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HYPERIDENTITIES IN TRANSITIVE GRAPH ALGEBRAS

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Abstract

Graph algebras establish a connection between directed graphs without multiple edges and special universal algebras of type (2,0). We say that a graph G satisfies an identity $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$. A graph $G = (V, E)$ is called a transitive graph if the corresponding graph algebra $\underline{A(G)}$ satisfies the equation $x(yz) \approx (xz)(yz)$. An identity $s \approx t$ of terms s and t of any type τ is called a hyperidentity of an algebra $\underline{A}$ if whenever the operation symbols occurring in s and t are replaced by any term operations of $\underline{A}$ of the appropriate arity, the resulting identities hold in $\underline{A}$.

In this paper we characterize transitive graph algebras, identities and hyperidentities in transitive graph algebras.

Keywords: identity, hyperidentity, term, normal form term, binary algebra, graph algebra, transitive graph algebra.

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1. Introduction

An identity $s \approx t$ of terms s, t of any type τ is called a *hyperidentity* of an algebra $\underline{A}$ if whenever the operation symbols occurring in s and t are replaced by any term operations of $\underline{A}$ of the appropriate arity, the resulting identity holds in $\underline{A}$. Hyperidentities can be defined more precisely using the concept of a hypersubstitution.

We fix a type $\tau = (n_i)_{i \in I}, n_i > 0$ for all $i \in I$, and operation symbols $(f_i)_{i \in I}$, where f_i is n_i-ary. Let $W_\tau(X)$ be the set of all terms of type τ over some fixed alphabet X, and let $Alg(\tau)$ be the class of all algebras of type τ. Then a mapping

$$\sigma : \{f_i |\ i \in I\} \longrightarrow W_\tau(X)$$

which assigns to every n_i-ary operation symbol f_i an n_i-ary term will be called a *hypersubstitution of type* τ (for short, a hypersubstitution). By $\hat{\sigma}$ we denote the extension of the hypersubstitution σ to a mapping

$$\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X).$$

The term $\hat{\sigma}[t]$ is defined inductively by

(i) $\hat{\sigma}[x] = x$ for any variable x in the alphabet X

and

(ii) $\hat{\sigma}[f_i(t_1, ..., t_{n_i})] = \sigma(f_i)^{W_\tau(X)}(\hat{\sigma}[t_1], ..., \hat{\sigma}[t_{n_i}])$.

Here $\sigma(f_i)^{W_\tau(X)}$ on the right hand side of (ii) is the operation induced by $\sigma(f_i)$ on the term algebra with the universe $W_\tau(X)$.

Graph algebras have been invented in [9] to obtain examples of non-finitely based finite algebras. To recall this concept, let $G = (V, E)$ be a (directed) graph with the vertex set V and the set of edges $E \subseteq V \times V$. Define the *graph algebra* $\underline{A(G)}$ corresponding to G with the underlying set $V \cup \{\infty\}$, where ∞ is a symbol outside V, and with two basic operations, namely a nullary operation pointing to ∞ and a binary one denoted by juxtaposition, given for $u, v \in V \cup \{\infty\}$ by

$$uv = \begin{cases} u, & \text{if} \quad (u,v) \in E, \\ \infty, & \text{otherwise.} \end{cases}$$

Graph identities were characterized in [3] by using the rooted graph of a term t, where the vertices correspond to the variables occurring in t. Since on a graph algebra we have one nullary and one binary operation, $\sigma(f)$ in this case is a *binary term* in $W_\tau(X)$, i.e. a term built up from variables of a two-element alphabet and a binary operation symbol f corresponding to the binary operation of the graph algebra.

In [7], R. Pöschel has shown that any term over the class of all graph algebras can be uniquely represented by a normal form term and that there is an algorithm to construct the normal form term to every given term t.

In [1], K. Denecke and T. Poomsa-ard characterized graph hyperidentities by using normal form graph hypersubstitutions.

In [6], T. Poomsa-ard characterized associative graph hyperidentities by using normal form graph hypersubstitutions.

We say that a graph $G = (V, E)$ is *transitive* if the corresponding graph algebra $\underline{A(G)}$ satisfies the equation $x(yz) \approx (xz)(yz)$. In this paper we characterize transitive graph algebras, identities and hyperidentities in transitive graph algebras.

2. Transitive graph algebras

We begin with a more precise definition of terms of the type of graph algebras.

Definition 2.1. The set $W_\tau(X)$ of all *terms* over the alphabet

$$X = \{x_1, x_2, x_3, ...\}$$

is defined inductively as follows:

(i) every variable $x_i, i = 1, 2, 3, ...,$ and ∞ are terms;

(ii) if t_1 and t_2 are terms, then $f(t_1, t_2)$ is a term; instead of $f(t_1, t_2)$ we will write t_1t_2, for short;

(iii) $W_\tau(X)$ is the set of all terms which can be obtained from (i) and (ii) in finitely many steps.

