ADJOINTNESS BETWEEN THEORIES
AND STRICT THEORIES

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Dedicated to Prof. Dr. habil. Klaus Denecke
on the occasion of his 60th birthday

Abstract

The categorical concept of a theory for algebras of a given type was
found by Lawvere in 1963 (see [8]). Hoehnke extended this concept
to partial heterogenous algebras in 1976 (see [5]). A partial theory is
a dhfts-category such that the object class forms a free algebra of type
(2,0,0) freely generated by a nonempty set $J$ in the variety determined
by the identities $ox \approx o$ and $xo \approx o$, where $o$ and $i$ are the elements
selected by the 0-ary operation symbols.

If the object class of a dhfts-category forms even a monoid with unit
element $I$ and zero element $O$, then one has a strict partial theory.

In this paper is shown that every $J$-sorted partial theory corre-
sponds in a natural manner to a $J$-sorted strict partial theory via a
strongly $d$-monoidal functor. Moreover, there is a pair of adjoint func-
tors between the category of all $J$-sorted theories and the category of
all corresponding $J$-sorted strict theories.

This investigation needs an axiomatic characterization of the fun-
damental properties of the category $\mathbf{Par}$ of all partial function
between arbitrary sets and this characterization leads to the concept of dhfts-
and dhth-$\Delta$-categories, respectively (see [5], [11], [13]).

Keywords: symmetric monoidal category, dhfts-category, partial
theory, adjoint functor.

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1. Introduction

Heterogeneous algebras (many-sorted algebras) are, as well-known, algebraic systems consisting of a family of carrier sets and a family of functions such that their definition domain are cartesian products of certain carrier sets and their values are elements of a distinguished carrier set. The concept of such algebraic systems was independently introduced and investigated by P.J. Higgins ([4]) and G. Birkhoff & J.D. Lipson ([1]).

The development of a functorial semantic of algebraic theories for heterogeneous partial algebras requires a good knowledge about diagonal-halfterminal-halfdiagonal-inversional-symmetric monoidal categories ($dhth\n\n\s\r\l\s$-categories).

The morphism class of a category $K$ will be denoted by $K$, the object class of $K$ by $|K|$, and the set of all morphisms in $K$ out of an object $A$ into an object $B$ by $K[A,B]$.

The concept of a symmetric monoidal category in the sense of ([3]) is of fundamental importance.

Definition 1.1 ([3]). A sequence

$$K^\bullet = (K, \otimes, I, a, r, l, s)$$

is called symmetric monoidal category, if $K$ is a category, $\otimes : K \times K \to K$ is a bifunctor, $I$ is a distinguished object of $K$, $a = (a_{A,B,C} \in K[A \otimes (B \otimes C)], (A \otimes B) \otimes C \mid A, B, C \in |K|), r = (r_A \in K[A \otimes I, A] \mid A \in |K|), l = (l_A \in K[I \otimes A, A] \mid A \in |K|), s = (s_{A,B} \in K[A \otimes B, B \otimes A] \mid A, B \in |K|)$ are families of isomorphisms in $K$ (associativity, right-identity, left-identity, symmetry) such that

(F1) $\forall \rho, \rho' \in K (\text{dom}(\rho \otimes \rho') = \text{dom} \rho \otimes \text{dom} \rho')$,

(F2) $\forall \rho, \rho' \in K (\text{cod}(\rho \otimes \rho') = \text{cod} \rho \otimes \text{cod} \rho')$,

(F3) $\forall A, B \in |K| (1_{A \otimes B} = 1_A \otimes 1_B)$,

(F4) $\forall A, B, C, A', B', C' \in |K| \forall \rho \in K[A,B], \sigma \in K[B,C], \rho' \in K[A',B'], \sigma' \in K[B',C'] ((\rho \otimes \rho')(\sigma \otimes \sigma') = \rho \sigma \otimes \rho' \sigma')$,

(M1) $\forall A, B, C, D \in |K|$

$$((a_{A,B,C,D} a_{A \otimes B, C,D} D = (1_A \otimes a_{B,C,D} a_{A,B,C,D} D (a_{A,B,C} \otimes 1_D)),$$

(M2) $\forall A, B \in |K| (a_{A,I,B} r_A \otimes 1_B = 1_A \otimes l_B)$,
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\((M3)\) \(\forall A, B, C \in |K| \ (a_{A,B,C} s_{A \otimes B \otimes C} c_{A,B,C} = (1_A \otimes s_{B,C}) a_{A,C,B} (s_{A \otimes C} \otimes 1_B)),\)

\((M4)\) \(\forall A, B \in |K| \ (s_{A \otimes B} s_{B,A} = 1_{A \otimes B}),\)

\((M5)\) \(\forall A \in |K| \ (s_{A,I} l_A = r_A),\)

\((M6)\) \(\forall A, B, C, A', B', C' \in |K| \ \forall \rho \in K[A, A'], \sigma \in K[B, B'], \tau \in K[C, C'] \)

\(\ (a_{A,B,C} ((\rho \otimes \sigma) \otimes \tau) = (\rho \otimes (\sigma \otimes \tau)) a_{A',B',C'}),\)

\((M7)\) \(\forall A, A' \in |K| \ \forall \rho \in K[A, A'] \ (r_A \rho = (1_I \otimes \rho) r_{A'}),\)

\((M8)\) \(\forall A, B \in |K| \ \forall \rho \in K[A, A'], \sigma \in K[B, B'] \ (s_{A,B} (\sigma \otimes \rho) = (\rho \otimes \sigma) s_{A'B'}).\)

A symmetric monoidal category is called symmetric strictly monoidal, if all associativity, right-identity, and all left-identity isomorphisms, are unit morphisms, i.e. identity morphisms in \(K\) (in the other terminology), only.

The defining conditions determine a lot of properties as follows.

**Corollary 1.2.** Let \(K^*\) be a symmetric monoidal category. Then

\((M9)\) \(\forall A, B \in |K| \ (a_{I,A,B} (l_A \otimes 1_B) = l_{A \otimes B}),\)

\((M10)\) \(\forall A, B \in |K| \ (a_{A,B,I} r_{A \otimes B} = 1_A \otimes r_B),\)

\((M11)\) \(r_I = l_I,\)

\((M12)\) \(s_{I,I} = 1_{I \otimes I},\)

\((M13)\) \(\forall A \in |K| \ (s_{I,A} r_A = l_A),\)

\((M14)\) \(\forall A, A' \in |K| \ \forall \rho \in K[A, A'] \ (l_A \rho = (1_I \otimes \rho) l_{A'}),\)

\((ASR)\) \(\forall A, B \in |K| \ (a_{A,B,I} r_{A \otimes B} = r_{A \otimes B} (r_A^{-1} \otimes 1_B)),\)

\((ASL)\) \(\forall A, B \in |K| \ (a_{I,A,B} s_{I,A} \otimes 1_B) a_{I,A,B}^{-1} = l_{A \otimes B} (1_A \otimes l_B^{-1})).\)

**Defining**

\((B1)\) \(b_{A,B,C,D} := a_{A \otimes B,C,D} (a_{A,B,C}^{-1} (1_A \otimes s_{B,C}) a_{A,C,B} \otimes 1_D) a_{A \otimes C,B,D}^{-1},\)

for arbitrary \(A, B, C, D \in |K|,\)

one obtains furthermore
the carrier is not a set.

