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# SEMIGROUPS DEFINED BY AUTOMATON EXTENSION MAPPINGS

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**Abstract**

We study semigroups generated by the restrictions of automaton extension (see, e.g., [3]) and give a characterization of automaton extensions that generate finite semigroups.

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## Introduction

In the set of all transformations of the set of finite words over given alphabet we distinguish a subset of the automaton mappings, i.e. transformations induced by (finite or infinite) Mealy automata . Although both sets are uncountable, not every function $f : X^* \to X^*$ is defined by certain automaton.

In sixties of the XX century has been indicated (e.g., in [3]) that after addition a new symbol to the alphabet, arbitrary transformation can be extended to an automaton mapping, that uniquely determines the initial transformation. Moreover, an effective method for such constructions has been established (see [3] and [1]).

Since mentioned extension is not unique, we define three different possibilities of the construction. The main result of this paper is

**Theorem 1.** *Let* $\mathsf{Sg}(\widehat{f})$ *be a semigroup generated by the restrictions of the automaton extension mapping* $\widehat{f}$*. The semigroup* $\mathsf{Sg}(\widehat{f})$ *is finite iff* $\widehat{f}$ *is finite-state and nilpotent.*

The contents are organized as follows. In Section 1 we list the basic notations and recall a notions of a Mealy automaton and a Rabin-Scott automaton. In Section 2 we define a notion of an automaton extension of the transformation. In Section 3 we introduce semigroups generated by extension mappings. Section 4 contains proof of the theorem.

## 1. Preliminaries

Let $X$ be an alphabet, and $X^*$ be the free monoid over $X$ with an empty word $\varepsilon$ as a neutral element. We shall write $uv$ for the product of $u, v \in X^*$ and $u^k$ for $\overbrace{u \dots u}^{k}$. The length of the word $u$ is denoted by $|u|$. The word $u$ is a *prefix* of the word $v$ (denoted by $u \leq v$) if $v = uw$ for the certain word $w$. The word $v$ is a *segment* of $u$ if there exist words $u_1, u_2 \in X^*$ such that $u = u_1 v u_2$.

An *initial Mealy-type automaton* (see, e.g., [4], [5]) over the alphabet $X$ is a tuple

$$\mathcal{A} = (Q, q_0, X, \delta, \lambda)$$

which consists of the following data:

- ▷ a set $Q$ of the *internal states*, $Q \neq \emptyset$;
- ▷ a distinguished state $q_0 \in Q$ called *initial* state;
- ▷ an alphabet of the automaton, $X \neq \emptyset$;
- ▷ a *next-state function* $\delta : Q \times X \to Q$;
- ▷ an *output function* $\lambda : Q \times X \to X$.

A tuple $\mathcal{A} = (Q, q_0, X, \delta, \lambda)$ is a *partial Mealy automaton* if either $\delta$ or $\lambda$ is a partial function. The automaton $\mathcal{A}$ is *finite* if the sets $Q$ and $X$ are finite.

We often use a notation

$$q_i \overset{x/y}{\longmapsto} q_j$$

instead of

$$\delta(q_i, x) = q_j\,, \qquad \lambda(q_i, x) = y\,.$$

The next-state function and the output function of the automaton $\mathcal{A} = (Q, q_0, X, \delta, \lambda)$ can be extended to the set $Q \times X^*$ by the following recurrent equalities:

$$\delta(q, \varepsilon) = q\,, \qquad \delta(q, ux) = \delta(\delta(q, u), x)\,,$$
$$\lambda(q, \varepsilon) = \varepsilon\,, \qquad \lambda(q, ux) = \lambda(\delta(q, u), x)\,,$$

where $x \in X$ and $u \in X^*$. An initial automaton $\mathcal{A}$ defines the mapping $f_{\mathcal{A}} : X^* \to X^*$ as follows:

$$f_{\mathcal{A}}(\varepsilon) = \varepsilon\,, \qquad f_{\mathcal{A}}(x_1 \dots x_k) = \lambda(q_0, x_1)\lambda(q_0, x_1x_2) \dots \lambda(q_0, x_1 \dots x_k)\,.$$

If $\mathcal{A}$ is a partial automaton, then $f_{\mathcal{A}}$ is a partial function.

