T-VARIETIES AND CLONES OF T-TERMS

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Abstract

The aim of this paper is to describe how varieties of algebras of type $\tau$ can be classified by using the form of the terms which build the (defining) identities of the variety. There are several possibilities to do so. In [3], [19], [15] normal identities were considered, i.e. identities which have the form $x \approx x$ or $s \approx t$, where $s$ and $t$ contain at least one operation symbol. This was generalized in [14] to $k$-normal identities and in [4] to $P$-compatible identities. More generally, we select a subset $T$ of $W_\tau(X)$, the set of all terms of type $\tau$, and consider identities from $T \times T$. Since any variety can be described by one heterogenous algebra, its clone, we are also interested in the corresponding clone-like structure. Identities of the clone of a variety $V$ correspond to $M$-hyperidentities for certain monoids $M$ of hypersubstitutions. Therefore we will also investigate these monoids and the corresponding $M$-hyperidentities.

Keywords: $T$-quasi constant algebra, $T$-identity, $j$-ideal, $T$-hyperidentity, clone of $T$-terms.

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1. Introduction

Let \( \tau = (n_i)_{i \in I} \) be a type of algebras, indexed by a set \( I \), with operation symbols \( f_i \) of arity \( n_i \). Let \( X = \{ x_1, x_2, x_3, \ldots \} \) be a countably infinite set of variables, and for each \( n \geq 1 \) let \( X_n = \{ x_1, x_2, \ldots, x_n \} \). We denote by \( W_\tau(X) \) and \( W_\tau(X_n) \) the sets of all terms, and of all \( n \)-ary terms of type \( \tau \), respectively. These two sets are the universes of two absolutely free algebras,

\[
F_\tau(X) := \left( W_\tau(X); (f_i)_{i \in I} \right)
\]

and

\[
F_\tau(X_n) := \left( W_\tau(X_n); (f_i)_{i \in I} \right),
\]

respectively. The operations \( f_i \) are defined by setting \( f_i(t_1, \ldots, t_{n_i}) := f_i(t_1, \ldots, t_{n_i}) \) for any variable \( x_j \in X_{n_i} \), and

\[
S_n^m := f_r(S_n^m(s_1, t_1, \ldots, t_n), \ldots, S_n^m(s_{n_r}, t_1, \ldots, t_n)).
\]
Using these operations, we form the heterogeneous or multi-based algebra

\[
\text{clone}(\tau) := ((W_\tau(X_n))_{n>0}; (S^m_m)_{n,m>0}, (x_i)_{i \leq n,n>0}).
\]

It is well-known and easy to check that this algebra satisfies the clone axioms

(C1) \(S^p_m(\tilde{Z}, S^n_m(\tilde{Y}_1, \tilde{X}_1, \ldots, \tilde{X}_n), \ldots, S^n_m(\tilde{Y}_p, \tilde{X}_1, \ldots, \tilde{X}_n)) \approx S^n_m(S^n_m(\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_p), \tilde{X}_1, \ldots, \tilde{X}_n), \text{ for } m, n, p = 1, 2, 3, \ldots,\)

(C2) \(S^m_m(\lambda_1, \tilde{X}_1, \ldots, \tilde{X}_n) \approx \tilde{X}_j, \text{ for } 1 \leq j \leq n \text{ and } m = 1, 2, 3, \ldots,\)

(C3) \(S^m_m(\tilde{X}_j, \lambda_1, \ldots, \lambda_m) \approx \tilde{X}_j, \text{ for } 1 \leq j \leq m \text{ and } m = 1, 2, 3, \ldots,\)

where \(S^p_m\) and \(S^n_m\) are operation symbols corresponding to the operations \(S^p_m\) and \(S^n_m\) of \(\text{clone}(\tau)\), where \(\lambda_1, \ldots, \lambda_m\) are nullary operation symbols and where \(\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_p, \tilde{X}_1, \ldots, \tilde{X}_m\) are variables. The algebra \(\text{clone}(\tau)\) is also called the clone of terms of type \(\tau\).

