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SOME DISTRIBUTIVITIES IN GBbi-QRs CHARACTERIZING BOOLEAN RINGS

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Abstract

This paper presents some manner of characterization of Boolean rings. These algebraic systems one can also characterize by means of some distributivities satisfied in GBbi-QRs.

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1. Introduction

Generalized Boolean bi-quasirings (GBbi-QRs for short) are generalizations of Boolean rings and arise as a result of considering systems of experimental propositions occuring in classical and quantum physics. On the one hand, the structure of such systems can be described in the framework of lattices which leads to so-called quantum logics. On the other hand, they can be represented by means of ring-like structures. These two possibilities of describing such propositional systems generalize the well-known analogy existing between the variety of Boolean algebras and that of Boolean rings. For more details concerning generalizations of Boolean rings readers are referred to [5], [4], [9] or [8]. Similar algebraic systems were also considered in the papers [1] - [3].

Recall the well-known (see, e.g., [10]) notion of a quasigroup:

Definition 1.1. A groupoid $(A; \circ)$ (where A is a non-empty set and o is a binary operation defined on A) is called a *quasigroup* if for an arbitrary pair of elements $x,\ y \in A$, there is a unique pair of elements $x_1,\ y_1 \in A$, so that $x \circ x_1 = y$ *and* $y_1 \circ x = y$ (unique solvability).

Every GBbi-QR can be treated as a quasigroup with an additional operation. More precisely, we introduce the following concept.

Definition 1.2. An algebra $(R; +, \cdot)$ of type $(2, 2)$ is called a *generalized Boolean bi-quasiring* (GBbi-QR, for short) if there are two distinct elements $0, 1 \in R$ such that for all $x, y, z \in R$ the following laws hold:

(1) For every pair $(x, y) \in R^2$ there is a unique pair $(x_1, y_1) \in R^2$ such that $x + x_1 = y$ and $y_1 + x = y$,

(2) $x + 0 = x = 0 + x$,

(3) $x + 1 = 1 + x$,

(4) $xy = yx$,

(5) $x1 = x$,

(6) $x0 = 0$,

(7) $xx = x$,

(8) $x(yz) = (xy)z$,

(9) $1 + (1 + xy)(1 + x) = x$.

We can notice that if we omit axiom (1) and replace axioms (2) and (3) by the axioms:

(2') $x + y = y + x$

and

(3') $x + 0 = x$

with the axioms (4) − (9) unchanged, then, according to [5], we obtain a generalized Boolean quasiring (GBQR, for short).

We denote by the symbols + and · the binary operations of addition and multiplication, respictively.

The axioms of a GBbi-QR imply $1 + (1 + x) = 1 + (1 + xx)(1 + x) = x$ and $(1 + xy)(1 + x) = 1 + (1 + (1 + xy)(1 + x)) = 1 + x$.

Examples of GBbi-QRs

1. Let $R = \{0, a, b, c, d, 1\}$, where the operations of + and · are defined by the following tables:

+	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	d	0	b	1	c
b	b	0	c	1	a	d
c	c	b	1	d	0	a
d	d	1	a	0	c	b
1	1	c	d	a	b	0

·	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	b	0	b	b
c	0	a	0	c	0	c
d	0	0	b	0	d	d
1	0	a	b	c	d	1

Then $(R; +, \cdot)$ is a GBbi-QR and the semilattice $(R; \cdot)$ is isomorphic to a meet-semi lattice of the bounded 6-element lattice with two atoms a and b.