**Definition 1** ([8]). A function $f : X^* \to X^*$ is called an *(finite-state) automaton mapping* if there exists an (finite) initial automaton $\mathcal{A}$ such that $f = f_{\mathcal{A}}$.

**Definition 2.** A function $f : X^* \to X^*$ is called a *partial automaton mapping* if there exists a partial initial automaton $\mathcal{A}$ such that $f = f_{\mathcal{A}}$.

A *Rabin-Scott automaton* is a tuple

$$\mathcal{A} = (Q, q_0, T, X, \delta)\,,$$

which is just as the Mealy automaton, except that the output function $\lambda$ is replaced by the set $T \subset Q$ which is set of *terminal nodes* (or *accept states*). A set of the words

$$L(\mathcal{A}) = \{u \in X^* : \delta(q_0, u) \in T\}$$

is known as the *language recognizable by the automaton* $\mathcal{A}$.

The language $L \subset X^*$ is said to be *regular* if there exists a finite automaton recognizing $L$. For more information refer to [9] and [2].

## 2. Automaton extension of mapping

Let $f : X^* \to X^*$ be a function that satisfies $f(\varepsilon) = \varepsilon$. Let $X_\alpha = X \cup \{\alpha\}$, $\alpha \notin X$ be the extended alphabet. Let $t : X_\alpha^* \to X^*$ be a homomorphism given by

$$t(\alpha) = \varepsilon, \qquad t(x) = x, \qquad x \in X\,.$$

**Definition 3.** An automaton mapping $\widehat{f} : X_\alpha^* \to X_\alpha^*$ is called an *automaton extension mapping* (or simply an *extension*) of $f : X^* \to X^*$ if there exists an embedding $\mu_f : X^* \to X_\alpha^*$ such that the following diagram is commutative:

$$\begin{array}{ccc} u \in X^* & \overset{f}{\longmapsto} & f(u) \in X^* \\ \mu_f \downarrow & & \uparrow t \\ u' \in X_\alpha^* & \overset{\widehat{f}}{\longmapsto} & \widehat{f}(u') \in X_\alpha^*\,. \end{array}$$

The extension $\widehat{f}$ of an arbitrary function $f : X^* \to X^*$ will be defined in two steps:

1. a partial extension $X_\alpha^* \to X_\alpha^*$ is defined on a certain fixed subset $M \subset X_\alpha^*$;
2. the domain of the obtained function is extended to the monoid $X_\alpha^*$.

For the construction related to the first step we will apply a method described in [3], p. 19.

**Definition 4.** For every $u \in X^*$, we define

$$\mu_f(u) = u\alpha^{|f(u)|}$$

and we introduce the set

$$M = \{v' \in X_\alpha^* :\ v' \le \mu_f(u),\ u \in X^*\}\,.$$

The mapping $\widehat{f} : M \to X_\alpha^*$ is defined as follows:

a) if $u' = u\alpha^{|f(u)|}$, $u \in X^*$, then

$$\widehat{f}(u') = \alpha^{|u|} f(u)\,,$$

b) if $u' \leq u\alpha^{|f(u)|}$, then

$$\widehat{f}(u') = w'\,,$$

where $w'$ is chosen to satisfy

$$w' \leq \alpha^{|u|} f(u) \quad \text{and} \quad |w'| = |u'|\,.$$

For $u' = \mu_f(u)$, the properties $t(u') = u$ and $t(\widehat{f}(u')) = f(u)$ hold. Thus, the diagram from the Definition 3 is commutative.

**Proposition 1.** *The extension $\widehat{f} : X_\alpha^* \to X_\alpha^*$ is a partial automaton mapping over the alphabet $X_\alpha$.*

***Proof.*** See [6]. ■

**Example 1.** Consider a function $f : X^* \to X^*$, $X = \{0, 1\}$ defined by

$$f(u) = \begin{cases} \varepsilon, & \text{if } \ u = \varepsilon, \\ 0, & \text{if } \ |u| \ \text{ is even}, \ \ |u| > 0, \\ 11, & \text{if } \ |u| \ \text{ is odd}. \end{cases}$$

We see that $f$ is not an automaton mapping since it does not preserve either lengths nor has the common prefix property (see [4]). Since $f(01) = 0$ and $f(101) = 11$, for the extension mapping we have

$$\widehat{f}(01\alpha) = \alpha\alpha 0 \qquad \text{and} \qquad \widehat{f}(101\alpha\alpha) = \alpha\alpha\alpha 11\,.$$

It can be seen that for the arguments of the extension the letter $\alpha$ is utilized to terminate a sequence of letters from the set $X$, whereas for the values it plays role of an "empty" symbol while the automaton waits for completing the input word.