Terms built up from the two-element set $X_2 = \{x_1, x_2\}$ of variables are thus binary terms. We denote the set of all binary terms by $W_\tau(X_2)$. The leftmost variable of a term t is denoted by $L(t)$ and rightmost variable of a term t is denoted by $R(t)$. A term, in which the symbol ∞ occurs, is called a *trivial term*.

Definition 2.2. To each non-trivial term t of type $\tau = (2,0)$ one can define a directed graph $G(t) = (V(t), E(t))$, where the vertex set $V(t)$ is the set $var(t)$ of all variables occurring in t, and where $E(t)$ is defined inductively by

$$E(t) = \phi \text{ if } t \text{ is a variable and } E(t_1t_2) = E(t_1) \cup E(t_2) \cup \{(L(t_1), L(t_2))\},$$

when $t = t_1t_2$ is a compound term and $L(t_1), L(t_2)$ are the leftmost variables in t_1 and t_2, respectively.

$L(t)$ is called the *root* of the graph $G(t)$ and the pair $(G(t), L(t))$ is the *rooted graph* corresponding to t. Formally, to every trivial term t we assign the empty graph ϕ.

Definition 2.3. We say that a graph $G = (V, E)$ *satisfies an identity* $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$ (i.e. we have $s = t$ for every assignment $V(s) \cup V(t) \to V \cup \{\infty\}$), and in this case, we write $G \models s \approx t$.

Definition 2.4. Let $G = (V, E)$ and $G' = (V', E')$ be graphs. A *homomorphism* h from G into G' is a mapping $h : V \to V'$ carrying edges to edges, that is, for which $(u, v) \in E$ implies $(h(u), h(v)) \in E'$.

In [3] it was proved:

Proposition 2.1. *Let s and t be non-trivial terms from $W_\tau(X)$ with variables $V(s) = V(t) = \{x_0, x_1, ..., x_n\}$ and $L(s) = L(t)$. Then a graph $G = (V, E)$ satisfies $s \approx t$ if and only if the graph algebra $\underline{A(G)}$ has the following property:*

A mapping $h : V(s) \longrightarrow V$ is a homomorphism from $G(s)$ into G iff it is a homomorphism from $G(t)$ into G.

Proposition 2.1 gives a method to check whether a graph $G = (V, E)$ satisfies the equation $s \approx t$. Hence, we can check whether a graph $G = (V, E)$ has a transitive graph algebra by the following proposition.

Proposition 2.2. *Let $G = (V, E)$ be a graph. Then G has a transitive graph algebra if and only if $(a, b), (b, c) \in E$ implies $(a, c) \in E$ for any edges $(a, b), (b, c) \in E$.*

Proof. Suppose $G = (V, E)$ has a transitive graph algebra. Let s and t be terms such that $s = x(yz)$, $t = (xz)(yz)$. Let (a, b), $(b, c) \in E$

and $h : V(s) \rightarrow V$ be the restriction of an evaluation function of the variables such that $h(x) = a,\ h(y) = b$, and $h(z) = c$. We see that h is a homomorphism from $G(s)$ into G. By Proposition 2.1, we have that h is a homomorphism from $G(t)$ into G. Since $(x, z) \in E(t)$, so $(h(x), h(z)) = (a, c) \in E$.

Conversely, suppose $G = (V, E)$ is a graph such that if $(a, c), (b, c) \in E$, then $(a, c) \in E$. Let s and t be non-trivial terms such that $s = x(yz)$, $t = (xz)(yz)$. Suppose that $h : V(s) \rightarrow V$ is a homomorphism from $G(s)$ into G. Since $(x, y),\ (y, z) \in E(s)$, we have $(h(x), h(y)),\ (h(y), h(z)) \in E$. By the assumption, we get $(h(x), h(z)) \in E$. Therefore, h is a homomorphism from $G(t)$ into G. Suppose that h is a homomorphism from $G(t)$ into G. Then, it is clear that h is also a homomorphism from $G(s)$ into G. Hence, by Proposition 2.1, we get that $\underline{A(G)}$ satisfies $s \approx t$. ■

3. IDENTITIES IN TRANSITIVE GRAPH ALGEBRAS

Graph identities were characterized in [3] by the following proposition:

Proposition 3.1. *A non-trivial equation $s \approx t$ is an identity in the class of all graph algebras iff either both terms s and t are trivial or none of them is trivial, $G(s) = G(t)$ and $L(s) = L(t)$.*

Further it was proved:

Proposition 3.2. *Let $G = (V, E)$ be a graph and let $h : X \longrightarrow V \cup \{\infty\}$ be an evaluation of the variables. Consider the canonical extension of h to the set of all terms. Then there holds: if t is a trivial term then $h(t) = \infty$. Otherwise, if $h : G(t) \longrightarrow G$ is a homomorphism of graphs, then $h(t) = h(L(t))$, and if h is not a homomorphism of graphs, then $h(t) = \infty$.*

In [6] the following lemma was proved:

Lemma 3.1. *Let $G = (V, E)$ be a graph, let t be a term and let*

$$h : X \longrightarrow V \cup \{\infty\}$$

be an evaluation of the variables. Then:

(i) *If h is a homomorphism from $G(t)$ into G with the property that the subgraph of G induced by $h(V(t))$ is complete, then $h(t) = h(L(t))$;*

(ii) *If h is a homomorphism from $G(t)$ into G with the property that the subgraph of G induced by $h(V(t))$ is disconnected, then $h(t) = \infty$.*

Now, we apply our results to characterize all identities in the class of all transitive graph algebras. Clearly, if s and t are trivial, then $s \approx t$ is an identity in the class of all transitive graph algebras and $x \approx x$ $(x \in X)$ is an identity in the class of all transitive graph algebras too. So we consider the case that s and t are non-trivial and different from variables. Then all identities in the class of all transitive graph algebras are characterized by the following theorem:

Theorem 3.1. *Let s and t be non-trivial terms and let $x_0 = L(s)$. Then $s \approx t$ is an identity in the class of all transitive graph algebras if and only if the following conditions are satisfied:*

(i) $L(s) = L(t)$,

(ii) $V(s) = V(t)$,

(iii) *for any $x \in V(s), x$ is on a dicycle in $G(s)$ iff x is on a dicycle in $G(t)$,*

(iv) *for any $x, y \in V(s), x \neq y$, $G(s)$ has a dipath from x to y iff $G(t)$ has a dipath from x to y.*

Proof. Suppose that $s \approx t$ is an identity in the class of all transitive graph algebras. Since any complete graph is transitive, it follows that $L(s) = L(t)$ and $V(s) = V(t)$.

For any $x \in V(s)$, suppose that x is on a dicycle in $G(s)$ but x is not on a dicycle in $G(t)$. Consider the graph $G = (V, E)$ such that $V = \{0, 1, 2\}$, $E = \{(0, 0), (0, 1), (0, 2), (1, 2), (2, 2)\}$. Then, by Proposition 2.2, $\underline{A(G)}$ has a transitive graph algebra. Let $h : V(t) \to V$ be the restriction of an evaluation of variables such that $h(x) = 1$, $h(w) = 2$ for all $w \in V(t)$ such that $G(t)$ has a dipath from x to w and $h(z) = 0$ for all other $z \in V(t)$. We see that $h(s) = \infty$, $h(t) = 0$ or $h(t) = 1$. Hence, $\underline{A(G)}$ does not satisfy $s \approx t$.

Now for any $x, y \in V(s), x \neq y$, suppose that $G(s)$ has a dipath from x to y but $G(t)$ has no a dipath from x to y. Consider the transitive graph $G = (V, E)$ such that $V = \{0, 1\}$, $E = \{(0, 0), (0, 1), (1, 1)\}$. Let $h : V(t) \to V$ be the restriction of an evaluation of the variables which $h(x) = 1$ and $h(w) = 1$ for all $w \in V(t)$ such that $G(t)$ has a dipath from

x to w and $h(y') = 0$ for all other $y' \in V(t)$. We see that $h(s) = \infty$ and $h(t) = 0$. Hence, $\underline{A(G)}$ does not satisfy $s \approx t$.

Conversely, suppose that s and t are non-trivial terms satisfying (i), (ii), (iii) and (iv). Let $G = (V, E)$ be a transitive graph and let $h : V(s) \to V$ be the restriction of an evaluation of the variables. Suppose h is a homomorphism from $G(s)$ into G and let $(x, y) \in E(t)$. If $x = y$, then by (iii), x is on a dicycle in $G(s)$. Let $(x, x_1), (x_1, x_2), (x_2, x_3), ..., (x_{n-1}, x_n), (x_n, x)$ be such the dicycle. Since h is a homomorphism from $G(s)$ into G, so

$$(h(x), h(x_1)), (h(x_1), h(x_2)), ..., (h(x_{n-1}), h(x_n)), (h(x_n), h(x)) \in E.$$

By transitivity of G, we get $(h(x), h(x)) \in E$. If $x \neq y$, then by (iv) $G(s)$ has a dipath from x to y. Let $(x, x_1), (x_1, x_2), (x_2, x_3), ..., (x_{n-1}, x_n), (x_n, y)$ be such the dipath. Since h is a homomorphism from $G(s)$ into G, so

$$(h(x), h(x_1)), (h(x_1), h(x_2)), ..., (h(x_{n-1}), h(x_n)), (h(x_n), h(y)) \in E.$$