2. Let $R = \{0, a, b, c, d, e, f, g, h, 1\}$, where the operations of $+$ and $\cdot$ are defined by the following tables:

$+$	0	a	b	c	d	e	f	g	h	1
0	0	a	b	c	d	e	f	g	h	1
a	a	g	0	b	c	h	1	d	f	e
b	b	c	h	d	a	g	e	0	1	f
c	c	f	d	h	e	b	0	1	a	g
d	d	0	e	1	b	f	c	a	g	h
e	e	1	c	0	g	d	h	f	b	a
f	f	d	g	e	1	0	a	h	c	b
g	g	h	a	f	0	1	d	b	e	c
h	h	b	1	a	f	c	g	e	0	d
1	1	e	f	g	h	a	b	c	d	0

$\cdot$	0	a	b	c	d	e	f	g	h	1
0	0	0	0	0	0	0	0	0	0	0
a	0	a	0	a	0	0	a	0	0	a
b	0	0	b	b	0	b	0	0	0	b
c	0	a	b	c	0	b	a	0	0	c
d	0	0	0	0	d	0	0	0	0	d
e	0	0	b	b	0	e	g	g	0	e
f	0	a	0	a	0	g	f	g	0	f
g	0	0	0	0	0	g	g	g	0	g
h	0	0	0	0	0	0	0	0	h	h
1	0	a	b	c	d	e	f	g	h	1

Then $(R; +, \cdot)$ is a GBbi-QR and the semilattice $(R; \cdot)$ is isomorphic to a meet-semilattice of the horizontal sum of 8-element and 2-element Boolean algebras.

2. Characterization of Boolean rings in the framework of generalized Boolean bi-quasirings

It is known that Boolean rings are derived from Boolean algebras. In this section it will be proved that Boolean rings can also be characterized by means of generalized Boolean bi-quasirings with some distributivities.

Let $\underline{R} = (R; +, \cdot)$ be an arbitrary GBbi-QR. We define in R the following relation:

$$xCy \overset{def}{\iff} y\,(1 + x) = y + xy$$

(xCy denotes: "x commutes with y", where $x, y \in R$).

The relation C has already been considered in the theory of GBQRs (see [5], Theorem 4.11, and [6], Lemma 4.1). Because for an arbitrary GBbi-QR we can determine the lattice induced by our system in the same way as the lattice induced by an arbitrary GBQR (see for instance [5]), so the commutativity of each pair of elements of GBbi-QR also implies the commutativity of these elements in the lattice induced by it. Lemma 4.1 occuring in [6] one can quote also in this case, whereas taking into consideration Theorem 5.1 in [8], we have genuinity of Theorem 4.11 of [5] in our situation, too.

The proof of the main theorem in this section is based on three technical lemmas. The following theorem (see [5]) is also used in the proof.

Theorem 2.1. *If* $\mathbf{R} = (R; +, \cdot)$ *is a GBQR in which for all* $x,\ y \in R$ *we have:*

a) $x(1 + y) = x + xy$,

b) $x + y = (1 + (1 + x)(1 + y))(1 + xy)$,

then $\mathbf{R}$ *is a Boolean ring.*

Now we state and prove the lemmas that have been mentioned earlier.

Lemma 2.1. *Let $\underline{R}$ be an arbitrary GBbi-QR, and let x, y, $z \in R$ satisfy the following conditions:*

(a) $x\ Cy$,

(b) $x\ Cz$,

(c) $xy = xz$,

(d) $(1+x)y = (1+x)z$,

then $y = z$.

Proof. If $(1+x)y = (1+x)z$, then $y+xy = z+xz$, therefore $y+xy = z+xy$ and it means (by (1)) that $y = z$. ∎

Lemma 2.2. *If $\underline{R}$ is a GBbi-QR and for each x, $y \in R$ we have:*

($\star$) $x(1+y) = x + xy$,

then $x + x = 0$, it means that $\underline{R}$ is of characteristic 2.

Proof. For $x = 0$ we obtain by (9) that $1 + 1 = 0$. For $y = 1$, we have $x(1+1) = x + x$, by ($\star$), and so $x + x = 0$, by (6). ∎

Lemma 2.3. *If $\underline{R}$ is a GBbi-QR in which for arbitrary x, $y \in R$ the following laws hold:*

1. $x(1+y) = x + xy$,
2. $x(x+y) = x + xy$ *and* $(y+x)x = xy + x$,
3. $(1+x)(x+y) = (1+x)(y+x)$,

then $xy + x = x + xy$.

