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CATEGORIES OF FUNCTORS BETWEEN CATEGORIES WITH PARTIAL MORPHISMS

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Dedicated to
Dr. habil. Hans-Jürgen Hoehnke
on the occasion of his 80th birthday.

Abstract

It is well-known that the composition of two functors between categories yields a functor again, whenever it exists. The same is true for functors which preserve in a certain sense the structure of symmetric monoidal categories. Considering small symmetric monoidal categories with an additional structure as objects and the structure preserving functors between them as morphisms one obtains different kinds of functor categories, which are even *dt*-symmetric categories.

Keywords: symmetric monoidal category, *dhts*-category, Hoehnke category, Hoehnke theory, monoidal functor, *d*-monoidal functor, *dht*-symmetric functor, functor composition, cartesian product.

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1. Introduction

Categories of "partial morphisms" have become a subject of stronger interest by several authors more than 25 years ago, since such categories are of importance in different branches of mathematics and computer science.

Hoehnke ([8]) introduced already in 1976 the basic concept of a "Hoehnke category", by himself named "diagonal-halfterminal-symmetric category", as a symmetric monoidal category in the sense of Eilenberg-Kelly ([4]) with additional properties.

It is easy to see that other approaches, given e.g. in [1], [2], [3], [13], [15], or [16], respectively, are more or less related to the concept of Hoehnke. More precisely, the concept of a Hoehnke category comprises the other concepts mentioned above and reflects best the properties of the category $\underline{Par}$ of all partial functions between arbitrary sets. A Hoehnke category $\underline{K}$, endowed with a morphism family $\nabla = (\nabla_A \in K[A \otimes A, A] \mid A \in |K|)$ characterized by two conditions, allows a category-theoretical characterization of $\underline{Par}$ (cf. [19], [9]). This observation leads to the introduction of *dht*-symmetric categories endowed with so called diagonal inversions ∇, see [22].

A *symmetric monoidal category* in the sense of Eilenberg-Kelly ([4]) is a sequence

$$K^{\bullet} = (K, \otimes, I, a, r, l, s)$$

consisting of a category K, a bifunctor $\otimes : K \times K \to K$, a distinguished object $I \in |K|$, and families $a = (a_{A,B,C} \in K[A \otimes (B \otimes C), (A \otimes B) \otimes C] \mid A, B, C \in |K|)$, $r = (r_A \in K[A \otimes I, A] \mid A \in |K|)$, $l = (l_A \in K[I \otimes A, A] \mid A \in |K|)$, $s = (s_{A,B} \in K[A \otimes B, B \otimes A] \mid A, B \in |K|)$ of isomorphisms in K (associativity, right-identity, left-identity, symmetry) such that the following conditions are fulfilled:

<u>Bifunctor conditions</u>:

(F1) $\forall \rho, \rho' \in K \ (\mathrm{dom}\,(\rho \otimes \rho') = \mathrm{dom}\,\rho \otimes \mathrm{dom}\,\rho')$,

(F2) $\forall \rho, \rho' \in K \ (\mathrm{cod}\,(\rho \otimes \rho') = \mathrm{cod}\,\rho \otimes \mathrm{cod}\,\rho')$,

(F3) $\forall A, B \in |K| \ (1_{A \otimes B} = 1_A \otimes 1_B)$,

(F4) $\forall A, B, C, A', B', C' \in |K| \ \forall \rho \in K[A, B], \sigma \in K[B, C],$

$$\rho' \in K[A', B'], \sigma' \in K[B', C'] \ ((\rho \otimes \rho')(\sigma \otimes \sigma') = \rho\sigma \otimes \rho'\sigma'),$$

Conditions of monoidality:

(M1) $\forall A, B, C, D \in |K|$

$$(a_{A,B,C\otimes D} a_{A\otimes B,C,D} = (1_A \otimes a_{B,C,D}) a_{A,B\otimes C,D} (a_{A,B,C} \otimes 1_D)),$$

(M2) $\forall A, B \in |K|\ (a_{A,I,B}(r_A \otimes 1_B) = 1_A \otimes l_B)$,

(M3) $\forall A, B, C \in |K|\ (a_{A,B,C} s_{A\otimes B,C} a_{C,A,B} = (1_A \otimes s_{B,C}) a_{A,C,B} (s_{A,C} \otimes 1_B))$,

(M4) $\forall A, B \in |K|\ (s_{A,B} s_{B,A} = 1_{A\otimes B})$,

(M5) $\forall A \in |K|\ (s_{A,I} l_A = r_A)$,

(M6) $\forall A, B, C, A', B', C' \in |K|\ \forall \rho \in K[A, A'], \sigma \in K[B, B'], \tau \in K[C, C']$

$$(a_{A,B,C}((\rho \otimes \sigma) \otimes \tau) = (\rho \otimes (\sigma \otimes \tau)) a_{A',B',C'}),$$

(M7) $\forall A, A' \in |K|\ \forall \rho \in K[A, A']\ (r_A \rho = (\rho \otimes 1_I) r_{A'})$,

(M8) $\forall A, B \in |K|\ \forall \rho \in K[A, A'], \sigma \in K[B, B'] (s_{A,B}(\sigma \otimes \rho) = (\rho \otimes \sigma) s_{A',B'})$.

The defining conditions of a symmetric monoidal category determine a lot of properties (see for example [22] or [26]), especially concerning the so-called "middle-exchange isomorphism"

$$b_{A,B,C,D} \in K[(A \otimes B) \otimes (C \otimes D), (A \otimes C) \otimes (B \otimes D)]$$

defined for arbitrary $A, B, C, D \in |K|$ by

(B1) $$b_{A,B,C,D} := a_{A\otimes B,C,D} \left(a^{-1}_{A,B,C} (1_A \otimes s_{B,C}) a_{A,C,B} \otimes 1_D \right) a^{-1}_{A\otimes C,B,D},$$

for instance:

(M15) $\forall A, B, C, D, A', B', C', D' \in |K|$

$\forall \rho \in K[A, A'] \; \forall \sigma \in K[B, B'] \; \forall \lambda \in K[C, C'] \; \forall \mu \in K[D, D']$

$(b_{A,B,C,D}((\rho \otimes \lambda) \otimes (\sigma \otimes \mu)) = ((\rho \otimes \sigma) \otimes (\lambda \otimes \mu))b_{A',B',C'D'}).$

(M19) $\forall A, B \in |K| \; (b_{A,I,I,B} = 1_{A\otimes I} \otimes 1_{I\otimes B}),$

Let $K^\bullet$ be a symmetric monoidal category as above. A sequence $(K^\bullet; d)$ is called *diagonal-symmetric monoidal category* or shortly, *ds-category* (in [7] considered in the strict case as a special Kronecker-category, in [22] "diagonal-symmetrische Kategorie") if $d = (d_A \in K[A, A \otimes A] \mid A \in |K|)$ is a family of morphisms of K such that the

Conditions of diagonality:

(D1) $\forall A, A' \in |K| \; \forall \varphi \in K[A, A'] \; (\varphi d_{[A'} = d_A(\varphi \otimes \varphi)),$

(D2) $\forall A \in |K| \; (d_A(d_A \otimes 1_A) = d_A(1_A \otimes d_A)a_{A,A,A}),$

(D3) $\forall A \in |K| \; (d_A s_{A,A} = d_A),$

(D4) $\forall A, B \in |K| \; ((d_A \otimes d_B)b_{A,A,B,B} = d_{A\otimes B})$

are fulfilled, where $b_{A,B,C,D}$ is the middle exchange isomorphism defined as above.

$(K^\bullet, d, t)$ is called *diagonal-terminal-symmetric monoidal category* or *dts-category* (cf. [7]) if $(K^\bullet, d)$ is a *ds*-category containing a family $t = (t_A \mid A \in |K|)$ of terminal morphisms $t_A \in K[A, I]$ such that the conditions

(T1) $\forall A, A' \in |K| \; \forall \varphi \in K[A, A'] \; (\varphi t_{A'} = t_A)$ and

(DTR) $\forall A \in |K| \; (d_A(1_A \otimes t_A)r_A = 1_A)$

are right.

$(K^\bullet; d, t)$ will be called *diagonal-halfterminal-symmetric monoidal category* or, shortly, *dhts-category* (cf. [8], [18]), if $K^\bullet$ is a symmetric monoidal category endowed with morphism families d and t as above, such that

(D1) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A']\ (d_A(\varphi \otimes \varphi) = \varphi d_{A'})$,

(DTR) $\forall A \in |K|\ (d_A(1_A \otimes t_A)r_A = 1_A)$,

(DTL) $\forall A \in |K|\ (d_A(t_A \otimes 1_A)l_A = 1_A)$,

(DTRL) $\forall A_1, A_2 \in |K|\ (d_{A_1 \otimes A_2}((1_{A_1} \otimes t_{A_2})r_{A_1} \otimes (t_{A_1} \otimes 1_{A_2})l_{A_2}) = 1_{A_1 \otimes A_2}))$,

(TT) $\forall A, B \in |K|\ (t_{A \otimes B} = (t_A \otimes t_B)t_{I \otimes I})$

are fulfilled. $(K^\bullet; d, \nabla)$ is called *diagonal-diagonalinversional-symmetric monoidal category* or *d*∇*s-category* (cf. [22]) if $(K^\bullet; d)$ is a *ds*-category such that there is a family $\nabla = (\nabla_A : A \otimes A \to A \mid A \in |K|)$ of morphisms in K (so-called *diagonal inversions*) fulfilling the conditions

(D_1^*) $\forall A \in |K|\ (d_A \nabla_A = 1_A)$,

(D_2^*) $\forall A \in |K|\ (\nabla_A d_A d_{A \otimes A} = d_{A \otimes A}(\nabla_A d_A \otimes 1_{A \otimes A}))$,

(D_3^*) $\forall A \in |K|\ (\nabla_A d_A = (1_A \otimes d_A)a_{A,A,A}(\nabla_A \otimes 1_A))$,

(D_4^*) $\forall A \in |K|\ (\nabla_A d_A = \Big(d_A \otimes 1_A)a^{-1}_{A,A,A}(1_A \otimes \nabla_A)\Big)$, and

(∇1) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A']\ ((\varphi \otimes \varphi)\nabla_{A'} = \nabla_A \varphi)$.

Let $(K^\bullet, d)$ be a *ds*-category.

Then $(K^\bullet, d, \nabla)$ is called *diagonal-halfdiagonalinversional-symmetric monoidal category* or *dh*∇*s-category* (cf. [22]) if $\nabla = (\nabla_A \in K[A \otimes A, A] \mid A \in |K|)$ is a family of morphisms of K (diagonal inversions) such that (D_1^*), (D_2^*), (D_3^*), (D_4^*), and

($\nabla\nabla$) $\forall A \in |K|\ ((\nabla_A \otimes \nabla_A)\nabla_A = \nabla_{A \otimes A} \nabla_A)$

hold.

A *diagonal-halfterminal-halfdiagonalinversional-symmetric monoidal category*, for short $dhth\nabla s$-category, is a sequence $(K^\bullet; d, t, \nabla)$ such that $(K^\bullet; d, t)$ is a $dhts$-category and $\nabla = (\nabla_A : A \otimes A \to A \mid A \in |K|)$ is a family of morphisms in K with the properties (D_1^*) and (D_2^*).

Example. The category consisting of one object I and one morphism 1_I forms the simplest model of the axioms above, where $I \otimes I = I,\ a_{I,I,I} = 1_I,\ r_I = l_I = s_{I,I} = 1_{I\otimes I} = 1_I, d_I = 1_I, t_I = 1_I, \nabla_I = 1_I, 1_I 1_I = 1_I, 1_I \otimes 1_I = 1_I$. This symmetric monoidal category will be denoted by Ω.

$(K^\bullet; d, t, o)$ is called *Hoehnke category* (in [8], [18], and [22] named *dhts*-category), if $(K^\bullet; d, t)$ is a $dhts$-category as above endowed with a distinguished object O and a distinguished morphism $o \in K[I, O]$ such that

(O1) $\quad \forall A \in |K|\ (A \otimes O = O \otimes A = O)$,

(o1) $\quad \forall A \in |K|\ \forall \varphi \in K[A, O]\ (t_A o = \varphi)$, and

(o2) $\quad \forall A \in |K|\ \forall \psi \in K[O, A]\ ((1_A \otimes t_O) r_A = \psi)$

are valid.

Finally, a *di-Hoehnke category* or *Hoehnke category with half-diagonalinversions* (in [22] denoted as *dht*∇*-symmetric category*) $(K^\bullet; d, t, \nabla, o)$ is defined by the conditions that $(K^\bullet; d, t, o)$ is a Hoehnke category and $(K^\bullet; d, t, \nabla)$ is a $dhth\nabla s$-category.