By transitivity of G again, we get $(h(x), h(y)) \in E$. Hence, h is also a homomorphism from $G(t)$ into G. By the same way, if h is a homomorphism from $G(t)$ into G, then we can prove that it is a homomorphism from $G(s)$ into G. By Proposition 2.1, we get that $\underline{A(G)}$ satisfies $s \approx t$. ■

4. Hyperidentities in transitive graph algebras

Let $\mathcal{TG}$ be the class of all transitive graph algebras and let $Id\mathcal{TG}$ be the set of all identities satisfied in $\mathcal{TG}$. Now we want to make precise the concept of a hypersubstitution for graph algebras.

Definition 4.1. A mapping $\sigma : \{f, \infty\} \to W_\tau(X_2)$, where f is the operation symbol corresponding to the binary operation of a graph algebra is called *graph hypersubstitution* if $\sigma(\infty) = \infty$ and $\sigma(f) = s \in W_\tau(X_2)$. The graph hypersubstitution with $\sigma(f) = s$ is denoted by σ_s.

Definition 4.2. An identity $s \approx t$ is a *transitive graph hyperidentity* iff for all graph hypersubstitutions σ, the equations $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities in $\mathcal{TG}$.

If we want to check that $s \approx t$ is a hyperidentity in $\mathcal{TG}$, we can restrict ourselves to a (small) subset of $Hyp\mathcal{G}$ – the set of all graph hypersubstitutions.

In [4] the following relation between hypersubstitutions was defined:

Definition 4.3. Two graph hypersubstitutions σ_1, σ_2 are *$\mathcal{TG}$-equivalent* iff $\sigma_1(f) \approx \sigma_2(f)$ is an identity in $\mathcal{TG}$. In this case we write $\sigma_1 \sim_{\mathcal{TG}} \sigma_2$.

In [2] (see also [4]) the following lemma was proved:

Lemma 4.1. *If $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in Id\mathcal{TG}$ and $\sigma_1 \sim_{\mathcal{TG}} \sigma_2$, then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{TG}$.*

Therefore, it is enough to consider the quotient set $Hyp\mathcal{G}/\sim_{\mathcal{TG}}$.

In [7] it was shown that any non-trivial term t over the class of graph algebras has a uniquely determined normal form term $NF(t)$ and there is an algorithm to construct the normal form term to a given term t. Now, we want to describe how to construct the normal form term. Let t be a non-trivial term. The *normal form term of* t is the term $NF(t)$ constructed by the following algorithm:

(i) Construct $G(t) = (V(t), E(t))$.

(ii) Construct for every $x \in V(t)$ the list $l_x = (x_{i_1}, ..., x_{i_{k(x)}})$ of all out-neighbors (i.e. $(x, x_{i_j}) \in E(t), 1 \leq j \leq k(x)$) ordered by increasing indices $i_1 \leq ... \leq i_{k(x)}$ and let s_x be the term $(...((x x_{i_1}) x_{i_2}) ... x_{i_{k(x)}})$.

(iii) Starting with $x := L(t), Z := V(t), s := L(t)$, choose the variable $x_i \in Z \cap V(s)$ with the least index i, substitute the first occurrence of x_i by the term s_{x_i}, denote the resulting term again by s and put $Z := Z \setminus \{x_i\}$. While $Z \neq \phi$ continue this procedure. The resulting term is the normal form $NF(t)$.

The algorithm stops after a finite number of steps, since $G(t)$ is a rooted graph. Without difficulties one shows $G(NF(t)) = G(t),\ L(NF(t)) = L(t)$.

In [2] the following definition was given:

Definition 4.4. The graph hypersubstitution $\sigma_{NF(t)}$, is called *normal form graph hypersubstitution.* Here $NF(t)$ is the normal form of the binary term t.

Since for any binary term t the rooted graphs of t and $NF(t)$ are the same, we have $t \approx NF(t) \in Id\mathcal{TG}$. Then for any graph hypersubstitution σ_t with $\sigma_t(f) = t \in W_\tau(X_2)$, one obtains $\sigma_t \sim_{\mathcal{TG}} \sigma_{NF(t)}$.