We introduce three different methods for extending $\widehat{f}$ on the set $X_\alpha^*$, which will be referred to as 'simple', 'plain' and 'cyclic'.

**Definition 5.** A *simple* extension of the transformation $f : X^* \to X^*$ is the mapping $\widehat{f}_0 : X_\alpha^* \to X_\alpha^*$ defined by:

(i) $\widehat{f}_0|_M = \widehat{f}$, where $\widehat{f}$ is the automaton extension of $f$ and $M$ is the set established in Definition 4,

(ii) $\widehat{f}_0(\alpha^k v') = \alpha^k \alpha^{|v'|}$,

(iii) $\widehat{f}_0(u\alpha^{|f(u)|}v') = \alpha^{|u|} f(u) \alpha^{|v'|}$,

(iv) if $m + |v'| \geq |f(u)|$, then

$$\widehat{f}_0(u\alpha^m v') = \alpha^{|u|} f(u) \alpha^n$$

and $n$ is chosen to satisfy

$$m + |v'| = |f(u)| + n\,,$$

(v) if $m + |v'| < |f(u)|$, then

$$\widehat{f}_0(u\alpha^m v') = \alpha^{|u|} v$$

and $v$ is chosen to satisfy

$$v \leq f(u), \quad |v| = m + |v'|\,,$$

where $u, v \in X^*$ and $v' \in X_\alpha^*$.

**Definition 6.** A *plain* extension of the transformation $f : X^* \to X^*$ is the mapping $\widehat{f}_1 : X_\alpha^* \to X_\alpha^*$ defined similarly to the simple extension, only with condition (ii) changed to

$$\widehat{f}_1(\alpha^k v') = \alpha^k \widehat{f}_1(v')\,.$$

The automaton mappings $\widehat{f}_0$ and $\widehat{f}_1$ translate only the first segment $u \in X^*$ from the input word and ignore appended word $v'$ by treating it as a sequence of "empty" symbols. Extension $\widehat{f}_0$ performes this translation only if $u$ is a prefix of the input word.

For the next definition recall that arbitrary word $v' \in X_\alpha^*$ can be uniquely written as

$$v' = \alpha^{k_0} u_1 \alpha^{k_1} u_2 \alpha^{k_2} \dots u_n \alpha^{k_n}, \qquad u_i \in X^*,$$

where $k_0, k_n \geq 0$ and $k_1, \dots, k_{n-1} \geq 1$.

**Definition 7.** A *cyclic* extension of the transformation $f : X^* \to X^*$ is the mapping $\widehat{f}_2 : X_\alpha^* \to X_\alpha^*$ defined by:

(i) $\widehat{f}_2|_M = \widehat{f}$, where $\widehat{f}$ is the automaton extension of $f$ and $M$ is the set established in Definition 4,

(ii) $\widehat{f}_2(\alpha^k) = \alpha^k$,

(iii) $\widehat{f}_2(u\alpha^{|f(u)|}\alpha^k) = \alpha^{|u|} f(u) \alpha^k$,

(iv) if $m < |f(u)|$, then

$$\widehat{f}_2(u\alpha^m) = \alpha^{|u|} v\,,$$

where $v$ is chosen to satisfy

$$v \leq f(u), \quad |v| = m\,,$$

(v) $\widehat{f}_2(v') = \alpha^{k_0} \widehat{f}_2(u_1\alpha^{k_1}) \widehat{f}_2(u_2\alpha^{k_2}) \dots \widehat{f}_2(u_n\alpha^{k_n})\,,$

where $v' = \alpha^{k_0} u_1 \alpha^{k_1} u_2 \alpha^{k_2} \dots u_n \alpha^{k_n}$.

The mapping obtained in this way translates independently every segment of the form $u_i \alpha^{k_i}$.