Since later on we have to consider subalgebras and congruences of heterogeneous algebras, we recall these concepts. A subalgebra of \(\text{clone}(\tau)\) consists of a sequence \((T^{(n)})_{n>0}\), where \(T^{(n)} \subseteq W_\tau(X_n)\) for all \(n > 0\) which is closed under all operations of \(\text{clone}(\tau)\). A congruence on \(\text{clone}(\tau)\) is a sequence \((\theta_n)_{n>0}\) of binary relations, where \(\theta_n \subseteq W_\tau(X_n) \times W_\tau(X_n)\), which is preserved by all operations from \(\text{clone}(\tau)\). For more background on heterogeneous algebras see \([16], [1]\).

Since the set \(W_\tau(X_n)\) of all \(n\)-ary terms of type \(\tau\) is closed under the superposition operation \(S^n := S^n_m\), there is a homogeneous analogue of this structure. The algebra \((W_\tau(X_n); S^n, x_1, \ldots, x_n)\) is an algebra of type \((n + 1, 0, \ldots, 0)\), which still satisfies the clone axioms above for the case that \(p = m = n\). Such an algebra is called a \textit{unitary Menger algebra of rank} \(n\) (see \([22]\)).
Let $n$-clone $(\tau) := (W_\tau(X_n); S^n)$ be the reduct of the unitary Menger algebra $(W_\tau(X_n); S^n, x_1, \ldots, x_n)$ of rank $n$. The algebra $n$-clone $(\tau)$ is called a Menger algebra of rank $n$.

If we consider the sequence $(W_\tau(X_n))_{n>0}$ together with the sequence of operations $(S^n_{m})_{m,n>0}$, we obtain a heterogeneous algebra $((W_\tau(X_n))_{n>0}; (S^n_{m})_{m,n>0})$ which we denote by $\text{Menger}(\tau)$. This heterogeneous algebra is called a Menger system (see [22]).

2. $T$- Identities

In [9] the authors studied the algebra $(W_{\tau/n}(X_n); S^n)$, the algebra of $n$-full terms of type $\tau$ and in [6], [14] the algebras of strongly full terms and of $k$-normal $n$-ary terms are studied. All of them are subalgebras of $(W_\tau(X_n); S^n)$. Now we generalize these results to an arbitrary subalgebra $T = (T; (S^n_{m})_{m,n>0})$ with $T := (T(n))_{n>0}$ and $T := \cup_{n>0} T(n)$ of the heterogeneous algebra $\text{Menger}(\tau)$. For any variety $V$ of type $\tau$, we define $Id^n_T V := \{ s \approx t \in IdV \mid s, t \in T(n) \}$, that is $Id^n_T V := IdV \cap T(n) \times T(n)$ for every $n > 0$, where $IdV$ is the set of all identities of $V$, and then $Id^T V := (Id^n_T V)_{n>0}$ is called the sequence of all $T$-identities of $V$. Then $Id^T V = \cup_{n>0} Id^n_T V$. We will also use the notation $Id^n_V := (Id^n_V)_{n>0}$, where $IdV$ is the union of the sets $Id^n V$. i.e. $IdV = \cup_{n>0} Id^n_V$.

Now we recall the following well-known facts:

**Lemma 2.1.** For any variety $V$ of type $\tau$, $IdV$ is a congruence on the algebra $\text{Menger}(\tau)$.

**Proof.** This follows from the fact that $IdV$ is a fully invariant congruence on the absolutely free algebra $F_\tau(X)$. ■

Now we consider subalgebras $T$ of the algebra $\text{Menger}(\tau)$ and will prove that the set of all $T$-identities of a variety $V$ is a congruence on $T$. We can use that $Id^n_T V = (Id^n_V)_{n>0}$ is a congruence on $\text{Menger}(\tau)$ and the well-known fact that for a congruence $\theta$ on an algebra $B$, and for a subalgebra $A \subseteq B$, the relation $\theta_A := \theta \cap (A \times A)$ is a congruence on $A$. Then we have

**Theorem 2.2.** For a subalgebra $T = ((T(n))_{n>0}; (S^n_{m})_{m,n>0})$ of the heterogeneous algebra $\text{Menger}(\tau)$ and for a variety $V$ of type $\tau$, the sequence $Id^T V$ is a congruence on $T$. ■
Because of the previous theorem, we can define the quotient algebra $T/Id^T V$ which we denote by $\text{clone}_T(V)$.