Example. A simple model of a Hoehnke category (di-Hoehnke category), denoted by Γ, is given as follows:

There are exactly 2 objects and 5 morphisms:

$$|\Gamma| = \{I, O \neq I\}, \quad \text{and} \quad \Gamma = \Gamma[O, O] \cup \Gamma[O, I] \cup \Gamma[I, O] \cup \Gamma[I, I],$$

where

$$\Gamma[O, O] = \{1_O\},\ \Gamma[O, I] = \{t_O\},\ \Gamma[I, O] = \{o\},\ \Gamma[I, I] = \{1_I, o_{I,I} \neq 1_I\},$$

the $\otimes$-operation for the objects is defined by

$$O \otimes O = I \otimes O = O \otimes I = O;\ I \otimes I = I,$$

the composition of morphisms by

$$1_O 1_O = 1_O,\ o1_O = o = 1_I o = o_{I,I}o,\ ot_O = o_{I,I},\ t_O o = 1_O,$$

$$t_O = t_O 1_I = 1_O t_O = t_O o_{I,I},\ 1_I o_{I,I} = o_{I,I} 1_I = o_{I,I},\ 1_I 1_I = 1_I,$$

the distinguished morphisms are

$$a_{I,I,I} = r_I = l_I = s_{I,I} = 1_I,$$

$$a_{X,Y,Z} = r_X = l_X = s_{X,Y} = 1_O \text{ if } X = O\ \vee\ Y = O\ \vee Z = O,$$

$$d_I = t_I = \nabla_I = 1_I\ ,\ \ d_O = t_O = \nabla_O = 1_O,$$

and the $\otimes$-operation for morphisms is defined by

$$\forall \varphi \in \Gamma\ (1_O \otimes \varphi = \varphi \otimes 1_O = 1_O),$$

$$o \otimes o = o,\ t_O \otimes t_O = t_O,\ o \otimes t_O = t_O \otimes o = 1_O,$$

$$1_I \otimes o = o \otimes 1_I = o \otimes o_{I,I} = o_{I,I} \otimes o = o,$$

$$1_I \otimes o_{I,I} = o_{I,I} \otimes 1_I = o_{I,I} \otimes o_{I,I} = o_{I,I},$$

$$1_I \otimes 1_I = 1_I.$$

It is easy to show that a $dhts$-category is a ds-category and each dts-category is a $dhts$-category too. Moreover, every Hoehnke category is a $dhts$-category, every $d\nabla s$-category is a $dh\nabla s$-category and each di-Hoehnke category is a $dhth\nabla s$-category. Altogether, there are the inclusions between the classes s-**C** of symmetric monoidal categories, ds-**C** of ds-categories, $dhts$-**C** of $dhts$-categories, $dh\nabla s$-**C** of $dh\nabla s$-categories, dts-**C** of dts-categories, $d\nabla s$-**C** of $d\nabla s$-categories, $dhth\nabla s$-**C** of $dhth\nabla s$-categories, Hoe-**C** of Hoehnke categories, and di-Hoe-**C** of di-Hoehnke categories, respectively, as described in Figure 1.

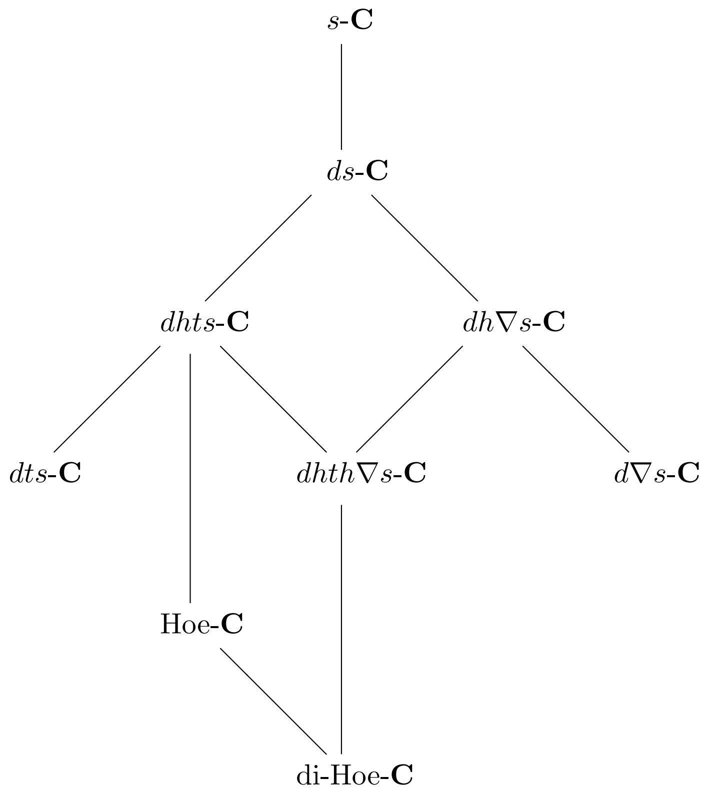

Figure 1.

Of importance is the fact that the relation $\leq$, defined by

$$\varphi \leq \psi :\Leftrightarrow \exists A, B \in |K| \ (\varphi, \psi \in K[A, B] \ \wedge \ d_A(\varphi \otimes \psi) = \varphi d_B)$$

is a nontrivial partial order relation in each $dhts$-category as well as in each $dh\nabla s$-category $\underline{K}$ ([18], [22]). Morphisms φ fulfilling $\varphi \leq \psi \quad \wedge \quad \varphi \neq \psi$ for any $\psi \in K$ are partial morphisms. In each Hoehnke category there exists the so-called zero morphism $o_{I,I} \in K[I, I]$, which is, because of $1_I \neq o_{I,I} \leq 1_I$, a proper partial morphism. Several important subcategories exist in every $dhts$-category $\underline{K}$ as follows (cf. [8], [18]):

$\mathbf{M}_K^{d,t}$, the *dts*-category generated by the families a, r, l, s, d, and t in $\underline{K}$,

$\mathbf{Iso}_K^{d,t}$, the *dts*-category generated by all isomorphisms and the families d and t in $\underline{K}$,

$\mathbf{Cor}_K^t$, the *dts*-category generated by all coretractions and the family t in $\underline{K}$,

$\mathbf{Tot}_K := \{\varphi \in K \mid \varphi t_{\mathrm{cod}\varphi} = t_{\mathrm{dom}\varphi}\}$, the *dts*-category of all "total morphisms" in $\underline{K}$, such that

$$\mathbf{M}_K^{d,t} \subseteq \mathbf{Iso}_K^{d,t} \subseteq \mathbf{Cor}_K^t \subseteq \mathbf{Tot}_K.$$

The classes $\mathbf{Cen}_K$ of all morphisms generated by the unit, associativity, right- and left-identity isomorphisms, and all their inverses in $\underline{K}$ ("central morphisms"), $\mathbf{Iso}_K$ of all isomorphisms of $\underline{K}$, and $\mathbf{Cor}_K$ of all coretractions of $\underline{K}$ form always symmetric monoidal subcategories of $\underline{K}$. Moreover, $\mathbf{Cor}_K$ is even a *ds*-category since all diagonal morphisms d_A are coretractions by (DTR).

Furthermore, of interest are functors between symmetric monoidal categories which preserve this structure in a certain sense ([24]). Such functors between *dt*-, *dht*-, $d\nabla$-, $dh\nabla$-, and $dhth\nabla$-symmetric categories, respectively, together with different kinds of "pseudonatural" transformations form certain symmetric monoidal categories ([24]).

Monoidal functors between different kinds of symmetric strictly monoidal categories $K^\bullet$ and $L^\bullet$ and their properties were investigated in [24], but the investigation is easely extendable to the general case.

If there is no danger of confusion, we will omit the index at the symbols $\otimes^{(K)}$ and $\otimes^{(L)}$, respectively, in the sequel.

A *monoidal functor* F from $K^\bullet$ into $L^\bullet$ is characterized by a family

$$\left(\widetilde{F}\langle A, B\rangle : AF \otimes BF \to (A \otimes B)F \mid A, B \in |K|\right)$$

of morphisms in L and a morphism $i_F : I^{(L)} \to I^{(K)}F \in L$ such that the following conditions are fulfilled.

(F$\sim$) $\forall A, B \in |K| \ \left(\widetilde{F}\langle A, B\rangle \in \mathbf{Iso}_L\right)$,

(FI) $i_F \in \mathbf{Iso}_L$,

(FA) $\forall A, B, C \in |K| \left(\left(1^{(L)}_{AF} \otimes \widetilde{F}\langle B, C\rangle\right) \widetilde{F}\langle A, B \otimes C\rangle \left(a^{(K)}_{A,B,C}F\right) = \right.$

$$\left. = a^{(L)}_{AF,BF,CF} \left(\widetilde{F}\langle A, B\rangle \otimes 1^{(L)}_{CF}\right) \widetilde{F}\langle A \otimes B, C\rangle\right),$$

(FR) $\forall A \in |K| \ \left(\widetilde{F}\langle A, I^{(K)}\rangle \left(r^{(K)}_A F\right) = \left(1^{(L)}_{AF} \otimes i_F^{-1}\right) r^{(L)}_{AF}\right)$,

(FS) $\forall A, B \in |K| \ \left(\widetilde{F}\langle A, B\rangle \left(s^{(K)}_{A,B}F\right) = s^{(L)}_{AF,BF}\widetilde{F}\langle B, A\rangle\right)$,

(FM) $\forall A, A', B, B' \in |K| \ \forall \varphi \in K[A, A'] \ \forall \psi \in K[B, B']$

$$\left((\varphi F \otimes \psi F)\, \widetilde{F}\langle A', B'\rangle = \widetilde{F}\langle A, B\rangle\, (\varphi \otimes \psi)\, F\right).$$

A monoidal functor F between *ds*-categories is called *d-monoidal* if in addition the condition

(FD) $\forall A \in |K| \ \left(d^{(K)}_A F = d^{(L)}_{AF}\widetilde{F}\langle A, A\rangle\right)$

is valid. $(F, \widetilde{F}, i_F)$ is called *strongly monoidal (strongly d-monoidal) functor* if $(F, \widetilde{F}, i_F)$ is a monoidal (d-monoidal) functor having the properties

$$\forall A, B \in |K| \ \left(\widetilde{F}\langle A, B\rangle = 1^{(L)}_{AF\otimes BF}\right) \qquad \text{and} \qquad i_F = 1^{(L)}_{I^{(L)}}.$$

Hoehnke proved in [8] the following fact:

Let $\underline{K}$ and $\underline{L}$ be at least *dhts*-categories and let $p_1^{A,B} = (1_A^{(K)} \otimes t_B^{(K)})r_A^{(K)}$, $p_2^{A,B} = (t_A^{(K)} \otimes 1_B^{(K)})l_B^{(K)}$, be the so-called canonical projections in $\underline{K}$. Then each functor $F : K \to L$ defines in a natural manner in $\underline{L}$ the morphism family

$$F^* := \Big(F^*\langle A, B\rangle := d^{(L)}_{(A\otimes B)F}\left(p_1^{A,B}F \otimes p_2^{A,B}F\right)$$

$$\in L[(A\otimes B)F, AF \otimes BF] \mid A, B \in |K|\Big),$$

satisfying the identities

$$\text{(FA}^*\text{)} \qquad \forall A, B, C \in |K| \ \left(\left(a^{(K)}_{A,B,C}F\right) F^*\langle A\otimes B, C\rangle \left(F^*\langle A, B\rangle \otimes 1^{(L)}_{CF}\right) = \right.$$

$$\left. = F^*\langle A, B\otimes C\rangle \left(1^{(L)}_{AF} \otimes F^*\langle B, C\rangle\right) a^{(L)}_{AF,BF,CF}\right),$$

$$\text{(FS}^*\text{)} \qquad \forall A, B \in |K| \ \left(\left(s^{(K)}_{A,B}F\right) F^*\langle B, A\rangle = F^*\langle A, B\rangle s^{(L)}_{AF,BF}\right),$$

$$\text{(FD}^*\text{)} \qquad \forall A \in |K| \ \left(\left(d^{(K)}_{A}F\right) F^*\langle A, A\rangle = d^{(L)}_{AF}\right),$$

$$\text{(FMT}^*\text{)} \quad \forall A, A', B, B' \in |K| \ \forall \varphi \in \mathbf{Tot}_K[A, A'] \ \forall \psi \in \mathbf{Tot}_K[B, B']$$

$$(F^*\langle A, B\rangle(\varphi F \otimes \psi F) = (\varphi \otimes \psi)FF^*\langle A', B'\rangle),$$

$$\text{(wFR}^*\text{)} \quad \forall A \in |K| \ \left(F^*\langle A, I\rangle \left(1^{(L)}_{AF} \otimes t^{(L)}_{IF}\right) r^{(L)}_{AF} \leq r^{(K)}_{A}F\right),$$

$$\text{(wFL}^*\text{)} \quad \forall A \in |K| \ \left(F^*\langle I, A\rangle \left(t^{(L)}_{IF} \otimes 1^{(L)}_{AF}\right) l^{(L)}_{AF} \leq l^{(K)}_{A}F\right),$$

(wFM*) $\forall A, A', B, B' \in |K|\ \forall \varphi \in K[A, A']\ \forall \psi \in K[B, B']$

$$((\varphi \otimes \psi)\, FF^*\langle A', B'\rangle \leq F^*\langle A, B\rangle(\varphi F \otimes \psi F)).$$

Moreover, there is the morphism $t^{(L)}_{I^{(K)}F} \in L[I^{(K)}F, I^{(L)}]$. In the case that F is a d-monoidal functor with respect to $\widetilde{F}$ and i_F, one obtains

$$\forall A, B \in |K| \left(F^*\langle A, B\rangle = \left(\widetilde{F}\langle A, B\rangle\right)^{-1}\right),$$

$$t^{(L)}_{I^{(K)}F} = (i_F)^{-1},$$

and

(FT) $\forall A \in |K| \left(t^{(K)}_A F t^{(L)}_{I^{(K)}F} = t^{(L)}_{AF}\right)$.

Conversely, let F be a functor between *dhts*-categories $\underline{K}$ and $\underline{L}$ such that all morphisms $F^*\langle A, B\rangle$ and $t^{(L)}_{I^{(K)}F}$ are isomorphisms in $\underline{L}$ and the condition

(FM*) $\forall A, A', B, B' \in |K|\ \forall \varphi \in K[A, A']\ \forall \psi \in K[B, B']$

$$((\varphi \otimes \psi)FF^*\langle A', B'\rangle = F^*\langle A, B\rangle(\varphi F \otimes \psi F))$$

is true.

Then $(F, (F^*)^{-1}, (t^{(L)}_{I^{(K)}F})^{-1}) : \underline{K} \to \underline{L}$ is a d-monoidal functor ([8]), [24]).

Moreover, let F be a functor fulfilling (FM*) such that

(sF$*$) $\forall A, B \in |K| \left(F^*\langle A, B\rangle = 1^{(L)}_{(A\otimes B)F}\right)$ and

(sFI$*$) $t^{(L)}_{I^{(K)}F} = 1^{(L)}_{I^{(L)}}$

are satified. Then $(F, (1^{(L)}_{(A\otimes B)F} \mid A, B \in |K|), 1^{(L)}_{I^{(L)}})$ is a strongly d-monoidal functor.

Example. A simple example of a monoidal functor between symmetric monoidal categories $K^\bullet$ and $L^\bullet$ is given by $(E, \widetilde{E}, i_E)$ with the properties $\forall A \in |K|\ (AE = I^{(L)})$, $\forall \varphi \in K\ (\varphi E = 1^{(L)}_{I^{(L)}})$, $\forall A, B \in |K|\ (\widetilde{E}\langle A, B\rangle = r^{(L)}_{I^{(L)}})$, $i_E = 1^{(L)}_{I^{(L)}}$.