By definition, the object class of a symmetric monoidal

\[ (b_{A,B,C,D} = a_{A,B,C,D}^{-1}(1_A \otimes a_{B,C,D}(s_{B,C} \otimes 1_D)a_{C,D}^{-1})a_{A,C,B,D}) \]

(B2) \[ \forall A, B, C, D \in |K| \]

\[ \forall A, B, C, D, A', B', C', D' \in |K| \forall \rho \in K[A, A'], \sigma \in K[B, B'], \lambda \in K[C, C'], \mu \in K[D, D'] \]

\[ (b_{A,B,C,D}((\rho \otimes \sigma) \otimes (\lambda \otimes \mu)) = ((\rho \otimes \lambda) \otimes (\sigma \otimes \mu)b_{A',B',C',D'}) \]

(M15) \[ \forall A, B, C, D \in |K| \]

\[ (b_{A,B,C,D}b_{A,B,C,D} = 1_{A \otimes B \otimes C \otimes D}) \]

(M16) \[ \forall A, B, C, D \in |K| \]

\[ (b_{A,B,C,D}(s_{A,C} \otimes s_{B,D}) = s_{A \otimes B,C \otimes D}b_{C,D,A,B}) \]

(M17) \[ \forall A, A', B, B', C, C' \in |K| \]

\[ (b_{(A \otimes B), (B' \otimes C')}((1_{A \otimes A'} \otimes b_{B,C,B'} \otimes 1_{C \otimes C'}))b_{(A' \otimes B'),(C' \otimes C')} \]

\[ = (a_{A,B,C} \otimes a_{A',B',C'})b_{(A \otimes B), (C \otimes C')}b_{(A' \otimes B' \otimes 1_{C \otimes C'})} \]

or equivalently,

\[ \forall A, A', B, B', C, C' \in |K| \]

\[ (a_{A \otimes A',B,B',C \otimes C'}(b_{A,A',B,B'} \otimes 1_{C \otimes C'}))b_{(A \otimes B),(A' \otimes B'),(C \otimes C')} \]

\[ = (1_{A \otimes A'} \otimes b_{B,B',C,C'})b_{A,A',B,B'}b_{(B \otimes C),(B' \otimes C')}(a_{A,B,C} \otimes a_{A',B',C'}) \]

(M19) \[ \forall A, B \in |K| \]

\[ (b_{A,I,I,B} = 1_{A \otimes B}) \]

(M20) \[ \forall A, B \in |K| \]

\[ (b_{A,I,B,I} = (r_A \otimes r_B)((1_{A \otimes B} \otimes r_I)r_{A \otimes B})^{-1}) \]

(M21) \[ \forall A, B \in |K| \]

\[ (b_{I,A,I,B} = (l_A \otimes l_B)((l_I \otimes 1_{A \otimes B})l_{A \otimes B})^{-1}) \]

(M22) \[ \forall A, B \in |K| \]

\[ (b_{I,A,B,I} = s_{A,A,B \otimes I}(s_{B,I} \otimes s_{I,A})) \]

(M23) \[ \forall A, B \in |K| \]

\[ (b_{A,B,I,I} = (1_{A \otimes B} \otimes r_I)r_{A \otimes B}(r_B^{-1} \otimes r_B^{-1})) \]

(M24) \[ \forall A, B \in |K| \]

\[ (b_{I,I,A,B} = (l_I \otimes 1_{A \otimes B})l_{A \otimes B}(l_A^{-1} \otimes l_B^{-1})) \]

Remark 1.3. By definition, the object class of a symmetric monoidal category \( K^* \) forms an illegitimate algebra \(|K|, \otimes, I \) of type \((2,0)\), because the carrier is not a set.

Especially, of interest are objects consisting of finitely many factors \( I \) in arbitrary brackets, namely objects of the subalgebra \( (I) \) generated by the one element set \( \{I\} \) as follows:

\[ \langle I \rangle^{(0)} := \{I\}, \quad \langle I \rangle^{(n+1)} := \langle I \rangle^{(n)} \cup \{X \otimes Y \mid X, Y \in \langle I \rangle^{(n)}\}, \]

\[ \langle I \rangle := \bigcup_{n \in \mathbb{N}} \langle I \rangle^{(n)}. \]
This is in fact an algebra of type $(2,0)$. The set $(I)$ determines in a natural manner a symmetric monoidal subcategory $(I)^\ast$ of $K^\ast$.

Moreover, every nonempty set $J \subseteq |K|$, $I \notin J$, determines a subalgebra $H$ of type $(2,0)$ as follows:

$$H^{(0)} := J \cup \{I\}, \quad H^{(n+1)} := H^{(n)} \cup \{X \otimes Y \mid X, Y \in H^{(n)}\},$$

$$H := \bigcup_{n \in \mathbb{N}} H^{(n)}.$$  

The symmetric monoidal subcategory of $K^\ast$ generated by $H$, respectively by $J$, will be denoted by $H^\ast$. Obviously, $H^\ast$ is a small category, since the carrier is a set.

If $K^\ast$ is a symmetric strictly monoidal category, then $(|K|, \otimes, I)$ is an illegitimate monoid, $(I)$ is a one element set and every set $J$ generates a monoid $S$ with unit $I$.

**Definition 1.4 ([10]).** Let $K^\ast$ be a symmetric monoidal category. The monoidal subcategory $C_K^\ast$ of $K^\ast$ generated by the morphism class

$$\{1_X \mid X \in |K|\} \cup \{a_{X,Y,Z} \mid X, Y, Z \in |K|\} \cup \{r_X \mid X \in |K|\} \cup \{l_X \mid X \in |K|\}$$

$$\cup \{a_{X,Y,Z}^{-1} \mid X, Y, Z \in |K|\} \cup \{r_X^{-1} \mid X \in |K|\} \cup \{l_X^{-1} \mid X \in |K|\}$$

is called central subcategory of $K^\ast$, its morphisms are called central morphisms of $K^\ast$.