If $A$ is an algebra of type $\tau$ and if $s \approx t$ is an equation consisting of terms of type $\tau$, then $A \models s \approx t$ means that $s \approx t$ is satisfied as an identity in $A$. Let $T$ be a subset of $W_\tau(X)$ and let $R_T$ be the relation between $\text{Alg}(\tau)$, the set of all algebras of type $\tau$, and $T^2$, which is defined by

$$R_T := \{(A, s \approx t) \mid A \in \text{Alg}(\tau), s, t \in T \ (A \models s \approx t)\}.$$ 

This relation induces a Galois connection $(\text{Mod}^T, \text{Id}^T)$ between $\text{Alg}(\tau)$ and $T^2$ where the operations $\text{Mod}^T$ and $\text{Id}^T$ are defined as follows: For $K \subseteq \text{Alg}(\tau)$ and for $\Sigma \subseteq T^2$,

$$\text{Id}^T(K) = \{ s \approx t \in T^2 \mid \forall A \in K (A \models s \approx t)\} \quad \text{and} \quad \text{Mod}^T(\Sigma) = \{ A \in \text{Alg}(\tau) \mid \forall s \approx t \in \Sigma (A \models s \approx t)\}.$$ 

Clearly, the operator $\text{Mod}^T$ is the restriction of the usual operator $\text{Mod}$ to $T^2$. From the properties of a Galois connection we obtain that the products $\text{Mod}^T \text{Id}^T$ and $\text{Id}^T \text{Mod}^T$ are closure operators on the power set of $\text{Alg}(\tau)$ and of $T^2$, respectively.

Now we consider the variety $T(V) := \text{Mod}^T \text{Id}^T V$ for a given variety $V$ and $\text{Id}^T V := \text{Id} V \cap T^2$. It is clear that if $\text{Id} V \subseteq T^2$, then $T(V) = V$. Since $T^2$ must not be an equational theory in general, the converse is not true.

**Proposition 2.3.** Let $T$ be a subset of $W_\tau(X)$ and let $\mathcal{L}(\tau)$ be the lattice of all varieties of type $\tau$. Then the operator $C_T : \mathcal{L}(\tau) \rightarrow \mathcal{L}(\tau)$ defined by $C_T(V) = T(V)$ is a closure operator.

**Proof.** From $\text{Id}^T V \subseteq \text{Id} V$ there follows $V = \text{Mod} \text{Id} V \subseteq \text{Mod}^T \text{Id}^T V = T(V) = C_T(V)$. Using the fact that $\text{Mod}^T \text{Id}^T$ is a closure operator, we obtain $C_T(C_T(V)) = T(T(V)) = \text{Mod}^T \text{Id}^T(\text{Mod}^T \text{Id}^T V) = \text{Mod}^T \text{Id}^T V = T(V) = C_T(V)$. Finally, from $V_1 \subseteq V_2$, we have $C_T(V_1) = T(V_1) = \text{Mod}^T \text{Id}^T V_1 = \text{Mod}^T (\text{Id} V_1 \cap T^2) \subseteq \text{Mod}^T (\text{Id} V_2 \cap T^2) = T(V_2) = C_T(V_2)$. Altogether, we obtain that $C_T$ is a closure operator. □
The set of all fixed points of $C_T$ forms a sublattice of the lattice $\mathcal{L}(\tau)$, in fact it is a complete lattice (see [12]). Now we are interested in the variety $\text{Mod}(T \times T)$.

**Definition 2.4.** Let $T$ be a subset of $W_\tau(X)$. An algebra $A$ of type $\tau$ is called $T$-quasi constant algebra if there exists a term $t_0 \in T$ such that $t^A = t^A_0$ for all $t \in T$.

Let $TQ$ be the class of all $T$-quasi constant algebras of type $\tau$. This definition generalizes the concept of a constant algebra introduced in [3], and that of a quasi-constant algebra introduced in [9].

**Proposition 2.5.** Let $T$ be a subset of $W_\tau(X)$. Then $\text{Mod}(T \times T) = TQ$.

**Proof.** Let $A \in TQ$. Then there exists a term $t_0 \in T$ such that $t^A = t^A_0$ for all $t \in T$. Let $t_1, t_2$ be arbitrary terms in $T$. Then $t^A_1 = t^A_0 = t^A_2$. This means $A \models t_1 \approx t_2$ and hence $A \in \text{Mod}(T \times T)$. Conversely, let $A \in \text{Mod}(T \times T)$. Then $t^A_1 = t^A_2$ for all $t_1, t_2 \in T$. Therefore, $A \in TQ$.