Let $\underline{K}$ and $\underline{L}$ be *dhts*-categories. Then the functor $E : K \to L$ is even d-monoidal, since:

$$\forall A, B \in |K|\ \left(E^*\langle A, B\rangle = d^{(L)}_{(A\otimes B)E}\left(p_1^{A,B}E \otimes p_2^{A,B}E\right) = d^{(L)}_{I^{(L)}} \in \mathbf{Iso}_L\right),$$

$$t^{(L)}_{I^{(K)}E} = t^{(L)}_{I^{(L)}} = 1^{(L)}_{I^{(L)}} \in \mathbf{Iso}_L, \text{ and}$$

$$\forall A, A', B, B' \in |K|\ \forall \varphi \in K[A, A']\ \forall \psi \in K[B, B']$$

$$\Big((\varphi \otimes \psi)EE^*\langle A', B'\rangle = 1^{(L)}_{I^{(L)}} d^{(L)}_{I^{(L)}}$$

$$= d^{(L)}_{I^{(L)}}\left(1^{(L)}_{I^{(L)}} \otimes 1^{(L)}_{I^{(L)}}\right) = E^*\langle A, B\rangle(\varphi E \otimes \psi E)\Big).$$

Remark. Hoehnke introduced in [8] the concept of a *dht*-symmetric functor between Hoehnke categories. This concept differs from that of a d-monoidal functor presented here as follows:

Instead of (FM) Hoehnke demands the weaker condition

(FMT) $\forall A, A', B, B' \in |K|\ \forall \varphi \in \mathbf{Tot}_K[A, A']\ \forall \psi \in \mathbf{Tot}_K[B, B']$

$$\left((\varphi F \otimes \psi F)\widetilde{F}\langle A', B'\rangle = \widetilde{F}\langle A, B\rangle(\varphi \otimes \psi)F\right)$$

and instead of (FI) the fact $t^{(L)}_{I^{(K)}F} F^1 = 1^{(L)}_{IF}$ for a suitable morphism F^1 in L.

A *Hoehnke functor* $F : \underline{K} \to \underline{L}$ is, by definition, a d-monoidal functor which preserves the zero object, i.e. $O^{(K)}F = O^{(L)}$, or it is one of the functors U $(\forall A \in |K| \ (AU = O^{(L)}, \ \ \forall \varphi \in K \ (\varphi U = 1^{(L)}_{O^{(L)}}))$ or E as above. ■

Functors between symmetric monoidal categories which preserve the whole symmetric monoidal structure directly are of importance for the further considerations.

Lemma 1.1 (cf. [24]). *Let $F : K^\bullet \to L^\bullet$ be an arbitrary functor between the symmetric monoidal categories $K^\bullet$ and $L^\bullet$ possessing the properties*

(sFI) $I^{(K)}F = I^{(L)}$,

(sFA) $\forall A, B, C \in |K| \left(a^{(K)}_{A,B,C}F = a^{(L)}_{AF,BF,CF}\right)$,

(sFR) $\forall A \in |K| \ \left(r^{(K)}_{A}F = r^{(L)}_{AF}\right)$,

(sFS) $\forall A, B \in |K| \ \left(s^{(K)}_{A,B}F = s^{(L)}_{AF,BF}\right)$,

(sFM) $\forall A, A', B, B' \in |K| \ \forall \varphi \in K[A, A'] \ \forall \psi \in K[B, B']$

$$((\varphi F \otimes \psi F) = (\varphi \otimes \psi)F).$$

Then $(F, (1^{(L)}_{AF\otimes BF} \mid A, B \in |K|), 1^{(L)}_{I^{(L)}})$ is a strongly monoidal functor and

(sFL) $\forall A \in |K| \ \left(l^{(K)}_{A}F = l^{(L)}_{AF}\right)$

is right.

If in addition $\underline{K}$ and $\underline{L}$ are ds-categories and F has the property (sFD) $\forall A \in |K| \ \left(d_A^{(K)} F = d_{AF}^{(L)}\right)$,

then $(F, (1^{(L)}_{AF\otimes BF} \mid A, B \in |K|), 1^{(L)}_{I^{(L)}})$ *is a strongly d-monoidal functor.*

Proof. First, for all $A, B \in |K|$,

$$1^{(L)}_{(A\otimes B)F} = 1^{(K)}_{A\otimes B} F = \left(1^{(K)}_A \otimes 1^{(K)}_B\right) F$$

$$= 1^{(K)}_A F \otimes 1^{(K)}_B F = 1^{(L)}_{AF} \otimes 1^{(L)}_{BF} = 1^{(L)}_{AF\otimes BF}$$

by the properties of symmetric monoidal categories and the usual functor properties, hence

$$\forall A, B \in |K| \ ((A \otimes B)F = AF \otimes BF).$$

All unit morphisms are isomorphism, therefore (F∼) is true for $\widetilde{F}\langle A, B\rangle := 1^{(L)}_{AF\otimes BF}$ and there is the morphism $i_F = 1^{(L)}_{I^{(L)}} \in L[I^{(L)}, I^{(K)}F]$. With respect to the suitable unit morphisms, the conditions (FI), (FA), (FR), (FS), and (FM) are fulfilled via (sFI), (sFA), (sFR), (sFS), and (sFM), respectively.

Let $\underline{K}$ and $\underline{L}$ be *ds*-categories. Then (FD) is a trivial consequence of (sFD). ■

Corollary 1.2. *Let F be a functor between $dh\nabla s$-categories which has the properties* (sFI), (sFA), (sFR), (sFS), (sFM), and (sFD). *Then F has the property*

(sF∇) $\forall A \in |K| \ \left(\nabla_A^{(K)} F = \nabla_{AF}^{(L)}\right)$.

Proof. For an arbitrary object A in K, the equations

$$d_A^{(K)} \nabla_A^{(K)} = 1_A^{(K)} \quad \text{and} \quad d^{(K)}_{A\otimes A}\left(\nabla_A^{(K)} d_A^{(K)} \otimes 1^{(K)}_{A\otimes A}\right) = \nabla_A^{(K)} d_A^{(K)} d^{(K)}_{A\otimes A}$$

are valid, hence

$$1_{AF}^{(L)} = 1_A^{(K)}F = \left(d_A^{(K)}\nabla_A^{(K)}\right)F = d_A^{(K)}F\nabla_A^{(K)}F = d_{AF}^{(L)}\left(\nabla_A^{(K)}F\right)$$

and

$$d_{AF\otimes AF}^{(L)}\left(\left(\nabla_A^{(K)}F\right)d_{AF}^{(L)} \otimes 1_{AF\otimes AF}^{(L)}\right) =$$

$$= \left(d_{A\otimes A}^{(K)}F\right)\left(\left(\nabla_A^{(K)}F\right)\left(d_A^{(K)}F\right) \otimes \left(1_{A\otimes A}^{(K)}F\right)\right) =$$

(by (sFD))

$$= \left(d_{A\otimes A}^{(K)}\left(\nabla_A^{(K)}\left(d_A^{(K)} \otimes 1_{A\otimes A}^{(K)}\right)\right)\right)F =$$

(by (sFM))

$$= \left(\nabla_A^{(K)}d_A^{(K)}d_{A\otimes A}^{(K)}\right)F =$$

(by (sFD))

$$= \left(\nabla_A^{(K)}F\right)d_{AF}^{(L)}d_{AF\otimes AF}^{(L)}.$$

Since there is at most one morphism family in any $dh\nabla s$-category which fulfils both identities with respect to the diagonal morphisms (cf. [18]), one receives the claim. ■

Lemma 1.3 ([24]). *Let $\underline{K}$ and $\underline{L}$ be at least dhts-categories and let $F : K \to L$ be a functor between the underlying categories fulfilling the conditions* (sFM) *and*

(sFT) $\forall A \in |K|\ (t_A^{(K)}F = t_{AF}^{(L)})$.

Then

(F$*$) $\forall A, B \in |K|\ (F^*\langle A, B\rangle = d_{(A\otimes B)F}^{(L)}(p_1^{A,B}F \otimes p_2^{A,B}F) \in \mathbf{Iso}_L)$ *and* (sFI*) $\quad t^{(L)}_{I^{(K)}F} = t^{(L)}_{I^{(L)}} = 1^{(L)}_{I^{(L)}}$

are right, where the properties (sFI*) *and* (sFI) *are equivalent.*

Moreover, the functor F possesses in addition even the properties (sF$*$), (sFA), (sFR), (sFL), and (sFS), *whenever F fulfils beside* (sFT) and (sFM) *the condition* (sFD).

In other words, $(F, (1^{(L)}_{AF\otimes BF} \mid A, B \in |K|), 1^{(L)}_{I^{(L)}})$ is a strongly d-monoidal functor between dhts-categories, whenever (sFM), (sFT), and (sFD) *are right.*

Proof. Putting $A = I^{(K)}$ in (sFT), one obtains $1^{(L)}_{I^{(K)}F} = 1^{(K)}_{I^{(K)}}F = t^{(K)}_{I^{(K)}}F = t^{(L)}_{I^{(K)}F}$, hence $I^{(K)}F = codom^{(L)}(1^{(L)}_{I^{(K)}F}) = codom^{(L)}(t^{(L)}_{I^{(K)}F}) = I^{(L)}$, thus $t^{(L)}_{I^{(K)}F} = t^{(L)}_{I^{(L)}} = 1^{(L)}_{I^{(L)}}$.

The equivalence of (sFI) and (sFI*) is obvious.

As already proved, $\forall A, B \in |K|$ $((A \otimes B)F = AF \otimes BF)$ (cf. Lemma 1.1), therefore via (sFT),

$$F^*\langle A, B\rangle = d^{(L)}_{AF\otimes BF}\left(\left(1^{(L)}_{AF}\otimes t^{(L)}_{BF}\right)\otimes\left(t^{(L)}_{AF}\otimes 1^{(L)}_{BF}\right)\right)\left(r^{(K)}_A F\otimes l^{(K)}_B F\right) \in \mathbf{Iso}_L,$$

since $\quad d^{(L)}_{AF\otimes BF}((1^{(L)}_{AF} \otimes t^{(L)}_{BF}) \otimes (t^{(L)}_{AF} \otimes 1^{(L)}_{BF}))(r^{(L)}_{AF} \otimes l^{(L)}_{BF}) = 1^{(L)}_{AF\otimes BF}$

and $r^{(K)}_A F$ and $l^{(K)}_B F$ are isomorphisms too.

Assuming the validity of (sFD), one receives

$$F^*\langle A, B\rangle = d^{(L)}_{AF\otimes BF}\left(\left(1^{(L)}_{AF}\otimes t^{(L)}_{BF}\right)\otimes\left(t^{(L)}_{AF}\otimes 1^{(L)}_{BF}\right)\right)\left(r^{(K)}_A F\otimes l^{(K)}_B F\right)$$

$$=\left(d^{(L)}_{AF}\otimes d^{(L)}_{BF}\right) b^{(L)}_{AF,AF,BF,BF}\left(\left(1^{(L)}_{AF}\otimes t^{(L)}_{BF}\right)\otimes\left(t^{(L)}_{AF}\otimes 1^{(L)}_{BF}\right)\right)\left(r^{(K)}_A F\otimes l^{(K)}_B F\right) =$$

(by (D4) in $\underline{L}$)

$$=\left(d^{(L)}_{AF}\otimes d^{(L)}_{BF}\right)\left(\left(1^{(L)}_{AF}\otimes t^{(L)}_{AF}\right)\otimes\left(t^{(L)}_{BF}\otimes 1^{(L)}_{BF}\right)\right)b^{(L)}_{AF,I^{(K)}F,I^{(K)}F,BF}\left(r^{(K)}_{A}F\otimes l^{(K)}_{B}F\right)=$$
(by (M15) in $\underline{L}$)

$$=\left(d^{(L)}_{AF}\left(1^{(L)}_{AF}\otimes t^{(L)}_{AF}\right)\otimes d^{(L)}_{BF}\left(t^{(L)}_{BF}\otimes 1^{(L)}_{BF}\right)\right)b^{(L)}_{AF,I^{(L)},I^{(L)},BF}\left(r^{(K)}_{A}F\otimes l^{(K)}_{B}F\right)=$$
(by (sFI))

$$=\left(d^{(L)}_{AF}\left(1^{(L)}_{AF}\otimes t^{(L)}_{AF}\right)\otimes d^{(L)}_{BF}\left(t^{(L)}_{BF}\otimes 1^{(L)}_{BF}\right)\right)\left(1^{(L)}_{AF,I^{(L)}}\otimes 1^{(L)}_{I^{(L)},BF}\right)\left(r^{(K)}_{A}F\otimes l^{(K)}_{B}F\right)=$$
(by (M19) in $\underline{L}$)

$$=\left(d^{(L)}_{AF}\left(1^{(L)}_{AF}\otimes t^{(L)}_{AF}\right)r^{(K)}_{A}F\otimes d^{(L)}_{BF}\left(t^{(L)}_{BF}\otimes 1^{(L)}_{BF}\right)l^{(K)}_{B}F\right)=$$

$$=\left(d^{(K)}_{A}F\left(1^{(K)}_{A}F\otimes t^{(K)}_{A}F\right)r^{(K)}_{A}F\otimes d^{(K)}_{B}F\left(t^{(K)}_{B}F\otimes 1^{(K)}_{B}F\right)l^{(K)}_{B}F\right)=$$
(by (sFT), (sFD))

$$=\left(\left(d^{(K)}_{A}F\left(1^{(K)}_{A}\otimes t^{(K)}_{A}\right)r^{(K)}_{A}\right)F\otimes\left(d^{(K)}_{B}\left(t^{(K)}_{B}\otimes 1^{(K)}_{B}\right)l^{(K)}_{B}\right)F\right)=$$
(by (sFM))

$$=\left(1^{(K)}_{A}F\otimes 1^{(K)}_{B}F\right)=1^{(L)}_{AF\otimes BF}.$$

Since $t^{(L)}_{I^{(K)}F}=1^{(L)}_{I^{(L)}}$ is an isomorphism, all $F^*\langle A,B\rangle=1^{(L)}_{AF\otimes BF}$ are isomorphisms and (sFM) is expected, $(F,(1^{(L)}_{AF\otimes BF}\mid A,B\in|K|),1^{(L)}_{I^{(L)}})$ is a d-monoidal functor between the *dhts*-categories $\underline{K}$ and $\underline{L}$, therefore the conditions (sFA), (sFR), (sFL), and (sFS) are fulfilled. ∎

Example. Let $K^\bullet$ be a symmetric monoidal category. Then $\Theta_K : K\to\Omega$, defined by $(A\mapsto I,\ \varphi\mapsto 1_I)$, is a strongly monoidal (d-monoidal) functor with respect to $\widetilde{\Theta}_K$ ($\widetilde{\Theta}_K\langle A,B\rangle := 1_I$) and $i_{\Theta_K} := 1_I$.