**Proposition 2.**

1. *For every function $f : X^* \to X^*$, $f(\varepsilon) = \varepsilon$ the following conditions hold:*
   a) *$\widehat{f}_0$, $\widehat{f}_1$, $\widehat{f}_2$ are full defined automaton mappings over the alphabet $X^*_\alpha$,*
   b) *$\widehat{f}_0$, $\widehat{f}_1$, $\widehat{f}_2$ are pairwise distinct unless $f$ is trivial (i.e., $f(u) = \varepsilon$ for all $u \in X^*$).*
2. *For every $f, g : X^* \to X^*$, such that $f \neq g$, we have*

$$\widehat{f}_0 \neq \widehat{g}_0\,, \qquad \widehat{g}_1 \neq \widehat{f}_1\,, \qquad \widehat{g}_2 \neq \widehat{f}_2\,.$$

***Proof.*** See [6]. ■

From now on, we consider extensions $\widehat{f}$ defined on infinite words as follows

$$\widehat{f} : X^*_\alpha \alpha^\omega \to X^*_\alpha \alpha^\omega,$$

$$\mu_f(u) = u\alpha^{|f(u)|}\alpha^\omega = u\alpha^\omega,$$

$$\widehat{f}(u') = \alpha^{|u|} f(u)\alpha^\omega, \quad u' = \mu_f(u)\,.$$

In this case we see that a sequence $\alpha^\omega = \alpha\alpha\ldots$ is the only fixed point.

**Example 2.** Let $X = \{1\}$ and

$$f(u) = \begin{cases} \varepsilon, & \text{if } \ u = \varepsilon, \\ 1, & \text{if } \ |u| \ \text{ is even, and } \ |u| > 0, \\ 11, & \text{if } \ |u| \ \text{ is odd.} \end{cases}$$

Extensions $\widehat{f}_0$, $\widehat{f}_1$, $\widehat{f}_2$ are depicted respectively on Figure 1, 2 and 3.

In case of $\widehat{f}_2$, state $f_\varepsilon$ on the left side is not removed in order to show similarity between extensions. States $f_\varepsilon, f_1, f_2$ are related to the Rabin-Scott part of the automaton as it is indicated in the proof of Lemma 8.

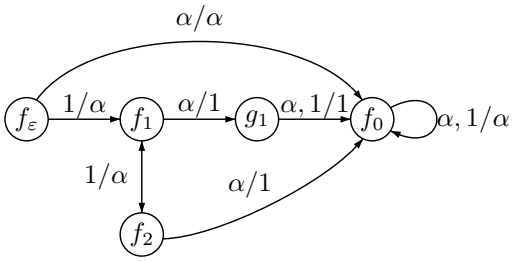


Figure 1. Automaton $\widehat{f_0}$

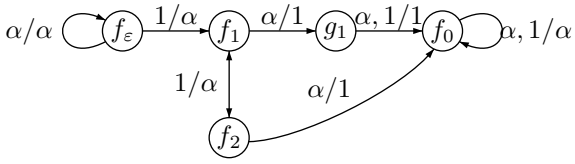


Figure 2. Automaton $\widehat{f_1}$

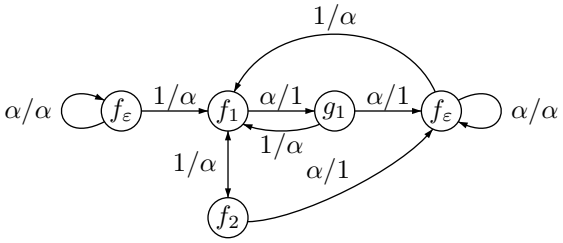


Figure 3. Automaton $\widehat{f_2}$

**2.1. Nilpotent extensions**
Fix $X$ and consider the monoid $T_\varepsilon(X^*)$ of all mappings $f : X^* \to X^*$ satisfying $f(\varepsilon) = \varepsilon$. A mapping $0 : X^* \to X^*$ defined by

$$0(u) = \varepsilon\,, \qquad \text{for } u \in X^*,$$

is a zero of the monoid.

Similarly, the monoid generated by all extensions $\widehat{f} : X^*_\alpha\alpha^\omega \to X^*_\alpha\alpha^\omega$, $f \in T_\varepsilon(X^*)$ contains a mapping $\widehat{0}$ defined by

$$\widehat{0}(u) = \alpha^\omega\,, \qquad \text{for } u \in X^*_\alpha\alpha^\omega,$$

which is a zero of the monoid. This follows from the fact that $\alpha^\omega$ is the fixed point of every $\widehat{f}$ (see [7]). Thus, the question whether the extension $\widehat{f}$ is nilpotent can be asked.