**Corollary 2.6.** For any variety $V$, we have $T(V) = V \lor TQ$.

**Proof.** $T(V) = \text{Mod}^T \text{Id}^T V = \text{Mod}(\text{Id}V \cap T \times T) = \text{Mod} \text{Id}V \lor \text{Mod}(T \times T)$

$= V \lor TQ$.

We notice that a similar approach is contained in [13].

3. **$T$-HYPERSUBSTITUTIONS AND $T$-HYPERIDENTITIES**

To study the identities in the algebra $T = (\langle T^{(n)} \rangle_{n>0}; \langle S^n_m \rangle_{m,n>0})$, we need the concepts of $T$-hypersubstitutions and $T$-hyperidentities.

A hypersubstitution $\sigma$ of type $\tau$ is a mapping which assigns to each operation symbol $f_i$ of type $\tau$ an $n_i$-ary term $\sigma(f_i)$ of type $\tau$. Any hypersubstitution $\sigma$ induces a mapping $\tilde{\sigma}$ on the set $W_\tau(X)$ of all terms of type $\tau$, given by the following inductive definition:

(i) $\tilde{\sigma}[x_j] := x_j$, if $x_j \in X$ is a variable,
(ii) \( \hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^n_{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]) \),

for compound terms \( f_i(t_1, \ldots, t_{n_i}) \).

Let \( Hyp(\tau) \) be the set of all hypersubstitutions of type \( \tau \). A binary operation \( \circ_h \) can be defined on this set, by

\[
\sigma_1 \circ_h \sigma_2 = \hat{\sigma}_1 \circ \sigma_2,
\]

where \( \circ \) is the usual composition of mappings. It is well-known that \((Hyp(\tau); \circ_h, \sigma_{id})\) is a monoid, where \( \sigma_{id} \) is the identity hypersubstitution which is defined by \( \sigma_{id}(f_i) = f_i(x_1, \ldots, x_{n_i}) \) for all \( i \in I \).

Let \( T = ((T^{(n)})_{n>0}; (S^n_{m})_{m,n>0}) \) be a subalgebra of the algebra \( Menger(\tau) \). We define \( Hyp^T(\tau) \), the set of all \( T \)-hypersubstitutions as follows;

\[
Hyp^T(\tau) = \{ \sigma \in Hyp(\tau) \mid \forall i \in I \, \sigma(f_i) \in T \text{ and } \forall t \in T \, \hat{\sigma}[t] \in T \}.
\]

Then we get:

**Proposition 3.1.** Let \( T \) be a subalgebra of the algebra \( Menger(\tau) \). Then \( (Hyp^T(\tau); \circ_h) \) is a subsemigroup of the semigroup \( (Hyp(\tau); \circ_h) \). Moreover, \( (Hyp^T(\tau) \cup \{\sigma_{id}\}; \circ_h, \sigma_{id}) \) is a submonoid of \( (Hyp(\tau); \circ_h, \sigma_{id}) \). \( \blacksquare \)

**Definition 3.2.** Let \( T \) be a subalgebra of \( Menger(\tau) \) and let \( V \) be a variety of type \( \tau \) and \( Id^T_V \) be the set of all identities of \( V \) consisting of terms from \( T \), i.e., \( Id^T_V = Id_V \cap T^2 \). Then \( s \approx t \in Id^T_V \) is called a \( T \)-hyperidentity in \( V \) if \( \hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^T_V \) for all \( \sigma \in Hyp^T(\tau) \). If every identity in \( Id^T_V \) is satisfied as a \( T \)-hyperidentity, then the variety \( V \) is called \( Hyp^T(\tau) \)-solid; for short, we will write \( T \)-solid.

4. \( T \)-HYPERSUBSTITUTIONS AND ENDOmorphisms of \( T \)

There is a close connection between extensions of \( T \)-hypersubstitutions and endomorphisms of \( T \). This connection will be used later on to describe identities in the quotient algebra \( T/Id^T_V \).