2. The cartesian product of categories

It is well-known that two categories K and L determine a new category $K \times L$, the so-called *cartesian product*, consisting of objects (A, B), $A \in |K|$, $B \in |L|$ and morphisms (φ, ψ), $\varphi \in K$, $\psi \in L$, where the structure of $K \times L$ is defined via the components in the ordered pairs by the structure in K and L, respectively.

The cartesian product $(K \times L)^{\bullet}$ of symmetric monoidal categories $K^{\bullet}$ and $L^{\bullet}$ is a symmetric monoidal category too. More precisely:

Proposition 2.1. *Let $K^{\bullet}$ and $L^{\bullet}$ be symmetric monoidal categories. Then all ordered pairs (A, B) of objects $A \in |K|$ and $B \in |L|$ together with all ordered pairs (φ, ψ) of morphisms $\varphi \in K$ and $\psi \in L$ form in a natural manner a symmetric monoidal category $(K \times L)^{\bullet}$, where the monoidal structure is defined componentvise:*

$$(A_1, B_1) \otimes (A_2, B_2) := (A_1 \otimes_K A_2, B_1 \otimes_L B_2),$$

$$(\varphi_1, \psi_1) \otimes (\varphi_2, \psi_2) := (\varphi_1 \otimes_K \varphi_2, \psi_1 \otimes_L \psi_2),$$

$$I := (I^{(K)}, I^{(L)}), \qquad a_{(A_1,A_2),(B_1,B_2),(C_1,C_2)} := \left(a^{(K)}_{A_1,B_1,C_1}, a^{(L)}_{A_2,B_2,C_2}\right),$$

$$r_{(A_1,A_2)} := \left(r^{(K)}_{A_1}, r^{(L)}_{A_2}\right), \quad l_{(A_1,A_2)} := \left(l^{(K)}_{A_1}, l^{(L)}_{A_2}\right),$$

$$s_{(A_1,A_2),(B_1,B_2)} := \left(s^{(K)}_{A_1,B_1}, s^{(L)}_{A_2,B_2}\right).$$

Moreover, if two symmetric monoidal categories possess additional properties concerning the monoidal structure, then the cartesian product $(K \times L)^{\bullet}$ has the same properties, especially:

Defining in addition

$$d_{(A_1,A_2)} := \left(d^{(K)}_{A_1}, d^{(L)}_{A_2}\right), \quad t_{(A_1,A_2)} := \left(t^{(K)}_{A_1}, t^{(L)}_{A_2}\right),$$

$$\nabla_{(A_1,A_2)} := \left(\nabla^{(K)}_{A_1}, \nabla^{(L)}_{A_2}\right),$$

$$O := \left(O^{(K)}, O^{(L)}\right), \quad o := \left(o^{(K)}, o^{(L)}\right), \textit{ respectively,}$$

one obtains a ds-, dts-, dhts-, d∇s-, dh∇s-, dhth∇s-category, Hoehnke category, and di-Hoehnke category $\underline{K \times L}$, respectively, whenever $\underline{K}$ and $\underline{L}$ are ds-, dts-, dhts-, d∇s-, dh∇s-, dhth∇s-categories, Hoehnke categories, and di-Hoehnke categories.

The necessary proofs of all the presented assertions are easy to do and will be left to the reader.

3. Composition of functors

Besides the $\otimes$-operation for functors between symmetric monoidal categories, investigated in [24] and already introduced in [8] by $A(F \otimes G) := AF \otimes AG$, $\varphi(F \otimes G) := \varphi F \otimes \varphi G$, there is the usual composition of functors $F : K^\bullet \to L^\bullet$ and $G : L^\bullet \to P^\bullet$.

Lemma 3.1 (cf. [8]). *Let $(F, \widetilde{F}, i_F) : K^\bullet \to M^\bullet$ and $(G, \widetilde{G}, i_G) : M^\bullet \to P^\bullet$ be monoidal functors between symmetric monoidal categories. Then $FG : K \to P$, defined by the usual functor composition, is a monoidal functor with respect to*

$$\widetilde{FG} := \left(\widetilde{FG}\langle A, B\rangle = \widetilde{G}\langle AF, BF\rangle(\widetilde{F}\langle A, B\rangle G) \mid A, B \in |K|\right)$$

and $i_{FG} := i_G(i_F G)$.

$(FG, \widetilde{FG}, i_{FG})$ is a strongly monoidal functor whenever both $(F, \widetilde{F}, i_F)$ and $(G, \widetilde{G}, i_G)$ are strongly monoidal functors.

Finally, $\mathbf{1}\langle K\rangle F = F = F\mathbf{1}\langle M\rangle$ for all monoidal functors F, where $\mathbf{1}\langle K\rangle$ is the identical functor of K.

Proof. Each morphism of the kind $\widetilde{G}\langle AF, BF\rangle(\widetilde{F}\langle A, B\rangle G)$ is an isomorphism in P, since $\forall A', B' \in |M|\ (\widetilde{G}\langle A', B'\rangle \in \mathbf{Iso}_P)$, $\forall A, B \in |K|\ (\widetilde{F}\langle A, B\rangle \in \mathbf{Iso}_M)$, and every functor preserves isomorphisms. For the same reason,

$$i_G \in \mathbf{Iso}_P \ \wedge \ i_F \in \mathbf{Iso}_M \Rightarrow i_{FG} = i_G(i_F G) \in \mathbf{Iso}_P \cap P[I^{(P)}, I(FG)].$$

To prove the conditions (FA), (FR), (FS), and (FM) for $(FG, \widetilde{FG}, i_{FG})$ one uses in a natural manner the properties of functors, the properties of symmetric monoidal categories, and the properties of the monoidal functors $(F, \widetilde{F}, i_F)$ and $(G, \widetilde{G}, i_G)$, respectively, as follows, where A, B, C, D are arbitrary objects of K:

Ad (FA):
Using the validity of the condition (FA) for the functors F and G one obtains

$$\left(1^{(P)}_{A(FG)} \otimes \widetilde{FG}\langle B, C\rangle\right) \widetilde{FG}\langle A, B \otimes C\rangle a^{(K)}_{A,B,C}(FG) =$$

$$= \left(1^{(P)}_{A(FG)} \otimes \widetilde{G}\langle BF, CF\rangle \left(\widetilde{F}\langle B, C\rangle\right) G\right) \widetilde{G}\langle AF, (B \otimes C)F\rangle$$
$$\left(\widetilde{F}\langle A, B \otimes C\rangle\right) G \left(a^{(K)}_{A,B,C} F\right) G =$$

$$= \left(1^{(P)}_{A(FG)} \otimes \widetilde{G}\langle BF, CF\rangle\right) \left(\left(1^{(M)}_{AF}\right) G \otimes \left(\widetilde{F}\langle B, C\rangle\right) G\right) \widetilde{G}\langle AF, (B \otimes C)F\rangle$$
$$\left(\widetilde{F}\langle A, B \otimes C\rangle\right) G \left(a^{(K)}_{A,B,C} F\right) G =$$

$$= \left(1^{(P)}_{A(FG)} \otimes \widetilde{G}\langle BF, CF\rangle\right) \widetilde{G}\langle AF, BF \otimes CF\rangle$$
$$\left(\left(1^{(M)}_{AF} \otimes \widetilde{F}\langle B, C\rangle\right) G\right) \left(\widetilde{F}\langle A, B \otimes C\rangle\right) G \left(a^{(K)}_{A,B,C} F\right) G =$$

$$= \left(1^{(P)}_{A(FG)} \otimes \widetilde{G}\langle BF, CF\rangle\right) \widetilde{G}\langle AF, BF \otimes CF\rangle$$
$$\left(\left(1^{(M)}_{AF} \otimes \widetilde{F}\langle B, C\rangle\right) \left(\widetilde{F}\langle A, B \otimes C\rangle\right) a^{(K)}_{A,B,C} F\right)\right) G =$$

$$= \left(1^{(P)}_{A(FG)} \otimes \widetilde{G}\langle BF, CF\rangle\right) \widetilde{G}\langle AF, BF \otimes CF\rangle$$
$$\left(a^{(M)}_{AF,BF,CF} \left(\widetilde{F}\langle A, B\rangle \otimes 1^{(M)}_{CF}\right) \widetilde{F}\langle A \otimes B, C\rangle\right) G =$$

$$= \left(1^{(P)}_{A(FG)} \otimes \widetilde{G}\langle BF, CF\rangle\right) \widetilde{G}\langle AF, BF \otimes CF\rangle \left(a^{(M)}_{AF,BF,CF}\right) G$$

$$\left(\left(\widetilde{F}\langle A, B\rangle \otimes 1^{(M)}_{CF}\right) \widetilde{F}\langle A \otimes B, C\rangle\right) G =$$

$$= a^{(P)}_{(AF)G,(BF)G,(CF)G} \left(\widetilde{G}\langle AF, BF\rangle \otimes 1^{(P)}_{(CF)G}\right) \widetilde{G}\langle AF \otimes BF, CF\rangle$$

$$\left(\left(\widetilde{F}\langle A, B\rangle \otimes 1^{(M)}_{CF}\right) \widetilde{F}\langle A \otimes B, C\rangle\right) G =$$

$$= a^{(P)}_{(AF)G,(BF)G,(CF)G} \left(\widetilde{G}\langle AF, BF\rangle \otimes 1^{(P)}_{(CF)G}\right)$$

$$\widetilde{G}\langle AF \otimes BF, CF\rangle \left(\widetilde{F}\langle A, B\rangle \otimes' 1'_{CF}\right) G \left(\widetilde{F}\langle A \otimes B, C\rangle\right) G =$$

$$= a^{(P)}_{(AF)G,(BF)G,(CF)G} \left(\widetilde{G}\langle AF, BF\rangle \otimes 1^{(P)}_{(CF)G}\right)$$

$$\left(\widetilde{F}\langle A, B\rangle\right) G \otimes \left(1^{(M)}_{CF}\right) G\widetilde{G}\langle (A \otimes B)F, CF\rangle \left(\widetilde{F}\langle A \otimes B, C\rangle\right) G =$$

$$= a^{(P)}_{(AF)G,(BF)G,(CF)G} \left(\widetilde{G}\langle AF, BF\rangle \left(\widetilde{F}\langle A, B\rangle\right) G \otimes 1^{(P)}_{(CF)G}\right)$$

$$\widetilde{G}\langle (A \otimes B)F, CF\rangle \left(\widetilde{F}\langle A \otimes B, C\rangle\right) G =$$

$$= a^{(P)}_{(AF)G,(BF)G,(CF)G} \left(\widetilde{FG}\langle A, B\rangle \otimes 1^{(P)}_{(CF)G}\right) \widetilde{FG}\langle A \otimes B, C\rangle.$$

Ad (FR):
Since F and G both fulfil (FR), the following is true:

$$\widetilde{FG}\langle A, I^{(K)}\rangle r^{(K)}_A (FG) =$$

$$= \widetilde{G}\langle AF, I^{(K)}F\rangle \left(\widetilde{F}\langle A, I^{(K)}\rangle G\right) \left(r^{(K)}_A F\right) G =$$

$$= \widetilde{G}\langle AF, I^{(K)}F\rangle \left(\widetilde{F}\langle A, I^{(K)}\rangle r^{(K)}_A F\right) G =$$

$$= \widetilde{G}\langle AF, I^{(K)}F\rangle \left(\left(1^{(M)}_{AF} \otimes i^{-1}_F\right) r^{(M)}_{AF}\right) G =$$

$$= \widetilde{G}\langle AF, I^{(K)}F\rangle \left(1^{(M)}_{AF} \otimes i_F^{-1}\right) G r^{(M)}_{AF} G =$$

$$= \left(1^{(M)}_{AF} G \otimes i_F^{-1} G\right) \widetilde{G}\langle AF, I^{(M)}\rangle r^{(M)}_{AF} G =$$

$$= \left(1^{(P)}_{A(FG)} \otimes (i_F G)^{-1}\right) \left(1^{(P)}_{(AF)G} \otimes i_G^{-1}\right) r^{(P)}_{(AF)G} =$$

$$= \left(1^{(P)}_{A(FG)} \otimes (i_G(i_F G)\right)^{-1} r^{(P)}_{A(FG)} = \left(1^{(P)}_{A(FG)} \otimes (i_{FG})^{-1}\right) r^{(P)}_{A(FG)}.$$

Ad (FS):
The functor FG has this property since

$$\widetilde{FG}\langle A, B\rangle s_{A,B}(FG) = \widetilde{G}\langle AF, BF\rangle \left(\widetilde{F}\langle A, B\rangle s_{A,B} F\right) G =$$

$$= \widetilde{G}\langle AF, BF\rangle \left(s^{(M)}_{AF,BF} \widetilde{F}\langle B, A\rangle\right) G = \widetilde{G}\langle AF, BF\rangle \left(s^{(M)}_{AF,BF} G\right)\left(\widetilde{F}\langle B, A\rangle\right) G =$$

$$= s^{(P)}_{(AF)G,(BF)G} \widetilde{G}\langle BF, AF\rangle \left(\widetilde{F}\langle B, A\rangle\right) G = s^{(P)}_{A(FG),B(FG)} \widetilde{FG}\langle B, A\rangle$$

via the definition of $\widetilde{FG}$ and the validity of (FS) for F and G.