For any nilpotent element $x$ of the semigroup with zero, the smallest number $k$ such that $x^k = 0$ will be denoted by $\mathrm{nil}(x)$.

**Proposition 3.** *For every mapping $f : X^* \to X^*$ with $f(\varepsilon) = \varepsilon$, the extension $\widehat{f}_0$ is nilpotent and* $\mathrm{nil}(\widehat{f}_0) \leq 2$.

***Proof.*** Any value of the mapping $\widehat{f}_0$ is of the form $\alpha v'$, where $v' \in X^*_\alpha\alpha^\omega$ and by definition we have $\widehat{f}(\alpha v') = \alpha^\omega$. Therefore, for $\widehat{f}_0 \neq \widehat{0}$, we have $\mathrm{nil}(\widehat{f}_0) = 2$. ■

**Proposition 4.** *The extension $\widehat{f}_1$ is nilpotent if and only if the mapping $f$ is nilpotent. If $f$ is nilpotent, then*

$$\mathrm{nil}(\widehat{f}_1) = \mathrm{nil}(f)\,.$$

***Proof.*** See [7]. ■

**Proposition 5.** *The extension $\widehat{f}_2$ is nilpotent if and only if*

1. *$f$ is nilpotent*

*and*

2. *the lengths of the sequences*

$$u_1, u_2, \ldots, u_m, \ \textit{where}\ u_k \leq f(u_{k-1})\ \textit{for}\ k = 2, ..., m,$$

*have a common upper bound.*

*If $\widehat{f}_2$ is nilpotent, then*

$$\mathrm{nil}(\widehat{f}_2) \geq \mathrm{nil}(f)\,.$$

***Proof.*** See [7]. ■

## 3. Semigroup defined by an automaton extension mapping

Recall that an automaton mapping $f : Y^* \to Y^*$ defines its *restrictions* $f_u$, $u \in Y^*$ (see [4], [8]) according to the equality

$$f(uv) = f(u)f_u(v), \quad v \in Y^* .$$

**Definition 8 .** For an automaton mapping $f : Y^* \to Y^*$, we define a transformation semigroup generated by its restrictions

$$\mathsf{Sg}(f) = \langle f_u : u \in Y^* \rangle .$$

Another equivalent definition can be established in terms of the functions related to states of the minimal initial automaton $\mathcal{A}(f)$ defining $f$. Therefore, in order to simplify notation, we shall not distinguish between state $\delta(q_0, u)$ of the automaton $\mathcal{A}(f)$ and the restriction $f_u$.

We introduce the following notations. For a subset $F$ of the semigroup, we define

$$F^1 = F, \qquad F^n = F^{n-1}F$$

and

$$F^{n+} = F^n \cup F^{n+1} \cup \dots .$$

The semigroup generated by $F$ will be denoted as $\langle F \rangle$. We also use the following notation $(f_i f_j)(x) = f_j(f_i(x))$.

**Lemma 6 .** *Let $f_1, \dots, f_k$ be partial or full defined transformations of some set $A$. If there exists a number $n$ such that every composition $f \in \{f_1, \dots, f_k\}^n$ has finite image $f(A)$, then the semigroup $\langle f_1, \dots, f_k \rangle$ is finite.*

***Proof.*** Let $F = \{f_1, \dots, f_k\}$ and

$$B = \bigcup \{f(A) : f \in F^n\}$$

From assumptions of the lemma, the sets $f(A)$ are finite and therefore so is $B$. Furthermore, every element $g \in F^{n+}$ admits a decomposition

$$g = fh, \qquad f \in F^n, \quad h \in \langle F \rangle \cup \{Id\} ,$$

$$g : A \xrightarrow{f} B \xrightarrow{h} B$$

and there are finite number of possibilities for $h$ since it is a mapping on a finite set. Therefore, $F^{n+}$ and $\langle f_1, \dots, f_k \rangle$ are finite sets. ■

## 4. Proof of the main theorem

Since we need to discuss the problem whether the semigroup $\mathsf{Sg}(\widehat{f})$ is finite, we assume $X$ is a finite set and we will use the criterion for $\widehat{f}$ to be a finite-state automaton mapping.