For a hypersubstitution \( \sigma \) in \( Hyp(\tau) \), it is well-known that the induced mapping \( \hat{\sigma} \), regarded as a sequence \( \hat{\sigma} := (\hat{\sigma}^{(n)})_{n>0} \) with

\[
\hat{\sigma}^{(n)} : W_\tau(X_n) \to W_\tau(X_n)
\]

is an endomorphism on \( clone(\tau) \). If we apply \( \hat{\sigma} \) on a subalgebra \( T \), instead of \( \hat{\sigma}/T \) we will simply write \( \hat{\sigma} \). Consequently, we have:
Theorem 4.1. For any hypersubstitution $\sigma \in \text{Hyp}_T^\tau$, the sequence $\hat{\sigma}$ is an endomorphism on the algebra $T$.

As a consequence of Theorem 4.1, the set $\text{Im}(\hat{\sigma}) := \{\hat{\sigma}[t] \mid t \in W_+(X)\}$ is the universe of a subalgebra of $T$.

Since it is not clear that every subalgebra $T$ has an independent generating set, we assume in addition that the algebra $T$ has an independent generating set $G := (G(n))_{n>0}$. That is, $T$ is free with respect to itself, freely generated by the set $G$. Then any substitution $\eta := (\eta(n))_{n>0}$ from $G$ into $T$ with $\eta(n) : G(n) \to T^{(n)}$ for all $n > 0$ can be uniquely extended to an endomorphism $\overline{\eta} := (\overline{\eta}(n) : T^{(n)} \to T^{(n)})_{n>0}$ of $T$. Such mappings are called $T$-substitutions with respect to $G$. Let $\text{Subst}_G(T)$ be the set of all such $T$-substitutions with respect to $G$. Together with a binary composition $\circ$ defined by $\eta_1 \circ \eta_2 := (\eta_1(n) \circ \eta_2(n))_{n>0} := (\overline{\eta_1}(n) \circ \overline{\eta_2}(n))_{n>0}$, where $\circ$ is the usual composition of functions, $(\text{Subst}_G(T); \circ)$ is a semigroup. In fact, together with the identity mapping $id_T$ it is a monoid. Let $\text{End}(T)$ be the monoid of all endomorphisms of the algebra $T$. In the next theorems, we describe the connection between the monoids $\text{Subst}_G(T)$, $\text{End}(T)$ and $\text{Hyp}_T^\tau \cup \{\sigma_{id}\}$.

Theorem 4.2. Let $T$ be a subalgebra of $\text{Menger}(\tau)$ and $G$ be an independent generating system of $T$. Then the monoids $\text{Subst}_G(T)$ and $\text{End}(T)$ are isomorphic.

Proof. We define a heterogeneous mapping $\psi : \text{Subst}_G(T) \to \text{End}(T)$ by $\psi(\eta) = \overline{\eta}$ for $\eta \in \text{Subst}_G(T)$. Clearly, $\psi$ is well-defined since $\overline{\eta}$ is uniquely determined by $\eta$. For any $\eta_1, \eta_2 \in \text{Subst}_G(T)$, we have

$$
\psi(\eta_1 \circ \eta_2) = \overline{\eta_1 \circ \eta_2} = \overline{\eta_1} \circ \overline{\eta_2} = \overline{\eta_1} \circ \eta_2 = \psi(\eta_1) \circ \psi(\eta_2),
$$

since $(\overline{\eta_1} \circ \overline{\eta_2})|_G = \overline{\eta_1 \circ \eta_2}|_G$ and using the uniqueness of $\overline{\eta_1} \circ \eta_2$. Therefore, $\psi$ is a homomorphism. For injectivity, let $\eta_1, \eta_2 \in \text{Subst}_G(T)$ such that $\overline{\psi(\eta_1)} = \psi(\eta_2)$. Then $\overline{\eta_1} = \overline{\eta_2}$ and so $\overline{\eta_1} = \overline{\eta_2}$. This means $\eta_1 = \eta_2$. Thus, $\psi$ is injective. Clearly, $\psi$ is surjective since for any endomorphism $\eta$ on $T$ we have $\eta|_G$ is a $T$-substitution. This proves the theorem.
By Theorem 4.1, we may consider the set \( \{ \hat{\sigma}/T \mid \sigma \in Hyp^T(\tau) \} \cup \{ \hat{id}/T \} \). Clearly, this set forms a submonoid of the monoid \( (End(T); \circ_n, id_T) \) of all endomorphisms of \( T \).

