (x^0 \vee z^0)^0$ thus $x \wedge y = \mathbf{0} = x \wedge z$ if and only if $(y \circ \mathbf{0}) \circ (x \circ \mathbf{0}) = (z \circ \mathbf{0}) \circ (x \circ \mathbf{0})$. Altogether, these yield (I) $\Leftrightarrow$ (III). ∎

Applying the cancellation law with comparable elements $y \leq z$, we obtain analogously:

Theorem 3. *Let $\mathcal{L} = (L; \vee, \wedge, \circ, \mathbf{0}, \mathbf{1})$ be a lattice with sectionally antitone involutions. The following conditions are equivalent:*

(I) *$\mathcal{L}$ is modular;*

(II) *$y \circ x = x$, $z \circ x = x$, $y \leq z$ and $(y \circ w) \circ (x \circ w) = (z \circ w) \circ (x \circ w)$ imply $y = z$;*

(III) *$y \circ x = z \circ x$, $y \leq z$ and $(y \circ \mathbf{0}) \circ (x \circ \mathbf{0}) = (z \circ \mathbf{0}) \circ (x \circ \mathbf{0})$ imply $y = z$.*

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