**Proposition 7.** *Let $\widehat{f} : X_\alpha^* \to X_\alpha^*$ be the extension of the transformation $f : X^* \to X^*$. The extension $\widehat{f}$ is a finite-state automaton mapping if and only if the following conditions are satisfied:*

1. *$f(X^*)$ is a finite set*

*and*

2. *the inverse image $f^{-1}(u)$ is a regular language for each $u \in f(X^*)$.*

***Proof.*** See [6]. ■

**Lemma 8.** *Let $f : X^* \to X^*$ be a mapping satisfying $f(\varepsilon) = \varepsilon$. The semigroup $\mathsf{Sg}(\widehat{f_0})$ is finite iff $\widehat{f_0}$ is finite-state.*

***Proof.***

($\Rightarrow$) Obviously, a finite number of generators is obtained iff $\widehat{f_0}$ is finite-state.

($\Leftarrow$) We may divide the set of the states of automaton $\mathcal{A}(\widehat{f_0})$ (that is generators of $\mathsf{Sg}(\widehat{f_0})$) in the following way:

$f_0$ is the state that satisfies a condition

$$f_0(v') = \alpha^\omega$$

for every $v' \in X_\alpha \alpha^\omega$,

$\{f_\varepsilon, f_1, \ldots, f_k\}$ are states related to the Rabin-Scott part of $\mathcal{A}(\widehat{f_0})$, i.e., satisfying

$$f_i(x) = \alpha$$

for all $x \in X$ and $f_\varepsilon$ is an initial state which corresponds to the extension $\widehat{f_0}$.

Moreover, $\{g_1, \ldots, g_m\}$ are remaining states of the automaton. It can be seen that they are right zeros of the semigroup $\mathsf{Sg}(\widehat{f_0})$.

It follows that:

1. the functions $f_0, g_1, \ldots, g_m$ (and consequently all compositions containing one or more of them) have finite images;

2. the functions $f_\varepsilon, f_1, \ldots, f_k$ may not have finite images, however from

$$f_\varepsilon(w) = \begin{cases} \alpha^\omega, & \text{if } w = \alpha^n u \alpha^\omega,\ u \in X^*,\ n \geq 1, \\ \alpha^{|u|} f(u) \alpha^\omega, & \text{if } w = u\alpha^\omega, \quad u \in X^*,\ u \neq \varepsilon; \end{cases}$$

$$f_i(w) = \begin{cases} v\alpha^\omega, & \text{if } w = \alpha^n u \alpha^\omega, u \in X^*, n \geq 1, \\ \alpha^{|u|} v \alpha^\omega, & \text{if } w = u\alpha^\omega, \quad u \in X^*,\ u \neq \varepsilon, \end{cases}$$

where all occurences of $v$ denote certain words from $f(X^*)$ that depend on $u$ and the particular generator $f_i$.

It can be seen that images of $f_\varepsilon f_\varepsilon$, $f_\varepsilon f_j$, $f_i f_\varepsilon$ and $f_i f_j$ are words of the form either $\alpha^\omega$ or

$$\alpha^n v \alpha^\omega\,, \qquad \text{with} \quad n \leq \max\{|f(u)| : u \in X^*\},$$

where $v$ is either a word from $f(X^*)$ or a prefix of such a word. Therefore, images of the mappings are finite.

Using the Lemma 6 with $n = 2$ we obtain $\mathsf{Sg}(\widehat{f_0})$ is finite. ■

**Lemma 9.** *Let $f : X^* \to X^*$ be a mapping satisfying $f(\varepsilon) = \varepsilon$. The semigroup $\mathsf{Sg}(\widehat{f_1})$ is finite iff $\widehat{f_1}$ is finite-state and nilpotent.*

***Proof.***

($\Rightarrow$) A finite number of generators is obtained iff $\widehat{f_0}$ is a finite-state. Also $\widehat{f_1}$ is nilpotent, for otherwise a semigroup $\langle \widehat{f_1} \rangle$ is infinite.