**Remark 1.5.** The class $C_K$ of all central morphisms of a symmetric monoidal category $K^\ast$ is given by the construction

$$C_K^{(0)} := \{1_X \mid X \in |K|\} \cup \{a_{X,Y,Z} \mid X, Y, Z \in |K|\} \cup \{r_X \mid X \in |K|\} \cup \{l_X \mid X \in |K|\}$$

$$\cup \{a_{X,Y,Z}^{-1} \mid X, Y, Z \in |K|\} \cup \{r_X^{-1} \mid X \in |K|\} \cup \{l_X^{-1} \mid X \in |K|\},$$

$$C_K^{(n+1)} := C_K^{(n)} \cup \{c_1 c_2 \mid c_1 \in K[X, Y] \land c_2 \in K[Y, P] \land c_1, c_2 \in C_K^{(n)} \land X, Y, P \in |K|\} \cup \{c_1 \otimes c_2 \mid c_1, c_2 \in C_K^{(n)}\},$$

$$C_K = \bigcup_{n \in \mathbb{N}} C_K^{(n)}$$

and forms a monoidal subcategory $C_K^\ast$ of $K^\ast$.  

$C_K$ consists of unit morphisms only, if $K^\bullet$ is symmetric strictly monoidal. The class of all unit morphisms of $K$ is denoted by $Un_K$.

**Coherence principle** ([9], [6], [7]). Let $K^\bullet$ be a symmetric monoidal category. Then every planar closed diagram of central morphisms is commutative.

**Corollary 1.6.** Let $K^\bullet$ be a symmetric monoidal category. Then, by the coherence principle, there is at most one central morphism between objects $X$ and $Y$ for every $X, Y \in |K|$. The central morphisms are isomorphisms only.

Let $X$ and $Y$ be arbitrary objects of $(I)^\bullet$. Then there is exactly one central morphism in the set $(I)[X,Y]$.

The isomorphisms

$$i^{(n)} : I^n \rightarrow I \text{ and } i^{* (n)} : \bigotimes_{k=1}^{n} I \rightarrow I,$$

where $I^n := \bigotimes_{k=1}^{n} I$ and $\bigotimes_{k=1}^{n+1} I := I \bigotimes \bigotimes_{k=1}^{n} I = I \bigotimes (\bigotimes_{K=1}^{n} I)$,

between the different powers of $I$ and the object $I$ are expressable in the following form:

$$i^{(1)} = 1_I, \quad i^{(n+1)} = (i^{(n)} \otimes 1_I) r_I, \quad n \geq 1, \text{ especially } i^{(2)} = r_I,$$

$$i^{* (1)} = 1_I, \quad i^{* (n+1)} = (1_I \otimes i^{* (n)}) l_I, \quad n \geq 1, \text{ especially } i^{* (2)} = l_I.$$

**Proof.** It remains to show the existence of an central morphism between arbitrary $X$ and $Y$ of $(I)$.

a) One proves by induction over the complexity of $X$: $\forall X \in (I) \exists c \in (I)[X,I] \ (c \in C_K)$:

$$\forall X \in (I)^{(0)} \ (X = I \ \land \ 1_I \in C_K);$$

$$\forall n \in \mathbb{N} \ [\forall X \in (I)^{(n)} \exists c \in (I)[X,I] \ (c \in C_K) \Rightarrow$$

$$\Rightarrow \forall X \in (I)^{(n+1)} \exists c \in (I)[X,I] \ (c \in C_K)],$$
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since

$$\forall X \in (\langle I \rangle^{(n+1)} \setminus \langle I \rangle^{(n)}) \exists X_1, X_2 \in \langle I \rangle^{(n)} \exists c_i \in \langle I \rangle[X_i, I] \cap C_K \ (i = 1, 2)$$

$$(X = X_1 \otimes X_2 \land c_1 \otimes c_2 \in C_K \Rightarrow (c_1 \otimes c_2)r_I \in \langle I \rangle[X, I] \cap C_K).$$

b) One proves by induction over the complexity of $Y$:

$$\forall X \in \langle I \rangle \forall Y \in \langle I \rangle \exists c \in \langle I \rangle[X, Y] \ (c \in C_K).$$

The truth of the assertion for an arbitrary $X \in \langle I \rangle$ and for $Y \in \langle I \rangle^{(0)}$ was shown in a).

$$\forall X \in \langle I \rangle \forall n \in \mathbb{N} \left( \forall Y \in \langle I \rangle^{(n)} \exists c \in \langle I \rangle[X, Y] \ (c \in C_K) \Rightarrow \forall Y \in \langle I \rangle^{(n+1)} \exists c \in \langle I \rangle[X, Y] \ (c \in C_K) \right),$$

since

$$\forall Y \in (\langle I \rangle^{(n+1)} \setminus \langle I \rangle^{(n)}) \exists Y_1, Y_2 \in \langle I \rangle^{(n)} \exists c_1 \in \langle I \rangle[X, Y_1] \cap C_K \exists c_2 \in \langle I \rangle[X, Y_2] \cap C_K$$

$$\exists c \in \langle I \rangle[Y_1, Y_2] \cap C_K$$

$$(Y = Y_1 \otimes Y_2 \land c_1 \otimes c_2 \in C_K \Rightarrow r^{-1}_X(c_1 \otimes c_2) \in \langle I \rangle[X, Y] \cap C_K).$$

Definitions 1.7. Let $K^\bullet$ be a symmetric monoidal category in the sense of [3].

A sequence $(K^\bullet, d)$ is called diagonal-symmetric monoidal category (shortly ds-category) (in [2] considered in the strict case as a special Kronecker-category, in [13] as “diagonal-symmetrische Kategorie”), if $d = (d_A \in K[A, A \otimes A] \mid A \in |K|)$ is a family of morphisms of $K$ such that

- **(D1)** $\forall A, A' \in |K| \forall \varphi \in K[A, A'] \ (\varphi d_{A'} = d_A(\varphi \otimes \varphi))$,

- **(D2)** $\forall A \in |K| \ (d_A(d_A \otimes 1_A) = d_A(1_A \otimes d_A)a_{A,A,A})$,

- **(D3)** $\forall A \in |K| \ (d_{A}s_{A,A} = d_A)$,

- **(D4)** $\forall A, B \in |K| \ ((d_A \otimes d_B)b_{A,A,B,B} = d_{A\otimes B})$

are fulfilled.
\((K^\bullet, d, t)\) is called **diagonal-terminal-symmetric monoidal category** (dts-category) ([2]), if \((K^\bullet, d)\) is a ds-category with a family \(t = (t_A \mid A \in |K|)\) of terminal morphisms \(t_A \in K[A, I]\) such that the conditions

\[(T1) \quad \forall A, A' \in |K| \forall \varphi \in K[A, A'] (\varphi t_A' = t_A)\]
and
\[(DTR) \quad \forall A \in |K| (d_A(1_A \otimes t_A)r_A = 1_A)\]

are right.