Proof. Note that $xy\ C(x+xy)$ and $xy\ C(xy+x)$. Now, $xy(x+xy) = xy+xy = 0$ and $xy(xy+x) = xy+xy = 0$, by Lemma 2.2, whereas $(xy+1)(x+xy) = (xy+1)(xy+x)$, therefore $x+xy = xy+x$, by Lemma 2.1. ■

Now we prove our main theorem.

Theorem 2.2. *If $\underline{R}$ is a GBbi-QR and $\underline{R}$ satisfies the following conditions:*

(I) $x(1+y) = x+xy$

(IIa) $x(x+y) = x+xy$ *and* (IIb) $(y+x)x = xy+x$,

(III) $(1+x)(x+y) = y(1+x) = (1+x)(y+x)$,

for arbitrary x, $y \in R$, then $\underline{R}$ is a Boolean ring.

Remark 2.1. The last condition is a special case of the distributive law since in the above Theorem should be written in the following form:

$$(1+x)(x+y) = x(1+x) + y(1+x)$$

and

$$(1+x)(y+x) = y(1+x) + x(1+x).$$

But $x\,(1+x) = 0$, by Lemma 2.2.

***Proof of Theorem* 2.2.** In order to prove Theorem 2.2, it is sufficient by Theorem 2.1 and Lemma 2.2 to show that

1) $x+y = y+x$

and

2) $x+y = (1+(1+x)(1+y))(1+xy)$

Concerning 1): We have $x\ C(x+y)$ and $x\ C(y+x)$, $x(x+y) = x+xy$ and $x(y+x) = xy+x = x+xy$, by Lemma 2.3, so $x(x+y) = x(y+x)$.

On the other hand, $(1+x)(x+y) = (1+x)(y+x)$, and so $x+y = y+x$, by Lemma 2.1.

Concerning 2): We have $x\ C(1+(1+x)(1+y))(1+xy)$ and $x(1+(1+x)(1+y))(1+xy) = (x+x(1+x)(1+y))(1+xy) = x+xy$ and $x+xy = x(x+y)$.

We can also notice that

$$(1+x)(1+(1+x)(1+y))(1+xy)$$

$$= ((1+x)+(1+x)(1+x)(1+y))(1+xy) \overset{by\ (7)}{=}$$

$$= ((1+x)+(1+x)(1+y))(1+xy)$$

$$= (1+x)(1+(1+y))(1+xy) = (1+x)y(1+xy)$$

$$= (1+x)(y+xy) = (y+xy)+x(y+xy)$$

$$= (y+xy)+(xy+xy) = (y+xy)+0 = y+xy = y(1+x).$$

We also have

$$(1+x)(x+y) = y(1+x),\ \text{hence}\ (1+x)(x+y)$$

$$= (1+x)(1+(1+x)(1+y))(1+xy).$$

Now by Lemma 2.1, we obtain that

$$x+y = (1+(1+x)(1+y))(1+xy).$$

■

Still open problem is the following:

if all conditions (I)-(III) *are necessary in Theorem* 2.2?

Let us consider the following examples:

Example 2.1. Consider $\underline{R} = (R; +, \cdot)$ defined by:

$+$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	0	f	e	b	c	1	d
b	b	f	0	1	c	d	a	e
c	c	e	d	0	1	a	b	f
d	d	1	c	b	0	f	e	a
e	e	c	1	a	f	0	d	b
f	f	b	a	d	e	1	0	c
1	1	d	e	f	a	b	c	0

$\cdot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	0	0	0	a	a	a
b	0	0	b	0	b	0	b	b
c	0	0	0	c	c	c	0	c
d	0	0	b	c	d	c	b	d
e	0	a	0	c	c	e	a	e
f	0	a	b	0	b	a	f	f
1	0	a	b	c	d	e	f	1

The semilattice $(R; \cdot)$ is isomorphic to the meet semilattice of an 8-element Boolean algebra. The above example shows that the equality: $x(1 + y) = x + xy$ holds in $\underline{R}$, but $\underline{R} = (R; +, \cdot)$ is not a Boolean ring, because $+$ is not commutative.

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