In [2] all rooted graphs with at most two vertices were considered. Then we formed the corresponding binary terms and used the algorithm to construct normal form terms. The result is given in the following table.

normal form term	graph hypers.	normal form term	graph hypers.
x_1x_2	σ_0	x_1	σ_1
x_2	σ_2	x_1x_1	σ_3
x_2x_2	σ_4	x_2x_1	σ_5
$(x_1x_1)x_2$	σ_6	$(x_2x_1)x_2$	σ_7
$x_1(x_2x_2)$	σ_8	$x_2(x_1x_1)$	σ_9
$(x_1x_1)(x_2x_2)$	σ_{10}	$(x_2(x_1x_1))x_2$	σ_{11}
$x_1(x_2x_1)$	σ_{12}	$x_2(x_1x_2)$	σ_{13}
$(x_1x_1)(x_2x_1)$	σ_{14}	$x_2((x_1x_1)x_2)$	σ_{15}
$x_1((x_2x_1)x_2)$	σ_{16}	$(x_2(x_1x_2))x_2$	σ_{17}
$(x_1x_1)((x_2x_1)x_2)$	σ_{18}	$(x_2((x_1x_1)x_2))x_2$	σ_{19}

By Theorem 3.1, we have the following relations:

(i) $\sigma_{12} \sim_{\mathcal{TG}} \sigma_{14} \sim_{\mathcal{TG}} \sigma_{16} \sim_{\mathcal{TG}} \sigma_{18}$,

(ii) $\sigma_{13} \sim_{\mathcal{TG}} \sigma_{15} \sim_{\mathcal{TG}} \sigma_{17} \sim_{\mathcal{TG}} \sigma_{19}$.

Let $M_{\mathcal{TG}}$ be the set of all normal form graph hypersubstitutions in $\mathcal{TG}$. Then we get,

$$M_{\mathcal{TG}} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}\}.$$

We define the product of two normal form graph hypersubstitutions in $M_{\mathcal{TG}}$ as follows.

Definition 4.5. The *product* $\sigma_{1N} \circ_N \sigma_{2N}$ *of two normal form graph hypersubstitutions* is defined by $(\sigma_{1N} \circ_N \sigma_{2N})(f) := NF(\hat{\sigma}_{1N}[\sigma_{2N}(f)])$.

The following table gives the multiplication of elements in $M_{\mathcal{TG}}$.

$\circ_N$	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_{13}
σ_0	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_{13}
σ_1	σ_1	σ_1	σ_2	σ_1	σ_2	σ_2	σ_1	σ_2	σ_1	σ_2	σ_1	σ_2	σ_1	σ_2
σ_2	σ_2	σ_1	σ_2	σ_1	σ_2	σ_1	σ_2	σ_2	σ_2	σ_1	σ_2	σ_2	σ_1	σ_2
σ_3	σ_3	σ_1	σ_2	σ_3	σ_4	σ_4	σ_3	σ_4	σ_3	σ_4	σ_3	σ_4	σ_3	σ_4
σ_4	σ_4	σ_1	σ_2	σ_3	σ_4	σ_3	σ_4	σ_4	σ_4	σ_3	σ_4	σ_4	σ_3	σ_4
σ_5	σ_5	σ_1	σ_2	σ_3	σ_4	σ_0	σ_9	σ_{13}	σ_7	σ_6	σ_{11}	σ_{13}	σ_6	σ_7
σ_6	σ_6	σ_1	σ_2	σ_3	σ_4	σ_7	σ_6	σ_7	σ_{10}	σ_{11}	σ_{10}	σ_{11}	σ_{12}	σ_{13}
σ_7	σ_7	σ_1	σ_2	σ_3	σ_4	σ_6	σ_{11}	σ_{13}	σ_7	σ_6	σ_{11}	σ_{13}	σ_6	σ_7
σ_8	σ_8	σ_1	σ_2	σ_3	σ_4	σ_9	σ_{10}	σ_{11}	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_{13}
σ_9	σ_9	σ_1	σ_2	σ_3	σ_4	σ_8	σ_9	σ_{13}	σ_{11}	σ_{10}	σ_{11}	σ_{13}	σ_{10}	σ_{11}
σ_{10}	σ_{10}	σ_1	σ_2	σ_3	σ_4	σ_{11}	σ_{10}	σ_{11}	σ_{10}	σ_{11}	σ_{10}	σ_{11}	σ_{12}	σ_{13}
σ_{11}	σ_{11}	σ_1	σ_2	σ_3	σ_4	σ_{10}	σ_{11}	σ_{13}	σ_{11}	σ_{10}	σ_{11}	σ_{13}	σ_{10}	σ_{11}
σ_{12}	σ_{12}	σ_1	σ_2	σ_3	σ_4	σ_{13}	σ_{12}	σ_{13}	σ_{12}	σ_{13}	σ_{12}	σ_{13}	σ_{12}	σ_{13}
σ_{13}	σ_{13}	σ_1	σ_2	σ_3	σ_4	σ_{12}	σ_{13}	σ_{13}	σ_7	σ_{12}	σ_{13}	σ_{13}	σ_{12}	σ_{13}

In [2] the concept of a leftmost normal form graph hypersubstitution was defined.

Definition 4.6. A graph hypersubstitution σ is called $leftmost$ hypersubstitution if $L(\sigma(f)) = x_1$.