\((K^\bullet; d, t, o)\) will be called **diagonal-halfterminal-symmetric monoidal category** or **Hoehnke category** (shortly dhts-category) ([5], [11], [13]), if \(d\) and \(t\) are morphism families as above and \(o : I \to O\) is a distinguished morphism in \(K\) related to a distinguished object \(O \in |K|\), such that

\[(D1) \quad \forall A, A' \in |K| \forall \varphi \in K[A, A'] (d_A(\varphi \otimes \varphi) = \varphi d_{A'})\]
\[(DTR) \quad \forall A \in |K| (d_A(1_A \otimes t_A)r_A = 1_A)\]
\[(DTL) \quad \forall A \in |K| (d_A(t_A \otimes 1_A)l_A = 1_A)\]
\[(DTRL) \quad \forall A_1, A_2 \in |K| ((d_A_1 \otimes 1_A_2)(1_A_1 \otimes t_{A_2})r_{A_1} \otimes (t_{A_1} \otimes 1_A_2)l_{A_2}) = 1_{A_1 \otimes A_2})\]
\[(TT) \quad \forall A, B \in |K| (t_{A \otimes B} = (t_A \otimes t_B)t_{I \otimes I})\]
\[(O1) \quad \forall A \in |K| (A \otimes O = O \otimes A = O)\]
\[(o1) \quad \forall A \in |K| \forall \varphi \in K[A, O] (t_Ao = \varphi)\]
and
\[(o2) \quad \forall A \in |K| \forall \psi \in K[O, A] ((1_A \otimes t_O)r_A = \psi)\]

are fulfilled.

\((K^\bullet; d, t, \nabla, o)\) is called **diagonal-halfterminal-halfdiagonal-inversional-symmetric monoidal category** or **Hoehnke category with halfdiagonalinversions** (for short dhth\(\nabla\)-s-category, in [13] named dhth\(\nabla\)-symmetric category), if \((K^\bullet; d, t, o)\) is a dhts-category endowed with a morphism family

\[\nabla = (\nabla_A \in K[A \otimes A, A] \mid A \in |K|)\]

fulfilling
(D_1^A) \forall A \in |K| (d_A \nabla_A = 1_A),

(D_2^A) \forall A \in |K| (\nabla_A d_A \otimes_A = d_{A \otimes} (\nabla_A \otimes 1_{A \otimes})).

Any ds-, dts-, dhts-, and dhth\n-s-category, respectively, is called strict, if the underlying symmetric monoidal category is strictly monoidal.

The zero morphisms \(o_{A,B}\) absorb all other morphisms at composition and \(\otimes\)-operation in any dhts-category, i.e.

(o3) \(\forall A, A', B, B' \in |K| \forall \rho \in K[A, A'], \sigma \in K[B, B']\)

\[
(\rho o_{A',B} = o_{A,B} \wedge o_{A,B} \sigma = o_{A,B'})
\]

(o4) \(\forall A, B, C, D \in |K| \forall \xi \in K[C, D]\)

\[
(o_{A,B} \otimes \xi = o_{A \otimes B, B \otimes D} \wedge \xi \otimes o_{A,B} = o_{C \otimes A, B \otimes D})
\]

(o5) \(\forall A \in |K| (o_{O,A} = (1_A \otimes t_O)r_A = (t_O \otimes 1_A)t_A)\).

Because of (o1) and (o2), the unit morphism \(1_O\) is identical with the zero morphism \(o_{O,O}\).

The category \(Par\) of all partial functions between arbitrary sets is an example for a dhth\n-s-category.

In view of the properties of the category \(Par\) we will consider mainly dhts-categories fulfilling the conditions

(N_1) \(\forall A, B \in |K| (A \otimes B = O \Rightarrow (A = O \vee B = O))\),

(N_2) \(\forall A, B, C, D \in |K| \forall \varphi \in K[A, B] \forall \psi \in K[C, D]\)

\[
(\varphi \otimes \psi = o_{A \otimes C, B \otimes D} \Rightarrow (\varphi = o_{A,B} \vee \psi = o_{C,D}))
\]

(N_3) \(I \neq O\),

(N_4) \(\forall A \in |K| \setminus \{\emptyset\} (1_A \neq o_{A,A})\).
Observe that \((K^\bullet; d)\) is a \(ds\)-category for each \(dhts\)-category \((K^\bullet; d, t, o)\) and \(\nabla\) is the only family in a \(dhth\nabla s\)-category with the properties \((D^*_1)\) and \((D^*_2)\), cf. [11].

Any \(dhts\)-category \(K = (K^\bullet; d, t, o)\) has the following properties:

- The class \(T_K := \{ \varphi \in K \mid \varphi \circ \text{cod} \varphi = \text{dom} \varphi \}\) of so-called total morphisms of \(K\) forms a \(dts\)-subcategory \(T_K\) of \(K\) ([12]).

- \((A \otimes B, (1_A \otimes t_B)r_A, (t_A \otimes 1_B)l_B)\)
is a categorical product in \(T_K\), but not in the whole category \(K\). The morphisms

\[ p_1^{A, B} := (1_A \otimes t_B)r_A \quad \text{and} \quad p_2^{A, B} := (t_A \otimes 1_B)l_B \]

are called the canonical projections concerning \(A\) and \(B\) ([5]).

- The class \(Iso_K\) of all isomorphisms of \(K\) forms a symmetric monoidal subcategory \(Iso^*_K\) and one has

\[ Un_K \subseteq C_K \subseteq Iso_K \subseteq Cor_K \subseteq T_K, \]

where \(Cor_K\) denotes the subcategory of all coretractions of \(K\).

- The relation \(\leq\) defined by

\[ \varphi \leq \psi \iff \exists A, A' \in [K] (\varphi, \psi \in K[A, A'] \land \varphi = d_A(\varphi \otimes \psi)p_2^{A', A'}) \]
is a partial order relation and it is compatible with composition and \(\otimes\)-operation of morphisms ([11]). Moreover, the following conditions are equivalent ([12]):

\[ \varphi = d_A(\varphi \otimes \psi)p_2^{A', A'}, \]

\[ \varphi = d_A(\psi \otimes \varphi)p_1^{A', A'}, \]

\[ \varphi d_{A'} = d_A(\varphi \otimes \psi), \]

\[ \varphi d_{A'} = d_A(\psi \otimes \varphi). \]
• Each morphism \( \phi \in K \) determines a so-called subidentity \( \alpha(\phi) \) as follows ([11]):

\[
\alpha(\phi) := d_{\dom \phi}(1_{\dom \phi} \otimes \phi) p^1_{\dom \phi, \cod \phi} \leq 1_{\dom \phi}.
\]

Moreover, each \( \text{dhth} \nabla s \)-category has the properties

\[
(h\nabla 1) \; \forall A, A' \in |K| \; \forall \varphi \in K[A, A'] \; (\nabla_A \varphi d_{A'} = d_A \otimes (\varphi \otimes \varphi) \nabla_{A'})
\]

\[
(hT 1) \; \forall A, A' \in |K| \; \forall \varphi \in K[A, A'] \; (\varphi t_{A'} d_{I} = d_A (\varphi t_{A'} \otimes t_A)),
\]

therefore \( \nabla_A \varphi \leq (\varphi \otimes \varphi) \nabla_A \) and \( \varphi t_{A'} \leq t_A \) for all morphisms \( \varphi \in K[A, A'] \) and all objects \( A, A' \in |K| \) ([15]).