Ad (FM):
Let $\varphi \in K[A, C]$, $\psi \in K[B, D]$ be arbitrary morphisms of K. Then

$$\widetilde{FG}\langle A, B\rangle (\varphi \otimes \psi)(FG) = \widetilde{G}\langle AF, BF\rangle \left(\widetilde{F}\langle A, B\rangle\right) G((\varphi \otimes \psi)F)G =$$

$$= \widetilde{G}\langle AF, BF\rangle \left(\left(\widetilde{F}\langle A, B\rangle\right) (\varphi \otimes \psi)\, F\right) G =$$

$$= \widetilde{G}\langle AF, BF\rangle \left((\varphi F \otimes \psi F)\, \widetilde{F}\langle C, D\rangle\right) G =$$

$$= \widetilde{G}\langle AF, BF\rangle (\varphi F \otimes \psi F) G \left(\widetilde{F}\langle C, D\rangle\right) G =$$

$$= ((\varphi F)G \otimes (\psi F)G)\, \widetilde{G}\langle CF, DF\rangle \left(\widetilde{F}\langle C, D\rangle\right) G =$$

$$= (\varphi(FG) \otimes \psi(FG)) \widetilde{FG}\langle C, D\rangle.$$

Now let $(F, \widetilde{F}, i_F)$ and $(G, \widetilde{G}, i_G)$ be strongly monoidal functors. Then

$$\forall A, B \in |K| \ \left(\widetilde{F}\langle A, B\rangle = 1^{(M)}_{AF\otimes BF}\right) \ \wedge \forall X, Y \in |M| \ \left(\widetilde{G}\langle X, Y\rangle = 1^{(P)}_{XG\otimes YG}\right),$$

therefore

$$\forall A, B \in |K| \ \left(\widetilde{FG}\langle A, B\rangle = 1^{(P)}_{(AF)G\otimes(BF)G}\left(1^{(M)}_{AF\otimes BF}\right) G = \right.$$

$$\left. = 1^{(P)}_{(AF)G\otimes(BF)G} 1^{(P)}_{(AF\otimes BF)G} = 1^{(P)}_{A(FG)\otimes B(FG)}\right).$$

Because of $i_F = 1^{(M)}_{I^{(M)}}$ and $i_G = 1^{(P)}_{I^{(P)}}$ one obtains

$$i_{FG} = i_G(i_F)G = 1^{(P)}_{I^{(P)}}\left(1^{(M)}_{I^{(M)}}\right)G = 1^{(P)}_{I^{(P)}}$$

Obviously, the validity of (sFA), (sFR), (sFS), and (sFM) is transmitted from F and G to the functor FG. ■

Theorem 3.2. *The class $|MON|$ of all small symmetric monoidal categories together with the monoidal functors between them forms a category* **MON**.

*All strongly monoidal functors establish a subcategory s***MON** *of* **MON**.

Proof. There is the identical functor $\mathbf{1}\langle K\rangle$ to each symmetric monoidal category $K^\bullet$ and $\mathbf{1}\langle K\rangle$ is a monoidal functor with respect to

$$\widetilde{\mathbf{1}\langle K\rangle} = \left\{1^{(K)}_{A\otimes B} \mid A, B \in |K|\right\} \ \text{ and } \ i_{\mathbf{1}\langle K\rangle} = 1^{(K)}_{I^{(K)}}.$$

Because of Lemma 3.1, the composition of two monoidal funtors is a monoidal functor too and

$$\left(1\langle K\rangle, \widetilde{1\langle K\rangle}, i_{1\langle K\rangle}\right)\left(F, \widetilde{F}, i_F\right) = \left(F, \widetilde{F}, i_F\right) = \left(F, \widetilde{F}, i_F\right)\left(1\langle M\rangle, \widetilde{1\langle M\rangle}, i_{1\langle M\rangle}\right)$$

for every monoidal functor $(F, \widetilde{F}, i_F) : K^\bullet \to M^\bullet$, since

$$\widetilde{1\langle K\rangle F}\langle A,B\rangle = \widetilde{1\langle K\rangle}\langle A,B\rangle\left(\widetilde{F}\langle A,B\rangle\right)1\langle K\rangle = \widetilde{F}\langle A,B\rangle =$$

$$= \widetilde{F}\langle A,B\rangle\left(\widetilde{1\langle M\rangle}\langle A,B\rangle\right)F = \widetilde{F1\langle M\rangle}\langle A,B\rangle$$

and

$$i_{1\langle K\rangle F} = i_F\left(i_{1\langle K\rangle}\right)F = (i_F)\left(1^{(K)}_{I^{(K)}}\right)F = i_F 1^{(M)}_{I^{(M)}} = i_F =$$

$$= 1^{(K)}_{I^{(K)}}(i_F)1\langle M\rangle = i_{1\langle M\rangle}(i_F)1\langle M\rangle = i_{F1\langle M\rangle}.$$

The usual functor composition is associative, i.e. $F(GH) = (FG)H$. Moreover, for all objects A and B of K the following is true:

$$\widetilde{F(GH)}\langle A,B\rangle = \widetilde{GH}\langle AF,BF\rangle\left(\widetilde{F}\langle A,B\rangle\right)(GH) =$$

$$= \widetilde{H}\langle (AF)G,(BF)G\rangle\left(\widetilde{G}\langle AF,BF\rangle\right)H\left(\left(\widetilde{F}\langle A,B\rangle\right)G\right)H =$$

$$= \widetilde{H}\langle A(FG),B(FG)\rangle\left(\widetilde{G}\langle AF,BF\rangle\left(\widetilde{F}\langle A,B\rangle\right)G\right)H =$$

$$= \widetilde{H}\langle A(FG),B(FG)\rangle\left(\widetilde{FG}\langle A,B\rangle\right)H =$$

$$= \widetilde{(FG)H}\langle A,B\rangle.$$

Therefore, $\widetilde{F(GH)} = \widetilde{(FG)H}$.

The assertion concerning strongly monoidal functors is obvious. ■

Corollary 3.3. *There are the following subcategories of* **MON**. *The class*
• $|dMON|$ *of all small ds-categories together with the d-monoidal functors between them forms a category* d**MON**,

• $|dhtMON|$ *of all small dhts-categories together with the d-monoidal functors between them forms a category* dht**MON**,

• $|HOE|$ *of all small Hoehnke categories together with the Hoehnke functors between them forms a category* **HOE**,

• $|dh\nabla MON|$ *of all small dh∇s-categories together with the d-monoidal functors between them forms a category dh∇***MON**,

• $|dtMON|$ *of all small dts-categories together with the d-monoidal functors between them forms a category dt***MON**,

• $|dhth\nabla MON|$ *of all small dhth∇s-categories together with the d-monoidal functors between them forms a category dhth∇***MON**,

• $|di\text{-}HOE|$ *of all small di-Hoehnke categories together with the Hoehnke functors between them forms a category di* − **HOE**,

• $|d\nabla MON|$ *of all small d∇s-categories together with the d-monoidal functors between them forms a category d∇***MON**.

Proof. By Theorem 3.2, it remains to show that the composition of two d-monoidal functors is d-monoidal too. This is true because of

$$d_A^{(K)}(FG) = \left(\left(d_A^{(K)}\right)F\right)G = \left(d_{AF}^{(M)}\widetilde{F}\langle A,A\rangle\right)G = \left(d_{AF}^{(M)}G\right)\left(\widetilde{F}\langle A,A\rangle\right)G =$$

$$= \left(d_{A(FG)}^{(P)}\widetilde{G}\langle AF,AF\rangle\right)\left(\widetilde{F}\langle A,A\rangle\right)G = d_{(AF)G}^{(P)}\widetilde{FG}\langle A,A\rangle.$$

One has for strongly monoidal functors F and G immediately:

$$d_A^{(K)}(FG) = \left(\left(d_A^{(K)}\right)F\right)G = \left(d_{AF}^{(M)}\right)G = d_{(AF)G}^{(P)} = d_{A(FG)}^{(P)}. \qquad \blacksquare$$

The diagram in Figure 2 illustrates the mutual inclusions in the general case.

Similarly, one has the subcategories sd**MON**, $sdht$**MON**, s**HOE**, $sdh\nabla$**MON**, $sdhth\nabla$**MON**, sdi-**HOE**, sdt**MON**, and $sd\nabla$**MON** of s**MON** in the case of strongly monoidal functors, i.e. a similar diagram for the subcategories of all strongly monoidal functors.

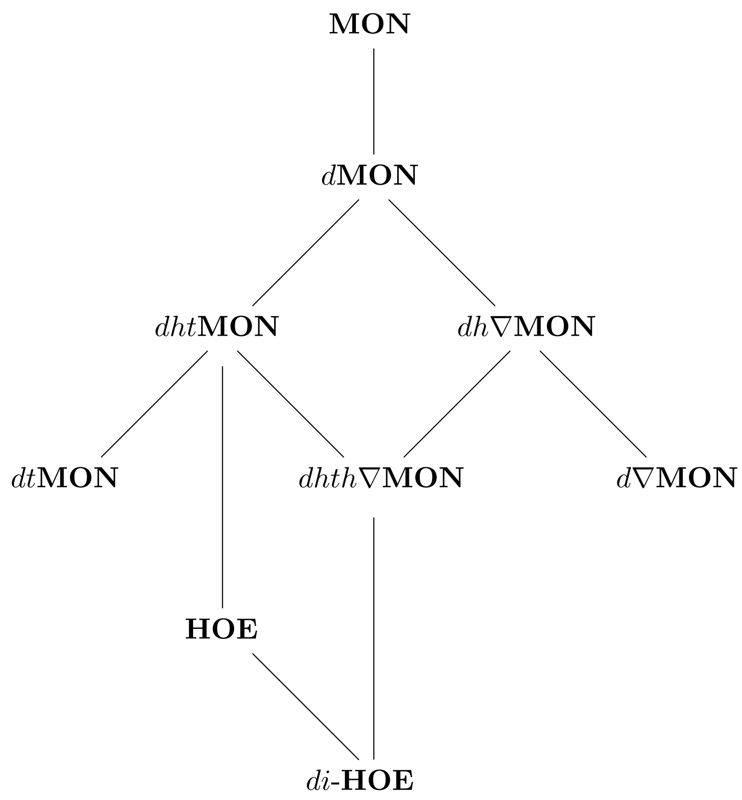

Figure 2.

Hoehnke proved in [8] (Theorem 6.1) that "the composition $FG : \mathcal{K} \to \mathcal{K}''$ of two *dht*-symmetric functors $F : \mathcal{K} \to \mathcal{K}'$, $G : \mathcal{K}' \to \mathcal{K}''$ is again *dht*-symmetric ...". In addition to this result one receives:

Lemma 3.4. *Let* $\underline{K}$, $\underline{M}$, $\underline{P}$ *be dhts-categories and let* $F : K \to M$, $G : M \to P$ *be functors. Then* FG *satisfies*

$$\text{(FC}^*) \quad \forall A, B \in |K| \; ((FG)^*\langle A, B\rangle = (F^*\langle A, B\rangle)GG^*\langle AF, BF\rangle)$$

and

$$\forall A, B \in |K| \ ((FG)^*\langle A, B\rangle \in \mathbf{Iso}_P),$$

whenever

$$\forall A, B \in |K| \ (F^*\langle A, B\rangle \in \mathbf{Iso}_M) \ \textit{and} \ \forall X; Y \in |M| \ (G^*\langle X, Y\rangle \in \mathbf{Iso}_P).$$

Moreover, let F and G fulfil (FT)*. Then FG also has the property* (FT)*.*

Proof. Ad (FC*):

$$(FG)^*\langle A, B\rangle = d^{(P)}_{(A\otimes B)(FG)} \left((p_1{}^{A,B}) (FG) \otimes (p_2{}^{A,B})(FG)\right) =$$

$$= d^{(P)}_{((A\otimes B)F)G} \left(\left((p_1{}^{A,B}\right) F\right) G \otimes \left(\left(p_2{}^{A,B}\right) F\right) G\right) =$$

(by functor property)

$$= \left(d^{(M)}_{(A\otimes B)F}\right) GG^*\langle (A\otimes B)F, (A\otimes B)F\rangle (((p_1{}^{A,B})F)G \otimes ((p_2{}^{A,B})F)G) =$$

(by (FD*) for G)

$$= \left(d^{(M)}_{(A\otimes B)F}\right) G((p_1{}^{A,B})F \otimes (p_2{}^{A,B})F)GG^*\langle AF, BF\rangle =$$

(by (FMT*) for G)

$$= \left(\left(d^{(K)}_{A\otimes B}\right) FF^*\langle A\otimes B, A\otimes B\rangle\right) G((p_1{}^{A,B})F \otimes (p_2{}^{A,B})F)GG^*\langle AF, BF\rangle =$$

(by (FD*) for F)

$$= \left(\left(d^{(K)}_{A\otimes B}\right) F\right) G((p_1{}^{A,B} \otimes p_2{}^{A,B})F)(F^*\langle A, B\rangle)GG^*\langle AF, BF\rangle =$$

(by (FMT*) for F)

$$= \left(\left(d^{(K)}_{A\otimes B} \left(p_1{}^{A,B} \otimes p_2{}^{A,B}\right)\right) F\right) G(F^*\langle A, B\rangle)GG^*\langle AF, BF\rangle =$$

(by functor property)

$$= (F^*\langle A, B\rangle)GG^*\langle AF, BF\rangle$$

(by (DTRL) in K).

The claim about the isomorphism property follows immediately by (FC*) and the fact that each functor maps isomorphisms onto isomorphisms.