The set $M_{L(\mathcal{TG})}$ of all leftmost normal form graph hypersubstitutions in $M_{\mathcal{TG}}$ contains exactly the following elements:

$$M_{L(\mathcal{TG})} = \{\sigma_0, \sigma_1, \sigma_3, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}\}.$$

In [5] the concept of a proper hypersubstitution of a class of algebras was introduced.

Definition 4.7. A hypersubstitution σ is called *proper with respect to a class* $\mathcal{K}$ *of algebras* if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}$ for all $s \approx t \in Id\mathcal{K}$.

A graph hypersubstitution with the property that $\sigma(f)$ contains both variables x_1 and x_2 is called regular. It is easy to check that the set of all regular graph hypersubstitutions forms a groupoid M_{reg}.

We want to prove that $\{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}\}$ is the set of all proper normal form graph hypersubstitutions with respect to $\mathcal{TG}$.

In [2] the following lemma was proved.

Lemma 4.2. *For each non-trivial term* $s, (s \neq x \in X_2)$ *and for all* $u, v \in X_2$*, we have:*

(i) $E(\hat{\sigma_6}[s]) = E(s) \cup \{(u,u) | (u,v) \in E(s)\}$,

(ii) $E(\hat{\sigma_8}[s]) = E(s) \cup \{(v,v) | (u,v) \in E(s)\}$,

and

(iii) $E(\hat{\sigma_{12}}[s]) = E(s) \cup \{(v,u) | (u,v) \in E(s)\}$.

Then we obtain:

Theorem 4.1. $\{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}\}$ *is the set of all proper graph hypersubstitution with respect to the class* $\mathcal{TG}$ *of transitive algebras.*

Proof. If $s \approx t \in Id\mathcal{TG}$ and $s,\ t$ are trivial terms, then for every graph hypersubstitution $\sigma \in \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}\}$ the term $\hat{\sigma}[s]$ and $\hat{\sigma}[t]$ are also trivial and thus $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{TG}$. In the same manner, we see that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{TG}$ for every $\sigma \in \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}\}$, if $s = t = x$.

Now, assume that s and t are non-trivial terms, different from variables, and $s \approx t \in Id\mathcal{TG}$. Then (i)–(iv) of Theorem 3.1 hold.

For $\sigma_6,\ \sigma_8,\ \sigma_{10},\ \sigma_{12}$, we obtain:

$$L(\hat{\sigma}_6[s]) = L(s) = L(t) = L(\hat{\sigma}_6[t]),\ L(\hat{\sigma}_8[s]) = L(s) = L(t) = L(\hat{\sigma}_8[t]),$$

$$L(\hat{\sigma}_{10}[s]) = L(s) = L(t) = L(\hat{\sigma}_{10}[t]),\ L(\hat{\sigma}_{12}[s]) = L(s) = L(t) = L(\hat{\sigma}_{12}[t]).$$

Since $\sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}$ are regular, we have:

$$V(s) = V(\hat{\sigma}_6[s]),\ V(t) = V(\hat{\sigma}_6[t]),\ V(s) = V(\hat{\sigma}_8[s]),\ V(t) = V(\hat{\sigma}_8[t]),$$

$$V(s) = V(\hat{\sigma}_{10}[s]),\ V(t) = V(\hat{\sigma}_{10}[t]),\ V(s) = V(\hat{\sigma}_{12}[s]),\ V(t) = V(\hat{\sigma}_{12}[t]).$$

Since $V(s) = V(t)$, we have

$$V(\hat{\sigma}_6[s]) = V(\hat{\sigma}_6[t]),\ V(\hat{\sigma}_8[s]) = V(\hat{\sigma}_8[t]),$$