Every morphism set \( K[A, B] \) of a \( \text{dhth} \nabla s \)-category \( K \) forms a meet-semilattice with respect to \( \varphi \land \psi = d_A (\varphi \otimes \psi) \nabla_B \). This semilattice has the minimum \( o_{A,B} \), maximal elements are the total morphisms. Especially, the morphism sets \( K[A, I] \) possess a maximum, namely \( t_I \).

The basic morphisms related to the distinguished object \( I \) in any symmetric monoidal category, any \( \text{dhts} \)-category, or even any \( \text{dhth} \nabla s \)-category have some interesting properties as follows:

**Lemma 1.8.** Let \( K^* \) be a symmetric monoidal category. Then one has

\[
a_{I,I,I} = r_I^{-1} \otimes r_I.
\]

Moreover, every \( \text{dhts} \)-category \( K \) has in addition the properties

\[
d_I = r_I^{-1}, \quad r_I d_I = 1_{I \otimes I}, \quad t_I = 1_I \quad ([11]), \quad t_{I \otimes I} = r_I,
\]

\[
i \in \text{Iso}_K[I, I] \Rightarrow i = t_I,
\]

\[
\forall X \in |K| \; \forall x \in K[I, X] \; (x \in \text{Iso}_K \Rightarrow x^{-1} = t_X).
\]

Finally, if \( K \) is a \( \text{dhth} \nabla s \)-category, then the additional property

\[
\nabla_I = r_I
\]

is true.
Proof. The identity $a_{A,I,B}(r_A \otimes 1_B) = 1_A \otimes l_B$ is one of the defining properties of monoidal-symmetric categories, hence $a_{I,I,I}(r_I \otimes 1_I) = 1_I \otimes r_I$ by $r_I = l_I$ and $a_{I,I,I} = (r_I^{-1} \otimes r_I)$, since all right-identity morphisms are isomorphisms.

In any dhts-category one has the defining identity $d_A(1_A \otimes t_A)r_A = 1_A$, hence $1_I = d_I(1_I \otimes t_I)r_I = d_I(1_I \otimes 1_I)r_I = d_Ir_I$, consequently $d_I = r_I^{-1}$ and $r_Id_I = 1_{I\otimes I}$.

Each coretraction $\varphi \in K[A,B]$ of a dhts-category has the property $\varphi t_B = t_A$. Because $d_I$ is even an isomorphism, one observes $d_I\varphi t_B = \varphi t_B = t_A$.

One of the characterizing conditions of the diagonal inversions in a dhts-category is $d_A \nabla_A = 1_A$. Therefore, $\nabla_I = 1_I \otimes \nabla_I = r_Id_I \otimes 1_I = r_I1_I = r_I$.

Let $x \in K[I,X]$ be an isomorphism in a dhts-category $K$. Then one obtains in the same manner as above $1_I = t_I = xt_X$, hence the assertion.

Remark 1.9. Let $K$ be a dhts-category. Then its object class $|K|$ forms an illegitimate algebra $(|K|, \otimes, I, O)$ of type $(2,0,0)$. Let $J$ be a nonempty set such that $J \cap \{I, O\} = \emptyset$. Then $J$ generates in $|K|$ a subalgebra $H^o$ of type $(2,0,0)$:

$$H^o(0) := J \cup \{I, O\}, \quad H^o(n+1) := H^o(n) \cup \{X \otimes Y \mid X, Y \in H^o(n)\},$$

$$H^o := \bigcup_{n \in \mathbb{N}} H^o(n).$$

The dhts-subcategory of $K$ generated by $H^o$, respectively by $J$, will be denoted by $H^o$. Obviously, $H^o$ is again a small category.

Let $K$ be a strict dhts-category. Then the algebra $S^o := (H^o, \otimes, I, O)$ generated by a set $J$ is a monoid with unit $I$ and zero $O$.

2. Hoehnke theories

Let $G$ denote the variety of all algebras of type $(2,0)$ (groupoids with a distinguished element $I$). Note that the distinguished element $I$ does not play the role of a unit element in general. By the principles of General Algebra, every set $J$ determines in $G$ a free $G$-algebra $F_G(J)$ freely generated
by \( J \). The algebra \( F_G(J) \) contains a subalgebra \( \langle I \rangle \) consisting of all possible products of \( I \) as follows:

\[
\langle I \rangle^{(0)} := \{ I \}, \quad \langle I \rangle^{(n+1)} := \langle I \rangle^{(n)} \cup \{ X \otimes Y \mid X, Y \in \langle I \rangle^{(n)} \}, \quad \langle I \rangle := \bigcup_{k \in \mathbb{N}} \langle I \rangle^{(k)}.
\]

Every algebra \( A = (A; \otimes, I, O) \in \mathcal{G} \) can be transfered into an algebra \( (A; \otimes, I, O) \) of type \((2,0,0)\) by addition of a distinguished element \( O \) with the property \( \forall X \in A \ (X \otimes O = O = O \otimes X) \).

By \( \mathcal{G}^0 \) shall be denoted the variety of all algebras \((A; \otimes, I, O)\) of type \((2,0,0)\) (groupoids with distinguished element \( I \) and zero element \( O \)) such that \( \forall X \in A \ (X \otimes O = O = O \otimes X) \). \( F_{\mathcal{G}^0}(J) \) denotes the free \( \mathcal{G}^0 \)-algebra freely generated by a set \( J \) such that \( J \cap \{I, O\} = \emptyset \). Clearly, \( F_{\mathcal{G}^0}(J) \) contains the trivial subalgebra \( \langle I \rangle^0 \) with the carrier set \( \langle I \rangle^0 = \langle I \rangle \cup \{O\} \).

Let \( \mathcal{M} \) be the variety of all monoids (algebras of type \((2,0)\)) and let \( \mathcal{M}^0 \) be the variety of all monoids with absorbing zero (algebras of type \((2,0,0)\) too).

The free \( \mathcal{M} \)-algebra (\( \mathcal{M}^0 \)-algebra) freely generated by \( J \) will be denoted by \( F_{\mathcal{M}}(J) \) (\( F_{\mathcal{M}^0}(J) \)). The trivial subalgebra \( \langle I \rangle^0 \) has the carrier set \( \langle I \rangle = \{I\} \) \((\langle I \rangle^0 = \{I, O\})\).

The identical embedding functions from \( J \) into the corresponding algebras will be denoted as follows:

\[
\iota_H: J \hookrightarrow F_G(J), \quad \iota_{H^0}: J \hookrightarrow F_{\mathcal{G}^0}(J),
\]
\[
\iota_S: J \hookrightarrow F_{\mathcal{M}}(J), \quad \iota_{S^0}: J \hookrightarrow F_{\mathcal{M}^0}(J).
\]

**Definition 2.1** ([5]). Let \( T \) be a \( dhts \)-category, a \( dhth \backslash s \)-category, or a \( dts \)-category and let \( J \) be a nonempty set of objects of \( T \) such that \( I, O \notin J \).