Because of functor properties and the property (FT) for F and G, we have

$$t_A^{(K)}(FG)t^{(P)}_{I^{(K)}(FG)} = \left(t_A^{(K)}F\right)Gt^{(P)}_{(I^{(K)}F)G} = \left(\left(t_A^{(K)}F\right)Gt^{(M)}_{I^{(K)}F}\right)Gt^{(P)}_{I^{(M)}G} =$$

$$= \left(t_A^{(K)}Ft^{(M)}_{I^{(K)}F}\right)Gt^{(P)}_{I^{(M)}G} = \left(t^{(M)}_{AF}G\right)t^{(P)}_{I^{(M)}G} = t^{(P)}_{(AF)G} = t^{(P)}_{A(FG)}. \qquad \blacksquare$$

Proposition 3.5. *Let $F : \underline{K} \to \underline{M}$ and $G : \underline{M} \to \underline{P}$ be functors between dhts-categories $\underline{K}$, $\underline{M}$, $\underline{P}$, such that both fulfil the conditions* (F∗)*,* (FI*)*, and* (FM*)*. Then the composition yields a functor $FG : \underline{K} \to \underline{P}$ between the dhts-categories $\underline{K}$ and $\underline{P}$ fulfilling* (F∗)*,* (FI*)*, and* (FM*) *too. If both are even Hoehnke functors between Hoehnke categories, then FG is also a Hoehnke functor.*

Moreover, if both functors have the property

$$\text{(FZ)} \quad O^{(K)}F = O^{(M)} \ \wedge \ \forall X \in |K| \ \left(XF = O^{(M)} \Rightarrow X = O^{(K)}\right),$$

then the functor FG has the same property.

Finally, let F and G be strongly d-monoidal functors between dhts-categories. Then FG is a strongly d-monoidal functor from $\underline{K}$ into $\underline{P}$.

Proof. Ad (F*):
The assertion is true because of

$$(FG)^*\langle A,B\rangle = d^{(P)}_{(A\otimes B)(FG)}\left(p_1^{A;B}(FG)\otimes p_2^{A,B}(FG)\right) =$$

$$= d^{(P)}_{((A\otimes B)F)G}\left(\left(p_1^{A;B}F\right)G\otimes\left(p_2^{A,B}F\right)G\right) =$$

$$= \left(d^{(M)}_{(A\otimes B)F}G\right)G^*\langle(A\otimes B)F,(A\otimes B)F\rangle\left(\left(p_1^{A;B}F\right)G\otimes\left(p_2^{A,B}F\right)G\right) =$$

(because G is a d-monoidal functor by the assumptions)

$$= \left(d^{(M)}_{(A\otimes B)F} G\right)\left(\left(p_1^{A;B}F\right)\otimes\left(p_2^{A,B}F\right)\right) GG^*\langle AF, BF\rangle =$$
(by (FM*) for G)

$$= \left(d^{(M)}_{(A\otimes B)F}\left(\left(p_1^{A;B}F\right)\otimes\left(p_2^{A,B}F\right)\right)\right) GG^*\langle AF, BF\rangle =$$
(since F and G fulfil (F$*$))

$$= (F^*\langle A, B\rangle)GG^*\langle AF, BF\rangle \in \mathbf{Iso}_P.$$

Ad (FI*):
Because of $t^{(M)}_{I^{(K)}F} \in \mathbf{Iso}_M$ and $t^{(P)}_{I^{(M)}G} \in \mathbf{Iso}_P$ one observes

$$t^{(P)}_{I^{(K)}(FG)} = \left(t^{(M)}_{I^{(K)}F}\right) G t^{(P)}_{I^{(M)}G} \in \mathbf{Iso}_P.$$

Ad (FM*):
Let $\varphi \in K[A, B]$, $\psi \in K[C, D]$. Then

$$(\varphi\otimes\psi)(FG)(FG)^*\langle B, D\rangle = ((\varphi\otimes\psi)F)G(F^*\langle B, D\rangle)GG^*\langle BF, DF\rangle =$$
(by (FC*))

$$= ((\varphi\otimes\psi)FF^*\langle B, D\rangle)GG^*\langle BF, DF\rangle =$$
(by (FM*) for F)

$$= (F^*\langle A, C\rangle(\varphi F\otimes\psi F))GG^*\langle BF, DF\rangle =$$

$$= (F^*\langle A, C\rangle)G((\varphi F)\otimes(\psi F))GG^*\langle AF, CF\rangle =$$
(by (FM*) for G)

$$= (F^*\langle A, C\rangle)GG^*\langle AF, CF\rangle((\varphi F)G\otimes(\psi F)G) =$$
(by (FC*))

$$= (FG)^*\langle A, C\rangle)(\varphi(FG)\otimes\psi(FG).$$

Now let $F \neq U$ and $G \neq U$ be even O-preserving functors between Hoehnke categories. Then the functor FG is an O-preserving functor between Hoehnke categories since

$$O^{(K)}(FG) = (O^{(K)}F)G = O^{(M)}G = O^{(P)}.$$

Moreover, if F and G both fulfil the condition (FZ), then

$$O^{(P)} = A(FG) = (AF)G \Rightarrow AF = O^{(M)} \Rightarrow A = O^{(K)}$$

shows that (FG) has the property (FZ) too.

If one of the functors F or G is the functor U, then obviously $FG = U$.
The functor FG satisfies the conditions (sFM), (sFT), and (sFD), since F and G have these properties.

Ad (sFM):

$$(\varphi\otimes\psi)(FG)=((\varphi\otimes\psi)F)G=(\varphi F\otimes\psi F)G=(\varphi F)G\otimes(\psi F)G=\varphi(FG)\otimes\psi(FG).$$

Ad (sFT):

$$t_A^{(K)}(FG)=\left(t_A^{(K)}F\right)G = t_{AF}^{(M)}G = t_{(AF)G}^{(P)} = t_{A(FG)}^{(P)}.$$

Ad (sFD):

$$d_A^{(K)}(FG)=\left(d_A^{(K)}F\right)G = d_{AF}^{(M)}G = d_{(AF)G}^{(P)} = d_{A(FG)}^{(P)}.$$

Therefore, FG is a strongly d-monoidal functor. ■

4. The cartesian product of monoidal functors

Furthermore, it will be of interest to investigate the "cartesian product" of functors between symmetric monoidal categories. In such a way one constructs functor categories with a symmetric monoidal structure.

Lemma 4.1. *Let* $(F, \widetilde{F}, i_F) : K^\bullet \to M^\bullet$ *and* $(G, \widetilde{G}, i_G) : P^\bullet \to Q^\bullet$ *be monoidal functors (strongly monoidal functors) between the symmetric monoidal categories* $K^\bullet$ *and* $M^\bullet$*,* $P^\bullet$ *and* $Q^\bullet$*, respectively.*

Then $(F \times G, \widetilde{F \times G}, i_{F\times G}) : (K \times P)^\bullet \to (M \times Q)^\bullet$ *is a monoidal functor* (*strongly monoidal functor*) *defined by*

$$(A, X)(F \times G) := (AF, XG), \quad (\varphi, \psi)(F \times G) := (\varphi F, \psi G),$$

$$\widetilde{F \times G}\langle(A, X), (B, Y)\rangle := \Big(\widetilde{F}\langle(A, B)\rangle, \widetilde{G}\langle(X, Y)\rangle\Big),$$

$$i_{F\times G} := (i_F, i_G).$$

Moreover, if the considered categories are ds-categories and F *and* G *are d-monoidal functors* (*strongly d-monoidal functors*), *then* $F \times G$ *is a d-monoidal functor* (*strongly d-monoidal functor*) *too.*

Finally, if the considered categories are even Hoehnke categories and F *as well as* G *fulfil the condition* (FO) *or* (FZ), *then* $F \times G$ *satisfies the same condition.*

Proof. All conditions for the fact that $(F \times G, \widetilde{F \times G}, i_{F\times G})$ is a monoidal functor follow from the relevant properties of the monoidal functors $(F, \widetilde{F}, i_F)$ and $(G, \widetilde{G}, i_G)$ via the definition above as well as the composition and $\otimes$-operation are defined componentwise. Altogether, one has to show the usual functor conditions and the validity of (F$\sim$), (FI), (FA), (FR), (FS), and (FM) for $F \times G$.

The functor properties are easy to verify, for instance:

$$\begin{aligned}((\varphi_1, \psi_1) \cdot (\varphi_2, \psi_2))(F \times G) &= (\varphi_1 \cdot \varphi_2, \psi_1 \cdot \psi_2)(F \times G) = \\ &= ((\varphi_1 \cdot \varphi_2)F, (\psi_1 \cdot \psi_2)G) = \\ &= ((\varphi_1 F) \cdot (\varphi_2 F), (\psi_1 G) \cdot (\psi_2 G)) = \\ &= ((\varphi_1 F), (\psi_1 G)) \cdot ((\varphi_2)F, (\psi_2 G)) = \\ &= (\varphi_1, \psi_1)(F \times G) \cdot (\varphi_2, \psi_2)(F \times G).\end{aligned}$$

Ad (F$\sim$):
The isomorphisms of $M \times Q$ are pairs of isomorphisms of M and Q, respectively, hence $\widetilde{F \times G}\langle(A, X), (B, Y)\rangle$ is an isomorphism in all cases.

Ad (FI):
$i_{F\times G}$ is an isomorphism since i_F and i_G are isomorphisms.

Ad (FA), (FR), (FS), and (FM):
By definition, all wished properties of $F \times G$ are immediate consequences of the relevant properties of F and G, for instance:

$$\begin{aligned}
&(\widetilde{F \times G})\langle (A_1, X_1), (A_2, X_2)\rangle ((\varphi_1, \psi_1) \otimes (\varphi_2, \psi_2))(F \times G) = \\
&\quad = \left(\widetilde{F}\langle A_1, A_2\rangle, \widetilde{G}\langle X_1, X_2\rangle\right) ((\varphi_1 \otimes \varphi_2)F, (\psi_1 \otimes \psi_2)G) = \\
&\quad = \left(\widetilde{F}\langle A_1, A_2\rangle (\varphi_1 \otimes \varphi_2)F, \widetilde{G}\langle X_1, X_2\rangle (\psi_1 \otimes \psi_2)G\right) = \\
&\quad = \left((\varphi_1 F \otimes \varphi_2 F)\widetilde{F}\langle B_1, B_2\rangle, (\psi_1 G \otimes \psi_2 G)\widetilde{G}\langle Y_1, Y_2\rangle\right) = \\
&\quad = ((\varphi_1 F \otimes \varphi_2 F), (\psi_1 G \otimes \psi_2 G)) \left(\widetilde{F}\langle B_1, B_2\rangle, \widetilde{G}\langle Y_1, Y_2\rangle\right) = \\
&\quad = ((\varphi_1 F, \psi_1 G) \otimes (\varphi_2 F, \psi_2 G)) \left(\widetilde{F}\langle B_1, B_2\rangle, \widetilde{G}\langle Y_1, Y_2\rangle\right) = \\
&\quad = ((\varphi_1, \psi_1)(F \times G) \otimes (\varphi_2, \psi_2)(F \times G))(\widetilde{F \times G})\langle (B_1, Y_1), (B_2, Y_2)\rangle.
\end{aligned}$$

Now let $\underline{K}$, $\underline{M}$, $\underline{P}$, $\underline{Q}$ be ds-categories. Then one has in addition:

$$\begin{aligned}
d^{(K\times P)}_{(A,X)}(F \times G) &= \left(d^{(K)}_A, d^{(P)}_X\right)(F \times G) = \\
&= \left(d^{(K)}_A F, d^{(P)}_X G\right) = \left(d^{(M)}_{AF} \widetilde{F}\langle A, A\rangle, d^{(Q)}_{XG} \widetilde{G}\langle X, X\rangle\right) = \\
&= \left(d^{(M)}_{AF}, d^{(Q)}_{XG}\right)\left(\widetilde{F}\langle A, A\rangle, \widetilde{G}\langle X, X\rangle\right) = \\
&= d^{(M\times Q)}_{(AF, XG)} \widetilde{F \times G}\langle (A, X), (A, X)\rangle,
\end{aligned}$$

i.e. $F \times G$ is a d-monoidal functor.

Ad (FO) and (FZ):

$$O^{(M \times Q)} = (O^{(M)}, O^{(Q)}) = (AF, XG) = (A, X)(F \times G)$$

$$\Leftrightarrow O^{(M)} = AF \ \wedge \ O^{(Q)} = XG \Leftrightarrow O^{(K)} = A \ \wedge \ O^{(P)} = X$$

$$\Leftrightarrow (A, B) = (O^{(K)}, O^{(P)}) = O^{(K \times P)}.$$

If F and G both are strongly d-monoidal functors, then the functor $F \times G$ satisfies the conditions (sFM), (sFT), (sFD), since F and G both have this properties, e.g.

$$((\varphi_1, \psi_1) \otimes (\varphi_2, \psi_2))(F \times G) = (\varphi_1 \otimes \varphi_2, \psi_1 \otimes \psi_2)(F \times G)$$

$$= ((\varphi_1 \otimes \varphi_2)F, (\psi_1 \otimes \psi_2)G) = ((\varphi_1 F \otimes \varphi_2 F), (\psi_1 G \otimes \psi_2 G))$$

$$= (\varphi_1 F, \psi_1 G) \otimes (\varphi_2 F, \psi_2 G) = (\varphi_1, \psi_1)(F \times G) \otimes (\varphi_2, \psi_2)(F \times G). \quad \blacksquare$$

Lemma 4.2. *Let $\underline{K}$, $\underline{M}$, $\underline{P}$ be arbitrary symmetric monoidal categories. Then one receives the strongly d-monoidal functors*

$$A_{K,M,P} : K \times (M \times P) \to (K \times M) \times P,$$

$$((A, (B, C)) \mapsto ((A, B), C), \ (\varphi, (\psi, \rho)) \mapsto ((\varphi, \psi), \rho))$$

(*associativity functor*);

$$R_K : K \times \Omega \to K, \quad ((A, I) \mapsto A, \ (\varphi, 1_I) \mapsto \varphi)$$

(*right-identity functor*);

$$L_K : \Omega \times K \to K, \quad ((I, A) \mapsto A, \ (1_I, \varphi) \mapsto \varphi)$$

(*left-identity functor*);

$S_{K,M} : K \times M \to M \times K, \quad ((A, B) \mapsto (B, A), \ (\varphi, \psi) \mapsto (\psi, \varphi))$

(*symmetry functor*);

$D_K : K \to K \times K, \quad (A \mapsto (A, A), \ \varphi \mapsto (\varphi, \varphi))$

(*diagonality functor*);

$\Theta_K : K \to \Omega, \quad (A \mapsto I, \ \varphi \mapsto 1_I)$ (*terminality functor*).

If $\underline{K}, \ \underline{M}, \ \underline{P}$ *are even Hoehnke categories, then the functors* $A_{K,M,P}, \ R_K, \ L_K, \ S_{K,M}, \ D_K$ *are strong Hoehnke functors.*

Proof. At first one has to prove the functor conditions for all the mappings defined above. Some selected examples shall demonstrate the argumentations.