$$V(\hat{\sigma}_{10}[s]) = V(\hat{\sigma}_{10}[t])$$

and

$$V(\hat{\sigma}_{12}[s]) = V(\hat{\sigma}_{12}[t]).$$

By Lemma 4.2, we have

$$E(\hat{\sigma}_6[s]) = E(s) \cup \{(u,u) \mid (u,v) \in E(s)\},$$

$$E(\hat{\sigma}_6[t]) = E(t) \cup \{(u,u) \mid (u,v) \in E(t)\},$$

$$E(\hat{\sigma}_8[s]) = E(s) \cup \{(v,v) \mid (u,v) \in E(s)\},$$

$$E(\hat{\sigma}_8[t]) = E(t) \cup \{(v,v) \mid (u,v) \in E(t)\},$$

$$E(\hat{\sigma}_{12}[s]) = E(s) \cup \{(v,u) \mid (u,v) \in E(s)\},$$

and

$$E(\hat{\sigma}_{12}[t]) = E(t) \cup \{(v,u) \mid (u,v) \in E(t)\}.$$

For any x, suppose that x is on a dicycle C in $G(\hat{\sigma}_6[s])$. If the dicycle C is not a loop, then $G(s)$ contains the dicycle C. By Theorem 3.1, x is on a dicycle in $G(t)$. It follows that x is on a dicycle in $G(\hat{\sigma}_6[t])$. Suppose that the dicycle C is a loop, i.e. $(x,x) \in E(\hat{\sigma}_6[s])$. If $(x,x) \in E(s)$, then x is on a dicycle in $G(t)$. Hence, x is on a dicycle in $G(\hat{\sigma}_6[t])$. If $(x,x) \notin E(s)$, then there exist $(x,x_j) \in E(s), x \neq x_j$ for some $x_j \in V(s)$. By Theorem 3.1, $G(t)$ has a dipath from x to x_j. Hence, $(x,x) \in E(\hat{\sigma}_6[t])$. By the same way, we can prove that, if x is on a dicycle in $G(\hat{\sigma}_6[t])$, then x is on a dicycle in $G(\hat{\sigma}_6[s])$.

For any $x,y \in V(s), x \neq y$, suppose that $G(\hat{\sigma}_6[s])$ has a dipath from x to y. Then $G(s)$ has a dipath from x to y. By Theorem 3.1, $G(t)$ has a dipath from x to y. It follows that $G(\hat{\sigma}_6[t])$ has a dipath from x to y and conversely. Hence, $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{TG}$. By the same way, we get $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{TG}$.

For σ_{10}, since $\sigma_6 \circ_N \sigma_8 = \sigma_{10}$ and σ_6, σ_8 are proper, then $\hat{\sigma}_{10}[s] = \hat{\sigma}_6[\hat{\sigma}_8[s]], \hat{\sigma}_{10}[t] = \hat{\sigma}_6[\hat{\sigma}_8[t]]$ and $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{TG}, \hat{\sigma}_6[\hat{\sigma}_8[s]] \approx \hat{\sigma}_6[\hat{\sigma}_8[t]] \in Id\mathcal{TG}$. Therefore, $\hat{\sigma}_{10}[s] \approx \hat{\sigma}_{10}[t] \in Id\mathcal{TG}$.

For σ_{12}, and for any $x \in V(s)$, suppose that x is on a dicycle in $G(\hat{\sigma}_{12}[s])$. Let $C = (x,x_1),(x_1,x_2),(x_2,x_3),...,(x_n,x)$ be such dicycle. If it is a dicycle in $G(s)$, then x is on a dicycle in $G(t)$ by Theorem 3.1(iii). Hence, x is on the dicycle in $G(\hat{\sigma}_{12}[t])$. If it is not, then we consider in two cases. The first case is such when all edges in dicycle C are not belong to $G(s)$. Then $(x,x_n),(x_n,x_{n-1}),(x_{n-1},x_{n-3}),...,(x_1,x)$ is the dicycle in $G(s)$. Hence, x is on a dicycle in $G(t)$ by Theorem 3.1(iii). By Lemma 4.2, we get that x is on a dicycle in $G(\hat{\sigma}_{12}[t])$. The second case is when there exists some edges in C not belong to $G(s)$. In this case the dicycle C is divided into subdipaths $P_1, P_2, ..., P_p$ and $Q_1, Q_2, ..., Q_q$ such that all edges of each $P_i, i = 1,2,...,p$ are belong to $G(s)$ and all edges of each $Q_j, j = 1,2,...,q$ are not belong to $G(s)$. Suppose $Q_j = (x_{j1},x_{j2}),(x_{j2},x_{j3}),...,(x_{jr_ij-1},x_{jr_j}), j = 1,2,...,q$. Then the dipaths $(x_{jr_j},x_{jr_j-1}),...,(x_{j3},x_{j2}),(x_{j2},x_{j1}),\ j = 1,2,...,q$ are in $G(s)$. Hence, there are the dipaths from x_{jr_j} to x_{j1} in $G(t)$ for all $j = 1,2,...,q$ by Theorem 3.1(iv). That is there are the dipaths from x_{j1} to x_{jr_j} in $G(\hat{\sigma}_{12}[t])$ for all $j = 1,2,...,j$. From these dipaths and the subdipaths $P_i, i = 1,2,...,p$ we can find the dicycle C' in $G(\hat{\sigma}_{12}[t])$ which contains x. By the same way, we can prove that, if x is on a dicycle in $G(\hat{\sigma}_{12}[t])$, then x is on a dicycle in $G(\hat{\sigma}_{12}[s])$.