Then \( T \) will be called

- \( J \)-sorted \( dhsts \)-theory or \( J \)-sorted Hochnke theory,
- \( J \)-sorted \( dhth \backslash s \)-theory or
- \( J \)-sorted Hochnke theory with halfdiagonal inversions,
- \( J \)-sorted \( dts \)-theory, respectively,

if \((|T|; \otimes, I, O)\) is a free \( \mathcal{G}^0 \)-algebra freely generated by \( J \) \((|T|; \otimes, I)\) is a free \( \mathcal{G} \)-algebra freely generated by \( J, I \notin J \).

The class of all \( J \)-sorted \( dhsts \)-theories \((J \text{-sorted } dhth \backslash s \text{-theories,}
\)
\( J \)-sorted \( dts \)-theories) will be denoted by \( |Th^0_{dhsts}(J)| \) \((|Th^0_{dhth \backslash s}(J)|, |Th_{dts}(J)|)\).
Besides the theory concept above we consider the following, more artifical, but simpler one, which arises in strict monoidal categories by replacing of the groupoid $\mathbf{F}_G^\ast(J)$ (\(\mathbf{F}_G(J)\)) by the monoid $\mathbf{F}_{M^+}(J)$ (\(\mathbf{F}_M(J)\)). So, one defines

**Definition 2.2.** Let $\mathbf{T}$ be a dhts-category, a dhth\(\nabla\)s-category, or a dts-category such that the underlying symmetric monoidal category $\mathbf{T}^\bullet$ is strictly monoidal, i.e. all the morphisms $a$, $r$, and $l$ are unit-morphisms only $\left(A \otimes (B \otimes C) = (A \otimes B) \otimes C, A \otimes I = A = I \otimes A, a_{A,B,C} = 1_{A \otimes B \otimes C}, r_A = 1_A = l_A \right.$ for all $A, B, C \in \{\mathbf{T}\}$.

Then $\mathbf{T}$ will be called

- \textit{J}-sorted strict dhts-theory or \textit{strict J}-sorted Hoehnke theory,
- \textit{J}-sorted strict dhth\(\nabla\)s-theory or
- \textit{strict J}-sorted Hoehnke theory with halfdiagonal inverions,
- \textit{J}-sorted strict dts-theory, respectively,

if there exists a nonempty set $J$ in $\{\mathbf{T}\}$ such that $I, O \notin J$ and $\left(|\mathbf{T}|; \otimes, I, O\right)$ is a free $\mathcal{M}^0$-algebra (\(|\mathbf{T}|; \otimes, I\) is a free $\mathcal{M}$-algebra) freely generated by $J$.

The class of all \textit{J}-sorted strict dhts-theories (\textit{J}-sorted strict dhth\(\nabla\)s-theories, \textit{J}-sorted strict dts-theories) will be denoted by

$$\left|\text{Th}_{dhth}^\ast(J)\right| \left|\text{Th}_{dhth\nabla}^\ast(J)\right| \left|\text{Th}_{dt}^\ast(J)\right|.$$  

The categories of the classes $\left|\text{Th}_{dhth}^\ast(J)\right|$, $\left|\text{Th}_{dhth\nabla}^\ast(J)\right|$, $\left|\text{Th}_{dt}^\ast(J)\right|$, and $\left|\text{Th}_{dh}^\ast(J)\right|$ shortly will called partial theories (Hoehnke theories) and categories of $\left|\text{Th}_{dt}^\ast(J)\right|$ and $\left|\text{Th}_{dh}^\ast(J)\right|$ are named total theories.

For a given set $J$ one has on the one hand the free algebra $\mathbf{F}_G^\ast(J)$ and on the other hand the free algebra $\mathbf{F}_{M^+}(J)$ and both are algebras of the variety $\mathcal{G}^0$ of type $\left(2,0,0\right)$. Therefore, there arises the question about a connection between the two algebras.

**Lemma 2.3.** Let $\mathbf{F}_G^\ast(J) =: (H^\circ; \otimes, I, O)$, $\mathbf{F}_{M^+}(J) =: (S^0; \otimes, I, O)$, $\mathbf{F}_G(J) =: (H; \otimes, I)$, and $\mathbf{F}_M(J) =: (S; \otimes, I)$ be the algebras defined as above. Then there is exactly one homomorphism $W^*: \mathbf{F}_G^\ast(J) \rightarrow \mathbf{F}_{M^+}(J)$ ($W^*: \mathbf{F}_G(J) \rightarrow \mathbf{F}_M(J)$) such that $t_H W^* = t_S$ ($t_H W^* = t_S$).

The mapping $W^*$ works as follows:

$I \mapsto I =: IW^*$, $O \mapsto O =: OW^*$, $J \ni A \mapsto A =: AW^*$,

$\forall X, Y \in H \left((X \otimes Y)W^* = XW^* \otimes YW^*\right)$. 
Proof. Let $T \in |Th_{dht}(J)|$. The algebra $F_{M^o}(J) = (S^o; \otimes, I, O)$, generated by $J$, belongs to $G^o$. Since $(H^o := |T|; \otimes, I, O)$ is a free $G^o$-algebra freely generated by $J$, there is exactly one homomorphism $W^*$ such that $\iota_{H^o}W^* = \iota_{S^o}$ and this homomorphism is surjective. The assertion about the working of the mapping becomes clear since $\iota_{S^o}$ is the identical embedding of $J$ into $S^o$.

The statement concerning groupoids and monoids without zero will be proved in the same manner.

Corollary 2.4. The mapping $W^* : H^o \to S^o$ has the following properties:

\[
\forall X \in \langle I \rangle (XW^* = I),
\]

\[
\forall Y \in H^o \forall X \in \langle I \rangle ((Y \otimes X)W^* = (X \otimes Y)W^* = YW^*),
\]

\[
\forall X, Y, Z \in H^o ((X \otimes (Y \otimes Z))W^* = ((X \otimes Y) \otimes Z)W^*),
\]

\[
\forall X \in H^o \setminus \langle I \rangle^o \exists!! A_1, A_2, ..., A_n (XW^* = A_1 \otimes A_2 \otimes \cdots \otimes A_n).
\]

Proof. The first assertion one proves by induction over the complexity of the elements of $\langle I \rangle$.

By Lemma 2.3, $IW^* = I$. Assume that for any $n \in \mathbb{N}$ the condition

\[
\forall Y \in \langle I \rangle^{(n)} (YW^* = I)
\]

is valid. Then

\[
\forall X \in \langle I \rangle^{(n+1)} \setminus \langle I \rangle^{(n)} \exists X_1, X_2 \in \langle I \rangle^{(n)}
\]

\[
(XW^* = (X_1 \otimes X_2)W^* = X_1W^* \otimes X_2W^* = I \otimes I = I),
\]

hence $\forall n \in \mathbb{N} \forall X \in \langle I \rangle^{(n)}(XW^* = I)$.