$A_{K,M,P}$ preserves the domains:

$$dom^{(K\times M)\times P)}((\varphi, (\psi, \rho))A_{K,M,P}) = dom^{(K\times M)\times P)}((\varphi, \psi), \rho)) =$$

$$= ((dom^{(K)}(\varphi), dom^{(M)}(\psi)), dom^{(P)}(\rho)) =$$

$$= ((dom^{(K)}(\varphi), (dom^{(M)}(\psi), dom^{(P)}(\rho))A_{K,M,P} =$$

$$= (dom^{(K\times(M\times P))}(\varphi, (\psi\rho)))A_{K,M,P}.$$

R_K preserves the codomains:

$$cod^{(K)}((\varphi, 1_I)R_K) = cod^{(K)}(\varphi) =$$

$$= (cod^{(K)}(\varphi), cod^{(\Omega)}(1_I))R_K = (cod^{(K\times\Omega)}(\varphi, 1_I))R_K.$$

$S_{K,M}$ preserves the units:

$$\left(1^{(K,M)}_{(A,B)}\right) S_{K,M} = \left(1^{(K)}_A, 1^{(M)}_B\right) S_{K,M} = \left(1^{(M)}_B, 1^{(K)}_A\right) = 1^{(M\times K)}_{(B,A)} = 1^{(M\times K)}_{(A,B)S_{K,M}}.$$

D_K is compatible with the composition:

$$(\varphi_1\varphi_2)D_K = (\varphi_1\varphi_2, \varphi_1\varphi_2) = (\varphi_1, \varphi_1)(\varphi_2, \varphi_2) = (\varphi_1 D_K)(\varphi_2 D_K).$$

The proof of the missing facts concerning the functor properties is left to the reader.

In a next step one has to verify the properties (sFI), (sFA), (sFR), (sFS), (sFM), and (sFD) for the functors introduced above. Several examples shall demonstrate the proofs.

$A_{K,M,P}$ satisfies (sFI), since

$$I^{(K\times(M\times P))}A_{K,M,P} = (I^{(K)}, (I^{(M)}, I^{(P)}))A_{K,M,P} =$$

$$= ((I^{(K)}, I^{(M)}), I^{(P)}) = I^{((K\times M)\times P)}.$$

R_K fulfils (sFA) as follows:

$$a^{(K\times\Omega)}_{(A,I),(B,I),(C,I)} R_K = \left(a^{(K)}_{A,B,C}, a^{(\Omega)}_{I,I,I}\right) R_K =$$

$$= \left(a^{(K)}_{A,B,C}, 1^{(\Omega)}_I\right) R_K = a^{(K)}_{A,B,C} = a^{(K)}_{(A,I)R_K,(B,I)R_K,(C,I)R_K}.$$

(sFS) for $S_{K,M}$:

$$s^{(K\times M)}_{(A,X),(B,Y)} S_{K,M} = \left(s^{(K)}_{(A,B)}, s^{(M)}_{(X,Y)}\right) S_{K,M} =$$

$$= \left(s^{(M)}_{(X,Y)}, s^{(K)}_{(A,B)}\right) = s^{(M\otimes K)}_{(X,A),(Y,B)} = s^{(M\otimes K)}_{(A,X)S_{K,M},(B,Y)S_{K,M}}.$$

(sFM) for D_K:

$$(\varphi_1 \otimes \varphi_2) D_K = (\varphi_1 \otimes \varphi_2, \varphi_1 \otimes \varphi_2) = (\varphi_1, \varphi_1) \otimes (\varphi_2, \varphi_2) = (\varphi_1 D_K) \otimes (\varphi_2 D_K).$$

(sFD) for L_K:

$$d^{(\Omega\times K)}_{(I,A)} L_K = \left(d^{(\Omega)}_I, d^{(K)}_A\right) L_K = \left(1^{(\Omega)}_I, d^{(K)}_A\right) L_K = d^{(K)}_A = d^{(K)}_{(I,A)L_K}.$$

(sFT) for $A_{K,M,P}$:

$$t^{(K\times(M\times P))}_{(A,(B,C))} A_{K,M,P} = \left(\left(t^{(K)}_A, t^{(M)}_B\right), t^{(P)}_C\right) = t^{((K\times M)\times P)}_{((A,B),C)} = t^{((K\times M)\times P)}_{(A,(B,C))A_{K,M,P}}.$$

Obviously, all mentioned functors excluding Θ_K fulfil the condition (FZ) by definition.

The corresponding morphisms to the d-monoidal functors above are the following:

$$\widetilde{A}_{K,M,P}\langle((A_1,(B_1,C_1)),(A_2,B_2),C_2))\rangle := 1^{((K\times M)\times P)}_{((A_1\otimes A_2, B_1\otimes B_2), C_1\otimes C_2)} =$$

$$= \left(\left(1^{(K)}_{A_1\otimes A_2}, 1^{(M)}_{B_1\otimes B_2}\right), 1^{(P)}_{C_1\otimes C_2}\right)$$

$$\text{and} \qquad i_{A_{K,M,P}} := 1^{((K\times M)\times P)}_{I^{((K\times M)\times P))}} = \left(\left(1^{(K)}_{I^{(K)}}, 1^{(M)}_{I^{(M)}}\right), 1^{(P)}_{I^{(P)}}\right);$$

$$\widetilde{R}_K\langle(A_1, I),(A_2, I)\rangle := 1^{(K)}_{A_1\otimes A_2} \quad \text{and} \quad i_{R_K} := 1^{(K)}_{I^{(K)}};$$

$$\widetilde{L}_K\langle(I, A_1),(I, A_2)\rangle := 1^{(K)}_{A_1\otimes A_2} \quad \text{and} \quad i_{L_K} := 1^{(K)}_{I^{(K)}};$$

$$\widetilde{S}_{K,M}\langle (A_1,B_1),(A_2,B_2)\rangle := 1^{(K\times M)}_{(B_1\otimes B_2,A_1\otimes A_2)} = \left(1^{(M)}_{B_1\otimes B_2}, 1^{(K)}_{A_1\otimes A_2}\right)$$

$$\text{and} \qquad i_{S_{K,M}} := 1^{(M\times K)}_{I^{(M\times K)}} = \left(1^{(M)}_{I^{(M)}}, 1^{(K)}_{I^{(K)}}\right);$$

$$\widetilde{D}_K\langle A_1,A_2\rangle := 1^{(K\times K)}_{(A_1\otimes A_2,A_1\otimes A_2)} = \left(1^{(K)}_{A_1\otimes A_2}, 1^{(K)}_{A_1\otimes A_2}\right)$$

$$\text{and} \qquad i_{D_K} := 1^{(K\times K)}_{I^{(K\times K)}} = \left(1^{(K)}_{I}, 1^{(K)}_{I}\right);$$

$$\widetilde{\Theta}_K\langle A,B\rangle := 1_I \quad \text{and} \quad i_{\Theta_K} := 1_I. \qquad \blacksquare$$

The different classes of distinguished functors will be denoted by $\mathcal{A}$ (associativity functors), $\mathcal{R}$ (right-identity functors), $\mathcal{L}$ (left-identity functors), $\mathcal{S}$ (symmetry functors), $\mathcal{D}$ (diagonality functors), and Θ (terminal functors), respectively.

The categories concidered in Corollary 3.3 have the following structure concerning the cartesian product:

Theorem 4.3. *All small symmetric monoidal categories as objects and all monoidal functors between them form a dts-category*

$$\underline{\mathbf{MON}} = (\mathbf{MON}; \times, \Omega, \mathcal{A}, \mathcal{R}, \mathcal{L}, \mathcal{S}, \mathcal{D}, \Theta).$$

There are the dts-subcategories of $\underline{\mathbf{MON}}$*:*

The dts-category of all d-monoidal functors between ds-categories

$$\underline{d\mathbf{MON}} = (d\mathbf{MON}; \times, \Omega, \mathcal{A}, \mathcal{R}, \mathcal{L}, \mathcal{S}, \mathcal{D}, \Theta),$$

the dts-category of all d-monoidal functors between dhts-categories

$$\underline{dht\mathbf{MON}} = (dht\mathbf{MON}; \times, \Omega, \mathcal{A}, \mathcal{R}, \mathcal{L}, \mathcal{S}, \mathcal{D}, \Theta),$$

the dts-category of all d-monoidal functors between dh∇s-categories

$$\underline{dh\nabla\mathbf{MON}} = (dh\nabla\mathbf{MON}; \times, \Omega, \mathcal{A}, \mathcal{R}, \mathcal{L}, \mathcal{S}, \mathcal{D}, \Theta),$$

the dts-category of all d-monoidal functors between dts-categories

$$\underline{dt\mathbf{MON}} = (dt\mathbf{MON}; \times, \Omega, \mathcal{A}, \mathcal{R}, \mathcal{L}, \mathcal{S}, \mathcal{D}, \Theta),$$

the dts-category of all d-monoidal functors between dhth∇s-categories

$$\underline{dhth\nabla\mathbf{MON}} = (dhth\nabla\mathbf{MON}; \times, \Omega, \mathcal{A}, \mathcal{R}, \mathcal{L}, \mathcal{S}, \mathcal{D}, \Theta),$$

the dts-category of all d-monoidal functors between d∇s-categories

$$\underline{d\nabla\mathbf{MON}} = (d\nabla\mathbf{MON}; \times, \Omega, \mathcal{A}, \mathcal{R}, \mathcal{L}, \mathcal{S}, \mathcal{D}, \Theta).$$

Proof. Since small categories and functors between them form functor categories, it remains to prove that the composition of monoidal functors (d-monoidal functors) yields a monoidal functor (d-monoidal functor). That was done in 3.1. Because of Lemma 4.1, $F \times G$ is a monoidal functor (d-monoidal functor), whenever F and G are monoidal (d-monoidal).

As already mentioned, $\underline{\Omega}$ is a $dhth\nabla s$-category. The mapping "$\times$" for objects and morphisms ($dhts$-categories and d-monoidal functors, respectively) defines a bifunctor from ($\mathbf{MON} \times \mathbf{MON}$) into $\mathbf{MON}$ since

$$dom(F \times G) = domF \times domG,$$

$$cod(F \times G) = codF \times codG,$$

$$1\langle F \times G\rangle = 1\langle F\rangle \times 1\langle G\rangle,$$

$$(F_1 \times G_1)(F_2 \times G_2) = F_1F_2 \times G_1G_2$$

by the definition above.

The families of the functors $A_{K,M,P}$, R_K, L_K, $S_{K,M}$ are obviously families of functor isomorphisms and the properties for a symmetric monoidal category are easy to verify by the following considerations. Note that two mappings are equal, if their immages coincide for all arguments, and it is sufficient to concider morphisms only in the computation.

Ad (M1):

$(\varphi,(\psi,(\rho,\sigma)))A_{K,M,P\times Q}A_{K\times M,P,Q} =$

$= ((\varphi,\psi),(\rho,\sigma))A_{K\times M,P,Q} = (((\varphi,\psi),\rho),\sigma) =$

$= ((\varphi,(\psi,\rho)),\sigma)(A_{K,M,P} \times 1\langle Q\rangle) =$

$= (\varphi,((\psi,\rho),\sigma))A_{K,M\times P,Q}(A_{K,M,P} \times 1\langle Q\rangle) =$

$= (\varphi,(\psi,(\rho,\sigma)))(1\langle K\rangle \times A_{M,P,Q})A_{K,M\times P,Q}(A_{K,M,P} \times 1\langle Q\rangle)$,
hence
$\forall K^\bullet, M^\bullet, P^\bullet, Q^\bullet \in |\mathbf{MON}|$

$(A_{K,M,P\times Q}A_{K\times M,P,Q} = (1\langle K\rangle \times A_{M,P,Q})A_{K,M\times P,Q}(A_{K,M,P} \times 1\langle Q\rangle))$,

Ad (M2):

$(\varphi,(1_I,\psi))A_{K,\Omega,M}(R_K \times 1\langle M\rangle) = ((\varphi,1_I),\psi)(R_K \times 1\langle M\rangle) = (\varphi,\psi) =$

$= (\varphi,(1_I,\psi))(1\langle K\rangle \times L_K)$,
hence
$\forall K^\bullet, M^\bullet \in |\mathbf{MON}|\ (A_{K,\Omega,M}(R_K \times 1\langle M\rangle) = (1\langle K\rangle \times L_K))$,

Ad (M3):

$(\varphi,(\psi,\rho))A_{K,M,P}S_{K\times M,P}A_{P,K,M} = ((\varphi,\psi),\rho)S_{K\times M,P}A_{P,K,M} =$

$= (\rho,(\varphi,\psi))A_{P,K,M} = ((\rho,\varphi),\psi) =$

$= ((\varphi,\rho),\psi)(S_{K,P}\times 1\langle M\rangle) = (\varphi,(\rho,\psi))A_{K,P,M}(S_{K,P}\times 1\langle M\rangle) =$

$= (\varphi,(\psi,\rho))(1\langle K\rangle\times S_{M,P})A_{K,P,M}(S_{K,P}\times 1\langle M\rangle),$
hence
$\forall K^\bullet, M^\bullet, P^\bullet \in |\mathbf{MON}|$

$(A_{K,M,P}S_{K\times M,P}A_{P,K,M} = (1\langle K\rangle\times S_{M,P})A_{K,P,M}(S_{K,P}\times 1\langle M\rangle)),$

Ad (M4):

$(\varphi,\psi)S_{K,M}S_{M,K} = (\psi,\varphi)S_{M,K} = (\varphi,\psi) = (\varphi,\psi)1\langle K\times M\rangle,$
hence
$\forall K^\bullet, M^\bullet \in |\mathbf{MON}|\ (S_{K,M}S_{M,K} = 1\langle K\times M\rangle),$

Ad (M5):

$(\varphi,1_I)S_{K,\Omega}L_K = (1_I,\varphi)L_K = \varphi = (\varphi,1_I)R_K,$
hence
$\forall K^\bullet \in |\mathbf{MON}|\ (S_{K,\Omega}L_K = R_K),$

Ad (M6):