5. \( T \)-hyperidentities and identities in \( T \)

We recall that for a subalgebra \( T = (\{(T^n)_{n>0}; (S^n_{m,n})_{m,n>0}\}) \) of the heterogeneous algebra \( Menger(\tau) = ((W_{\tau}(X^n))_{n>0}; (S^n_{m,n})_{m,n>0}) \) and for any variety \( V \) of type \( \tau \), by Theorem 2.2, the set of all \( T \)-identities of \( V \), \( Id^T_V \), is a congruence on the algebra \( T \). This allows us to define \( \text{clone}_T(V) = T/Id^T_V \).

Now we will use the following "translation mechanism" between elements of \( T \) and elements of a subalgebra of the absolutely free algebra \( F_\tau(X) \). The components of the sequence from the generating system \( G \) form a set \( G \) of terms of type \( \tau \) and the elements of \( T \) produced by application of the operation \( S^n_{m} \) from \( G \) correspond to elements of \( W_{\tau}(X) \) which arise by application of the operations \( f_i \) to elements from \( G \). This gives a one-to-one mapping \( \varphi \) (see [18]) between terms of type \( \tau \) and so-called operator terms formulated in the language of the heterogeneous algebra \( Menger(\tau) \).

To consider identities in \( \text{clone}_\tau(V) \), we need to build up the free heterogeneous algebra in a variety defined by \( (C_1) \) generated by the new variable system \( G^* \) which has the same cardinality as the system \( G \). This free heterogeneous algebra is denoted by \( F_\tau(G^*) \). This gives a one-to-one mapping \( \varphi \) from the system \( G \) onto the system \( G^* \). The extension of this mapping assigns to arbitrary elements from \( T \) the corresponding terms over the heterogeneous algebra \( T \).

**Theorem 5.1.** Let \( T \) be a subalgebra of the algebra \( Menger(\tau) \) which has an independent generating system \( G \), let \( V \) be a variety of type \( \tau \) and let \( s \approx t \in Id^T_V \). If \( \varphi(s) \approx \varphi(t) \) is an identity in \( \text{clone}_T(V) \), then it is a \( T \)-hyperidentity in \( V \). (That is \( \hat{\sigma}[s] \approx \hat{\sigma}[t] \) is an identity in \( V \) for all \( \sigma \in Hyp^T(\tau) \).)

**Proof.** Let \( \varphi(s) \approx \varphi(t) \) be an identity in \( \text{clone}_T(V) \). Then for every valuation \( \nu \) we have \( \nu(\varphi(s)) = \nu(\varphi(t)) \). The composition

\[
\text{nat } Id^T_V \circ \hat{\sigma} \circ \varphi^{-1}
\]

is the extension of a valuation mapping into \( \text{clone}_\tau(V) \), and so we have
\[ \varphi(s) \approx \varphi(t) \in Id(\text{clone}_T(V)) \Rightarrow (\text{natId}_V^T \circ \hat{\sigma} \circ \varphi^{-1})\varphi(s) \]

\[ = (\text{natId}_V^T \circ \hat{\sigma} \circ \varphi^{-1})\varphi(t) \]

\[ \Rightarrow \text{natId}_V^T \circ \hat{\sigma}(s) = \text{natId}_V^T \circ \hat{\sigma}(t) \]

\[ \Rightarrow [\hat{\sigma}^{(n)}[s]]_{Id}^T V = [\hat{\sigma}^{(n)}[t]]_{Id}^T V \quad \text{for every } n > 0 \]

\[ \Rightarrow \hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^T V \]

for every \( \sigma \in \text{Hyp}^T(\tau) \). Therefore \( s \approx t \) is satisfied as a \( T \)-hyperidentity in \( V \).

\textbf{Definition 5.2.} Let \( T \) be a nonempty subset of the universe of the algebra \( \text{Menger}(\tau) \). We call \( T \) a \( j \)-ideal of \( \text{Menger}(\tau) \) if there is an integer \( j \) with \( 1 \leq j \leq n + 1 \) and for any terms \( t_1, t_2, \ldots, t_{n+1} \), such that \( t_j \in T \), imply \( S_{n}^n(t_1, \ldots, t_j, \ldots, t_{n+1}) \in T \). A set \( T \) is called an ideal if it is a \( j \)-ideal for all \( 1 \leq j \leq n + 1 \) and for all \( n \).