($\Leftarrow$) We can divide states of $\mathcal{A}(\widehat{f_1})$ similarly as in the proof of Lemma 8. Our situation differs only with

$$\delta(f_\varepsilon, \alpha) = f_\varepsilon$$

instead of

$$\delta(f_\varepsilon, \alpha) = f_0\,.$$

Therefore, possible images of the generators are

$$f_\varepsilon(\alpha^n u\alpha^\omega) = \alpha^{n+|u|} f(u)\alpha^\omega, \quad n \geq 0;$$

$$f_i(w) = \begin{cases} v\alpha^\omega, & \text{if } \ w = \alpha^n u\alpha^\omega, u \in X^*, n \geq 1, \\ \alpha^{|u|} v\alpha^\omega, & \text{if } \ w = u\alpha^\omega, u \in X^*, u \neq \varepsilon, \end{cases}$$

where occurences of $v$ represent certain words from $f(X^*)$ that depend on $u$ and the particular generator $f_i$. Omitted generators $g_i$ have images of the form $v\alpha^\omega$, where $v \leq w$ for certain $w \in f(X^*)$.

Let $S = \mathsf{Sg}(\widehat{f_1})$, $M = \max\{|f(u)| : u \in X^*\}$ and $K = \mathrm{nil}(\widehat{f_1})$. From above scheme it follows that values of the functions $h \in S$ are of the form

$$\alpha^n v\alpha^\omega\,, \qquad \text{for } \ n \geq 0,$$

where $v$ is either a word from $f(X^*)$ or a prefix of such a word.

In case of $h \in Sf_i$, we have $n \leq M$, thus, all compositions containing $f_i$ have finite images.

In case of $h \in Sg_i$, we have $n = 0$ and $|v| \leq M$, thus, all compositions containing $g_i$ have finite images.

The remaining cases are $h = f_\varepsilon^k$ for $k = 1, 2, \ldots$. From assumption $f_\varepsilon^K = \widehat{0}$, it is understood that the mappings $f_\varepsilon^k$, $k \geq K$ have finite images.

We proved that every mapping $h \in S^{K+}$ has finite image, therefore, using the Lemma 6 with $n = K$, we obtain that the semigroup $\mathsf{Sg}(\widehat{f_1})$ is finite. ■

**Lemma 10.** *Let $f : X^* \to X^*$ be a mapping satisfying $f(\varepsilon) = \varepsilon$. The semigroup $\mathsf{Sg}(\widehat{f_2})$ is finite iff $\widehat{f_2}$ is a finite-state and nilpotent.*

***Proof.***

($\Rightarrow$) A finite number of generators is obtained iff $\widehat{f_0}$ is a finite-state. Also $\widehat{f_2}$ is nilpotent, for otherwise a semigroup $\langle\widehat{f_2}\rangle$ is infinite.

($\Leftarrow$) From Definition 7, every segment $u_i\alpha^{k_i}$ is translated to $\alpha^{|u_i|}v_i\alpha^{n_i}$, where $v \leq f(u_i)$ and $|v_i| + n_i = k_i$. This property concerns not only $\widehat{f_2}$ but also every its restriction.

Let $S = \mathsf{Sg}(\widehat{f_2})$, $M = \max\{|f(u)| : u \in X^*\}$, $K = \mathrm{nil}(\widehat{f_2})$ and $u' \in X_\alpha^*\alpha^\omega$. Denoting the set of the restrictions of $\widehat{f_2}$ by F, mapping $h \in S$ is a composition

$$h = h_1 h_2 \ldots h_m, \qquad \text{where } \ h_k \in F\,.$$

Every segment $u_i$ of $u'$ produces the occurence of $u_i^{(1)}$ in $h_1(u')$, then the occurence of $u_i^{(2)}$ in $h_2(h_1(u'))$, and so on. We obtain a sequence

$$u_i^{(1)},\ u_i^{(2)},\ \ldots,\ \text{where } u_i^{(k)} \leq f\left(u_i^{(k-1)}\right)$$

which satisfies $u_i^{(k)} = \varepsilon$ starting from $k = K$, since $\widehat{f_2}$ is nilpotent.

Therefore, for every $h \in S^{K+}$ the value $h(u')$ does not depend on segments $u_i$ of the word $u'$. It is easy to see that $h(u')$ depends only on the length of the prefix $\alpha^n$ of the word $u'$. It follows that in a word $h(u')$ the positions starting from $K \cdot M$ are occupied by the symbol $\alpha$ and, thus, $h$ has a finite image.

Using Lemma 6 with $n = K$, we obtain $\mathsf{Sg}(\widehat{f_2})$ is finite. ■

**Proof of the main theorem.**
Desired results are established in Lemmas 8–10. The necessary properties of the nilpotent extensions are given in Propositions 3–5. ■

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