$(\varphi,(\psi,\rho))A_{K_1,M_1,P_1}((F\times G)\times H) =$

$= ((\varphi,\psi),\rho)((F\times G)\times H) = ((\varphi F,\psi G),\rho H) =$

$= (\varphi F, (\psi G, \rho H))A_{K_2,M_2,P_2} = (\varphi, (\psi, \rho))(F \times (G \times H))A_{K_2,M_2,P_2}$,

hence

$\forall K_1^\bullet, M_1^\bullet, P_1^\bullet, K_2^\bullet, M_2^\bullet, P_2^\bullet \in |\mathbf{MON}|$

$\forall F \in (\mathbf{MON})[K_1^\bullet, K_2^\bullet]\ \forall G \in (\mathbf{MON})[M_1^\bullet, M_2^\bullet]\ \forall H \in (\mathbf{MON})[P_1^\bullet, P_2^\bullet]$

$(A_{K_1,M_1,P_1}((F \times G) \times H) = (F \times (G \times H))A_{K_2,M_2,P_2})$,

Ad (M7):

$(\varphi, 1_I)R_{K_1}F = \varphi F = (\varphi F, 1_I)R_{K_2} = (\varphi, 1_I)(F \times 1\langle\Omega\rangle)R_{K_2}$,

hence

$\forall K_1^\bullet, K_2^\bullet \in |\mathbf{MON}|\ \forall F \in (\mathbf{MON})[K_1^\bullet, K_2^\bullet]\ (R_{K_1}F = (F \times 1\langle\Omega\rangle)R_{K_2})$,

Ad (M8):

$(\varphi, \psi)S_{K_1,M_1}(G\times F) = (\psi G, \varphi F) = (\varphi F, \psi G)S_{K_2,M_2} = (\varphi, \psi)(F\times G)S_{K_2,M_2}$,

hence

$\forall K_1^\bullet, K_2^\bullet, M_1^\bullet, M_2^\bullet \in |\mathbf{MON}|\ \forall F \in (\mathbf{MON})[K_1^\bullet, K_2^\bullet]\ \forall G \in (\mathbf{MON})[M_1^\bullet, M_2^\bullet]$

$(S_{K_1,M_1}(G \times F) = (F \times G)S_{K_2,M_2})$.

$(D_K \mid K^\bullet \in |\mathbf{MON}|)$ is a $|\mathbf{MON}|$-indexed family of monoidal functors fulfilling the necessary conditions, namely:

Ad (D1):

$\varphi D_{K_1}(F \times F) = (\varphi F, \varphi F) = \varphi F D_{K_2}$,

hence

$\forall K_1^\bullet, K_2^\bullet \in |\mathbf{MON}|\ \forall F \in (\mathbf{MON})[K_1^\bullet, K_2^\bullet]\ (D_{K_1}(F \times F) = F D_{K_2})$, Ad (D2):

$\varphi D_K(D_K \times 1\langle K\rangle = ((\varphi,\varphi),\varphi) =$

$= (\varphi,(\varphi,\varphi))A_{K,K,K} = \varphi D_K(1\langle K\rangle \times D_K)A_{K,K,K}$,
hence
$\forall K^\bullet \in |\mathbf{MON}|\ (D_K(D_K \times 1\langle K\rangle) = D_K(1\langle K\rangle \times D_K)A_{K,K,K})$,

Ad (D3):

$\varphi D_K S_{K,K} = (\varphi,\varphi)S_{K,K} = (\varphi,\varphi) = \varphi D_K$,
hence
$\forall K^\bullet \in |\mathbf{MON}|\ (D_K S_{K,K} = D_K)$,

Ad (D4):

$(\varphi,\psi)(D_K \times D_M)B_{K,K,M,M} =$

$= ((\varphi,\varphi),(\psi,\psi))B_{K,K,M,M} = ((\varphi,\psi),(\varphi,\psi)) = (\varphi,\psi)D_{K\times K}$,

where

$B_{K,M,P,Q} = A_{K\times M,P,Q}\Big(A^{-1}_{K,M,P}\Big(1\langle K\rangle \times S_{M,P}A_{K,P,M} \times 1\langle Q\rangle\Big)\Big)A^{-1}_{K\times P,M,Q}$,
hence
$\forall K^\bullet, M^\bullet \in |\mathbf{MON}|\ ((D_K \times D_M)B_{K,K,M,M} = D_{K\times K})$.

Finally, $(\Theta_K \mid K^\bullet \in |\mathbf{MON}|)$ is a family of monoidal functors which is indexed by the class of all symmetric monoidal categories and, because of

$$\varphi(F\Theta_{K_2})) = (\varphi F)\Theta_{K_2} = 1_I = \varphi\Theta_{K_1}$$

$$\Rightarrow \forall K_1^\bullet, K_2^\bullet \in |\mathbf{MON}|\ \forall F : K_1 \to K_2\ (F\Theta_{K_2} = \Theta_{K_1}),$$

$$\varphi(D_K(1\langle K\rangle \times \Theta_K)R_K) = (\varphi, 1_I)R_K = \varphi = \varphi 1\langle K\rangle$$

$$\Rightarrow \forall K^\bullet \in |\mathbf{MON}|\ (D_K(1\langle K\rangle \times \Theta_K)R_K) = 1\langle K\rangle),$$

$$\varphi(D_K(\Theta_K \times 1\langle K\rangle)L_K) = (1_I, \varphi)L_K = \varphi = \varphi 1\langle K\rangle$$

$$\Rightarrow \forall K^\bullet \in |\mathbf{MON}|\ (D_K(\Theta_K \times 1\langle K\rangle)L_K = 1\langle K\rangle),$$

$$(\varphi,\psi)(D_{K\times M}((1\langle K\rangle \times \Theta_M)R_K \times (\Theta_K \times 1\langle M\rangle)L_M$$

$$= ((\varphi, 1_I)R_K, (1_I, \psi)L_M) = (\varphi,\psi) = (\varphi,\psi)1\langle K \times M\rangle$$

$$\Rightarrow \forall K^\bullet, M^\bullet \in |\mathbf{MON}|(D_{K\times M}((1\langle K\rangle \times \Theta_M)R_K \times (\Theta_K \times 1\langle M\rangle)L_M) =$$

$$= 1\langle K \times M\rangle),$$

the conditions (T1), (DTR), (DTL), (DTRL) are satisfied. ■

Corollary 4.4. *All strongly monoidal functors between symmetric monoidal categories establish a dts-subcategory* $\underline{\mathbf{sMON}}$ *of* $\underline{\mathbf{MON}}$*. All strongly d-monoidal functors between small ds-categories (small dhts-categories, small dh∇s-categories, small dts-categories, small dhth∇s-categories, small d∇s-categories) establish a dts-subcategory*

$\underline{s\mathbf{dMON}}$ $(\underline{s\mathbf{dhtMON}}, \underline{s\mathbf{dh\nabla MON}}, \underline{s\mathbf{dtMON}}, \underline{s\mathbf{dhth\nabla MON}}, \underline{s\mathbf{d\nabla MON}})$ *of* $\underline{s\mathbf{dMON}}$*.* ■

The mutual inclusions of the considered *dts*-categories are illustrated in the diagram in Figure 3, where $\underline{\mathbf{M}}$ shortly stands for $\underline{\mathbf{MON}}$.

Hoehnke proved in [8] (Theorem 6.1) that all "*dht*-symmetric categories (as objects) and the *dht*-symmetric functors between them (as morphisms) form an illegitime category, denoted by *dht-Sym*".

The statements presented in the theorem above are connected with the result of Hoehnke, but there are differences in the following aspects:

1. Each O-preserving nontrivial d-monoidal functor bewteen *dhts*-categories is a *dht*-symmetric functor in the sense of Hoehnke, but not conversely, since a *dht*-symmetric functor need not have the property (FM).
2. All the considered categories possess an additional structure which is not mentioned in the paper by Hoehnke.
3. The objects of the categories in this volume are small monoidal categories such that there are not necessary distinguished zero objects, whereas the objects of *dht-Sym* are Hoehnke categories only.
4. The distinguished $dhth\nabla s$-category Ω is not a Hoehnke category and the d-monoidal functor E does not preserve the zero object O.

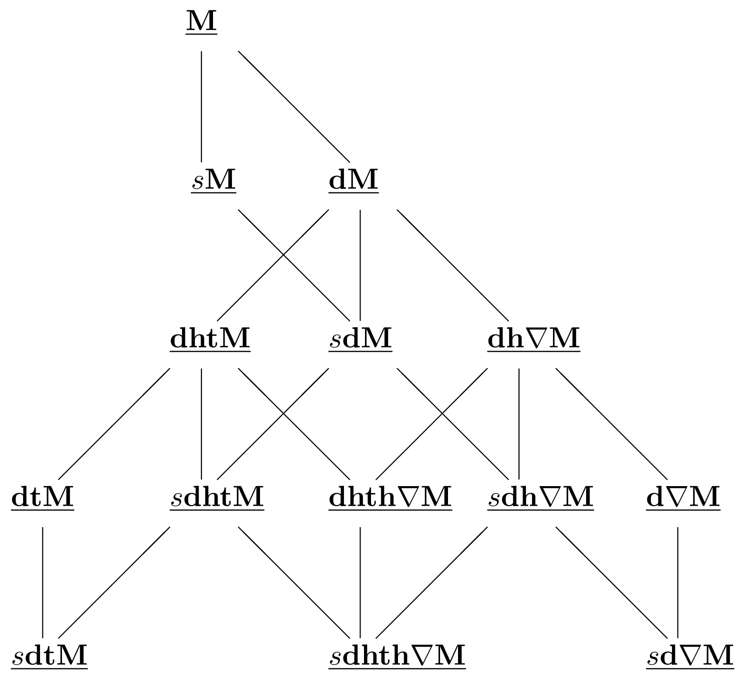

Figure 3.

Specific *dhts*-categories are of particular interest, namely *dhts*-theories, defined as follows.

A *dhts*-category $\underline{T}$ is called *J-sorted dhts-theory*, iff there exists a set $J \in |T|$ such that $I \notin J$ and $(|T|; \otimes, I)$ is a free algebra of type $(2, 0)$ freely generated by J.

By this definition, $\underline{T}$ is a small category since $|T|$ is a set. The algebra $(|T|; \otimes, I)$ contains a subalgebra $< I >$ of the same type consting of all possible $\otimes$-products of I with itself in arbitrary brackets. J-sorted *dts*-theories and J-sorted *dhth*∇*s*-theories will be defined in the same manner.

A Hoehnke category (di-Hoehnke category) $\underline{T}$ is called *J-sorted Hoehnke theory* (*J-sorted di-Hoehnke theory*), iff there is a set $J \in |T|$ such that $J \cap \{O, I\} = \emptyset$ and $(|T|; \otimes I, O)$ is the free algebra of type $(2, 0, 0)$ freely generated by J in the variety of type $(2, 0, 0)$ defined by the identity $X \otimes O = O = O \otimes X$.

It is well-known that each class of objects of a given category determines together with all possible morphisms between them a subcategory and the defined J-sorted theories are objects of the related functor categories.

Unfortunately, the cartesian product of a J_1-sorted theory $\underline{T_1}$ and a J_2-sorted theory $\underline{T_2}$ is not necessary a $(J_1 \times J_2)$-sorted theory, because of

$$A_1, A_2 \in J_1 \ \wedge \ B \in J_2 \Rightarrow |T_1| \times |T_2| \ni (A_1 \otimes A_2, B) \notin < J_1 \times J_2 >,$$

that means, that objects of the form $(A_1 \otimes A_2, B)$ are not generated by elements of $J_1 \times J_2$. Therefore, the J-sorted theories do not form symmetric monoidal subcategories of the suitable symmetric monoidal functor categories.

Corollary 4.5. *All dhts-theories (dts-theories, dhth∇s-theories, Hoehnke theories, di-Hoehnke theories) together with all d-monoidal functors (Hoehnke functors) between them in a natural manner form a subcategory dht***Th** *of dht***Mon** *(dt***Th** *of dt***Mon***, dhth∇***Th** *of dhth∇***Mon***, Hoe***Th** *of* **HOE***, di-Hoe***Th** *of di-***HOE***).* ■

The mutual inclusions of the subcategories mentioned above are presented in Figure 4.

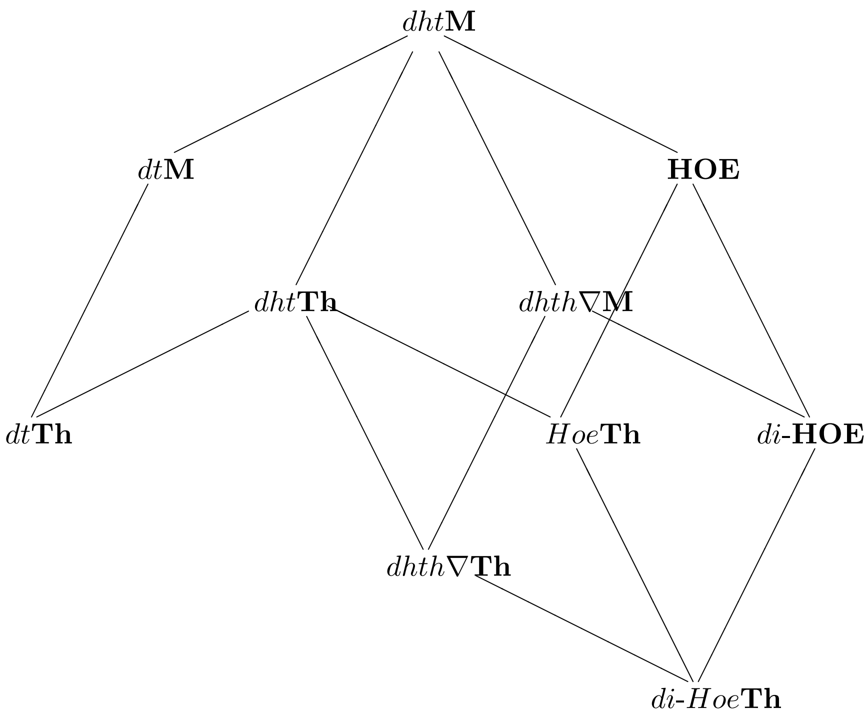

Figure 4.

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