It is clear that every \( j \)-ideal is a subalgebra of \( \text{Menger}(\tau) \).

Let \( \tau = (1) \) and \( f \) be a unary operation symbol. We consider the algebra \( (W_{(1)}(X_1); S_{f}^1) \). It is easy to see that the set \( N^k := \{ t \in W_{(1)}(X_1) \mid \text{op}(t) \geq k \} \) is an ideal of the algebra \( (W_{(1)}(X_1); S_{f}^1) \). \( \text{op}(t) \) is the number of occurrences of the operation symbol \( f \) in the term \( t \).

\textbf{Theorem 5.3.} Let \( T \) be an ideal of the algebra \( \text{Menger}(\tau) \) which has an independent generating system \( G \). Then \( T \times T \cup \Delta_{W_{\tau}(X)} \) is a fully invariant congruence on the absolutely free algebra \( F_{\tau}(X) \).

\textbf{Proof.} It is clear that \( T \times T \cup \Delta_{W_{\tau}(X)} := \rho \) is an equivalence relation on \( W_{\tau}(X) \). To prove the compatibility, we let \( (s_1, t_1), \ldots, (s_n, t_n) \in \rho \). If there exists \( s_i \in T \) for some \( 1 \leq i \leq n_i \), then \( t_i \) is also in \( T \) and we obtain
\( \mathcal{F}(s_1, \ldots, s_n) = S^n_m(f(x_1, \ldots, x_m), s_1, \ldots, s_n) \in T \) and similarly, we also have
\( \mathcal{F}(t_1, \ldots, t_n) = S^n_m(f(x_1, \ldots, x_m), t_1, \ldots, t_n) \in T \), since \( T \) is an ideal of \( \text{Menger}(\tau) \). If \( s_i \notin T \) for all \( 1 \leq i \leq n_i \), then \( s_i = t_i \) for all \( i \), and therefore
\( \mathcal{F}(s_1, \ldots, s_n) = \mathcal{F}(t_1, \ldots, t_n) \). Hence \( (\mathcal{F}(s_1, \ldots, s_n), \mathcal{F}(t_1, \ldots, t_n)) \in \rho \).

Next we will show that \( \rho \) is closed under any endomorphism on \( \mathcal{F}_\tau(X) \). Let \( (s, t) \in \rho \) and let \( \varphi \) be any endomorphism on \( \mathcal{F}_\tau(X) \). If \( s = t \), then clearly \( \varphi(s) = \varphi(t) \), and so \( (\varphi(s), \varphi(t)) \in \rho \). In the case \( s, t \in T \) it is easy to see that \( \varphi(s) = S^n_m(s, s_1, s_2, \ldots, s_n) \) and \( \varphi(t) = S^n_m(t, s_1, s_2, \ldots, s_n) \), where \( s_i = \varphi(x_i) \) for all \( 1 \leq i \leq n \) and \( x_i \) are variables occurring in terms \( s \) and \( t \). Since \( T \) is an ideal and \( s, t \in T \), we have \( S^n_m(s, s_1, s_2, \ldots, s_n) \) and \( S^n_m(t, s_1, s_2, \ldots, s_n) \) belong to \( T \). Thus \( (\varphi(s), \varphi(t)) \in \rho \). ■

Let \( V \) be a variety \( V \) of type \( \tau \). If \( \text{Id} V \cap T^2 \) is closed under all endomorphisms of \( \text{Menger}(\tau) \), then, by Theorem 4.1, it is closed under \( \hat{\sigma} \) for any \( T \)-hypersubstitution \( \sigma \), and therefore the variety \( T(V) = \text{Mod}(\text{Id} V \cap T^2) \) is \( T \)-solid. Then we have

**Proposition 5.4.** Let \( V \) be a variety of type \( \tau \). If \( \text{Id}^T V \) is a fully invariant congruence on \( \text{Menger}(\tau) \), then the variety \( T(V) \) is \( T \)-solid.

**References**


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