For any $x,y \in V(s), x \neq y$, suppose that $G(\hat{\sigma}_{12}[s])$ has a dipath from x to y. If it is the dipath in $G(s)$, then $G(t)$ has a dipath from x to y. Hence,

$G(\hat{\sigma}_{12}[t])$ has a dipath from x to y. If it is not, then by using the similar method of proof as above, we can find the dipath from x to y in $G(\hat{\sigma}_{12}[t])$ and we can prove the converse. Therefore, $\hat{\sigma}_{12}[s] \approx \hat{\sigma}_{12}[t] \in Id\mathcal{TG}$.

For any $\sigma \notin \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}\}$, we give an identity $s \approx t$ in $\mathcal{TG}$ such that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin Id\mathcal{TG}$. Clearly, if s and t are trivial terms with different leftmost and different rightmost, then $\hat{\sigma_1}[s] \approx \hat{\sigma_1}[t] \notin Id\mathcal{TG}, \hat{\sigma_3}[s] \approx \hat{\sigma_3}[t] \notin Id\mathcal{TG}$ and $\hat{\sigma_2}[s] \approx \hat{\sigma_2}[t] \notin Id\mathcal{TG}, \hat{\sigma_4}[s] \approx \hat{\sigma_4}[t] \notin Id\mathcal{TG}$.

Now, let $s = x_1(x_2x_1)$, $t = x_1((x_2x_1)x_2)$. By Theorem 3.1, we get $s \approx t \in Id\mathcal{TG}$. If $\sigma \in \{\sigma_5, \sigma_7, \sigma_9, \sigma_{13}\}$, then $L(\sigma(f)) = x_2$. We see that $L(\hat{\sigma}[s]) = x_1$ and $L(\hat{\sigma}[t]) = x_2$ for $\sigma \in \{\sigma_5, \sigma_7, \sigma_9, \sigma_{13}\}$. Thus $\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin Id\mathcal{TG}$. ∎

Now, we apply our results to characterize all hyperidentities in the class of all transitive graph algebras. Clearly, if s and t are trivial terms, then $s \approx t$ is a hyperidentity in $\mathcal{TG}$ if and only if they have the same leftmost and the same rightmost and $x \approx x (x \in X)$ is a hyperidentity in $\mathcal{TG}$ too. So we consider the case that s and t are non-trivial and different from variables.

In [2] the concept of a *dual term* s^d of the non-trivial term s was defined in the following way:

If $s = x \in X$, then $x^d = x$; if $s = t_1t_2$, then $s^d = t_2^d t_1^d$. The dual term s^d can be obtained by application of the graph hypersubstitution σ_5, namely, $\hat{\sigma}_5[s] = s^d$.

Theorem 4.2. *An identity $s \approx t$ in $\mathcal{TG}$, where s, t are non-trivial and $s \neq x, t \neq x$, is a hyperidentity in $\mathcal{TG}$ if and only if the dual equation $s^d \approx t^d$ is also an identity in $\mathcal{TG}$.*

Proof. If $s \approx t$ is a hyperidentity in $\mathcal{TG}$, then $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$ is an identity in $\mathcal{TG}$, i.e., $s^d \approx t^d$ is an identity in $\mathcal{TG}$. Conversely, assume that $s \approx t$ is an identity in $\mathcal{TG}$ and that $s^d \approx t^d$ is an identity in $\mathcal{TG}$ too. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from $M_{\mathcal{TG}}$.

If $\sigma \in \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}\}$, then σ is proper and we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{TG}$. By assumption, $\hat{\sigma}_5[s] = s^d \approx t^d = \hat{\sigma}_5[t]$ is an identity in $\mathcal{TG}$.

For $\sigma_1, \sigma_2, \sigma_3$ and σ_4, we have $\hat{\sigma}_1[s] = L(s) = L(t) = \hat{\sigma}_1[t]$, $\hat{\sigma}_2[s] = L(s^d) = L(t^d) = \hat{\sigma}_2[t]$, $\hat{\sigma}_3[s] = L(s)L(s) = L(t)L(t) = \hat{\sigma}_3[t]$ and $\hat{\sigma}_4[s] = L(s^d)L(s^d) = L(t^d)L(t^d) = \hat{\sigma}_4[t]$.

Because of $\sigma_6 \circ_N \sigma_5 = \sigma_7$, $\sigma_8 \circ_N \sigma_5 = \sigma_9$, $\sigma_{10} \circ_N \sigma_5 = \sigma_{11}$ $\sigma_{12} \circ_N \sigma_5 = \sigma_{13}$ and $\hat{\sigma}[\hat{\sigma}_5[t']] = \hat{\sigma}[t'^d]$ for all $\sigma \in M_{\mathcal{TG}}$, $t' \in W_\tau(X_2)$, we have that $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$, $\hat{\sigma}_9[s] \approx \hat{\sigma}_9[t]$, $\hat{\sigma}_{11}[s] \approx \hat{\sigma}_{11}[t]$, $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t]$ are identities in $\mathcal{TG}$. ∎

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