Because of $(X \otimes Y)W^* = XW^* \otimes YW^*$, $XW^* = I$ for every $X \in \langle I \rangle$ and $I$ is the unit element in the monoid, the second claim becomes true.

Let $X, Y,$ and $Z$ be elements of $|T| = H^o$. Then $XW^*, YW^*$, and $ZW^*$ are elements of the monoid $S^o$ and

\[
(X \otimes (Y \otimes Z))W^* = XW^* \otimes YW^* \otimes ZW^* = ((X \otimes Y) \otimes Z)W^*.
\]
Because of

\[ H = \bigcup_{k \in \mathbb{N}} H^{(n)} \], \quad H^{(0)} := J \cup \{ I \}, \]

\[ H^{(n+1)} := H^{(n)} \cup \{ X \otimes Y \mid X, Y \in H^{(n)} \}, \quad n \in \mathbb{N}, \]

one shows the existence of such a representation by induction over the complexity of \( X \).

Assuming that for any \( n \in \mathbb{N} \) each \( X \in H^{(n)} \) fulfills the assertion one investigates an arbitrary \( Y \in H^{(n+1)} \) \( \setminus H^{(n)} \setminus \{ I \} \). Then there are \( X_1, X_2 \in H^{(n)} \setminus \{ I \} \) such that \( YW^* = (X_1 \otimes X_2)W^* = X_1W^* \otimes X_2W^* \), hence there are \( A_1, \ldots, A_j, B_1, \ldots, B_k \in J \) such that \( YW^* = (X_1 \otimes X_2)W^* = X_1W^* \otimes X_2W^* = A_1 \otimes \cdots \otimes A_j \otimes B_1 \otimes \cdots \otimes B_k. \)

The uniqueness of the factors of a \( \otimes \)-product which are elements of \( J \) is a consequence of the fact that \( (S^\circ; \otimes, I, O) \) is a free \( \mathcal{M}^\circ \)-algebra freely generated by \( J \).

\[ X \in H^{(0)} \setminus \{ I \} \Rightarrow X = A \in J \Rightarrow XW^* = AW^* = A. \]

\section*{Lemma 2.5.}

Let be given \( H^\circ \) and \( S^\circ \) as above related to a fixed set \( J \). Then there is a function \( W : S^\circ \rightarrow H^\circ \) such that

- (W1) \( WW^* = 1_{S^\circ} \) and
- (W2) \( \forall A, B \in S^\circ \ (A \otimes B = (AW \otimes BW)W^*) \).

The function \( \Phi : H^\circ \rightarrow H^\circ \) defined by \( \Phi := W^*W \) has the properties

- (W3) \( \forall X \in \{ I \} \ (X\Phi = I) \),
- (W4) \( \forall X \in H \setminus \{ I \} \ \exists A_1, \ldots, A_n \in J \ (X\Phi = \bigotimes_{j=1}^{n} A_j) \),
- (W5) \( \forall X_1, X_2, Y_1, Y_2 \in H^\circ \)

\[ ((X_1 \otimes X_2)\Phi = (Y_1 \otimes Y_2)\Phi \iff (X_1\Phi) \otimes (X_2\Phi) = (Y_1\Phi) \otimes (Y_2\Phi)) \].
Proof. Ad (W1): Defining

\[ OW := O, \quad IW := I, \quad \forall A_1, \ldots, A_n \in J \left( (A_1 \otimes \cdots \otimes A_n)W := \bigotimes_{j=1}^{n} A_j \right) \]

one gets immediately \( WW^* = 1_{S^*} \).

Ad (W2): The assertion is trivial for \( A = O \) or \( B = O \). The same is true if \( A = I \) or \( B = I \). Now let \( A, B \in S \setminus \{I\} \). Then, by definition,

\[ A \otimes B = A_1 \otimes \cdots \otimes A_n \otimes B_1 \otimes \cdots \otimes B_m = \left( \bigotimes_{k=1}^{n} A_k \right) W^* \otimes \left( \bigotimes_{j=1}^{m} B_j \right) W^* = (AW)W^* \otimes (BW)W^* = (AW \otimes BW)W^*. \]

Ad (W3): The condition is valid for \( X \in \{ I, O \} \), since

\[ I\Phi = IW^*W = IW = I \quad \text{and} \quad O\Phi = OW^*W = OW = O. \]

Let \( X \) be an arbitrary element of \( \langle I \rangle \). Then

\[ X\Phi = (XW^*)W = IW = I. \]

Ad (W4): For all \( X \in H \setminus \langle I \rangle \) one has

\[ X\Phi = (XW^*)W = (A_1 \otimes \cdots \otimes A_n)W = \bigotimes_{j=1}^{n} A_j \]

and, by the properties of a free algebra,

\[ \bigotimes_{j=1}^{n} A_j = \bigotimes_{k=1}^{m} A'_k \Rightarrow n = m \quad \text{and} \quad A_j = A'_j \quad \text{for all} \ j \in \{1, \ldots, n\}. \]

Ad (W5):

\[ (X_1 \otimes X_2)\Phi = (Y_1 \otimes Y_2)\Phi \iff (X_1 \otimes X_2)W^*W = (Y_1 \otimes Y_2)W^*W \]

\[ \iff (X_1 \otimes X_2)W^* = (Y_1 \otimes Y_2)W^* \]
\( \Leftrightarrow X_1 W^* \otimes X_2 W^* = Y_1 W^* \otimes Y_2 W^* \)

\( \Leftrightarrow X_1 W^* = Y_1 W^* \land X_2 W^* = Y_2 W^* \)

\( (S^o \text{ is a free algebra}) \)

\( \Leftrightarrow X_1 W^* W = Y_1 W^* W \land X_2 W^* W = Y_2 W^* W \)

\( \Leftrightarrow X_1 \Phi = Y_1 \Phi \land X_2 \Phi = Y_2 \Phi \)

\( \Leftrightarrow X_1 \Phi \otimes X_2 \Phi = Y_1 \Phi \otimes Y_2 \Phi \) \((H^o \text{ is a free algebra})\). 

Observe that the function \( \Phi : H^o \rightarrow H^o \) maps \( O \) onto \( O \), all elements of \((I) \subseteq H \) onto \( I \), and all elements \( X \in H \setminus \langle I \rangle \) onto an \( \otimes \)-product of elements of \( J \) in canonical brackets consisting exactly of the factors of \( X \) which are different from \( I \) in the same order.

**Lemma 2.6.** Let be \( H^o \), \( S^o \), \( \Phi : H^o \rightarrow H^o \) as above. Then

\[ \forall X, Y, Z \in H^o \ ((X \otimes (Y \otimes Z)) \Phi = ((X \otimes Y) \otimes Z) \Phi), \]

\[ \forall n \in \mathbb{N} \setminus \{0\} \ \forall A_1, \ldots, A_n \in J \left( \left( \otimes_{j=1}^n A_j \right) \Phi = \otimes_{j=1}^n A_j \right), \]

\[ \forall X \in \langle I \rangle \ \forall Y \in H^o \ ((Y \otimes X) \Phi = (X \otimes Y) \Phi = Y \Phi). \]

**Proof.**

\[ (X \otimes (Y \otimes Z)) \Phi = (X \otimes (Y \otimes Z)) W^* W = (X W^* \otimes (Y W^* \otimes Z W^*)) W \]

\[ = ((X W^* \otimes Y W^*) \otimes Z W^*) W = ((X \otimes Y) \otimes Z) \Phi. \]

\[ \left( \otimes_{j=1}^n A_j \right) \Phi = \left( \otimes_{j=1}^n A_j \right) W^* W = \left( \otimes_{j=1}^n A_j \right) W = \otimes_{j=1}^n A_j, \]
(Y \otimes X)\Phi = (Y \otimes X)W^*W = (YW^* \otimes XW^*)W = (YW^* \otimes I)W = YW^*W = Y\Phi,

(X \otimes Y)\Phi = (X \otimes Y)W^*W = (XW^* \otimes YW^*)W = (I \otimes YW^*)W = YW^*W = Y\Phi.

Corollary 2.7. Let T be any J-sorted Hoehnke theory and let \Phi : H^0 \to H^0 be defined as above. Then there is exactly one central morphism c_X := c_{X,X\Phi} in CT for every X ∈ |T|. The same statement is true, if T is a J-sorted dts-theory and \Phi : H \to H.

Moreover, ∀X, Y ∈ |T| (X\Phi = Y\Phi ⇒ ∃c_{X,Y} ∈ CT[X, Y]).

Proof. The proof is organized by induction over the complexity of the objects X ∈ |T| = H^0.

Because of X\Phi = X for every X ∈ J∪\{I, O\} = H^0(0), 1_X ∈ CT[X, X\Phi], hence the start of induction is verified.

Let c_X exist in CT for any X ∈ H^0(n) and an arbitrary n ∈ N. Let be X ∈ H^0(n+1) \ H^0(n). Then there are X_1, X_2 ∈ H^0(n) such that X = X_1 \otimes X_2 and c_{X_1} ∈ CT[X_1, X_1\Phi], c_{X_2} ∈ CT[X_2, X_2\Phi], hence (c_{X_1} \otimes c_{X_2}) ∈ CT[X_1 \otimes X_2\Phi].

Since X_1\Phi = \bigotimes_{j=1}^{n} A_j and X_2\Phi = \bigotimes_{j=n+1}^{n+m} A_j for suitable A_j ∈ J, 1 ≤ j ≤ n + m, there is the canonical associativity isomorphism

a^{(n,m)}(X_1, X_2) : X_1\Phi \otimes X_2\Phi → (X_1 \otimes X_2)\Phi = X\Phi in CT,

therefore,

c_X := (c_{X_1} \otimes c_{X_2})a^{(n,m)}(X_1, X_2) ∈ CT[X, X\Phi].

So, the existence of a central morphism c_X for every X ∈ |T| = H^0 is proved.

Moreover, X\Phi = Y\Phi is sufficient for c_{X,Y} := c_X(c_Y)^{-1} ∈ CT[X, Y].

The uniqueness is again a consequence of the coherence principle.

The claim concerning the dts-case will be proved similarly.

The function \Phi defined as above induces in a natural manner a functor from a J-sorted theory T into itself with additional interesting properties. This properties concern the monoidal structur of T.
3. Structure preserving functors

Considering different symmetric monoidal categories $K^\bullet$ and $K'^\bullet$ one has to distinguish between the operations and the basic morphisms of $K^\bullet$ and those of $K'^\bullet$, respectively, for instance between $r^{(K)}_A$ and $r^{(K')}_X$. If there is not danger of confusion, the upper index will be omitted.

**Definition 3.1** ([14]). A functor $F : K^\bullet \to K'^\bullet$ between symmetric monoidal categories $K^\bullet$ and $K'^\bullet$ is called *monoidal*, iff there exists in $K'^\bullet$ a family of morphisms

$$\tilde{F} = (\tilde{F}(X,Y) : XF \otimes YF \to (X \otimes Y)F \mid X,Y \in |K|)$$

and a morphism

$$i_F : I' \to IF,$$

such that the following conditions are fulfilled:

(F) $\forall X,Y \in |K| \ (\tilde{F}(X,Y) \in Iso_{K'})$,

(FI) $i_F \in Iso_{K'}$,

(FA) $\forall X,Y,Z \in |K| \ \left( \left( 1^{(K')}_{XF} \otimes \tilde{F}(Y,Z) \right) \tilde{F}(X,Y \otimes Z) \left( a^{(K)}_{X,Y,Z}F \right) \right.$

$= a^{(K')}_{X,Y,Z} \left( \tilde{F}(X,Y) \otimes 1^{(K')}_{ZF} \right) \tilde{F}(X \otimes Y,Z) \left. \right),$\n
(FR) $\forall X \in |K| \ \left( \tilde{F}(X,I) \left( r^{(K')}_{XF}F \right) = \left( 1^{(K')}_{XF} \otimes i^{-1}_F \right) r^{(K')}_{XF} \right),$\n
(FS) $\forall X,Y \in |K| \ \left( \tilde{F}(X,Y) \left( s^{(K')}_{XY}F \right) = s^{(K')}_{XY} \tilde{F}(Y,X) \right),$

(FM) $\forall \varphi : X \to Y, \psi : U \to V \in K \ (\varphi F \otimes \psi F)\tilde{F}(Y,V) = $ $\tilde{F}(X,U)(\varphi \otimes \psi)F.$

A monoidal functor $F : K^\bullet \to K'^\bullet$ is called *strictly monoidal*, iff all morphisms of the family $\tilde{F}$ as well as the morphism $i_F$ are unit morphisms only.

**Corollary 3.2** ([14]). Let $F : K^\bullet \to K'^\bullet$ be a monoidal functor between symmetric monoidal categories with reference to $\tilde{F}, i_F$. Then

(FL) $\forall X \in |K| \ \left( \tilde{F}(I,X) \left( l^{(K')}_{X}F \right) = \left( i^{-1}_F \otimes 1^{(K')}_{XF} \right) l^{(K')}_{XF} \right).$
In applications to theories of algebraic structures, functors $F : K \to K'$ between $dhts$-categories, $dhth\nabla s$-categories, or $dts$-categories are of interest which preserve in addition to the functor properties the $dhts$, $dhth\nabla s$, and the $dts$-structure, respectively, with respect to a family $\tilde{F} = \langle \tilde{F}(X,Y) | X, Y \in |K| \rangle$ of isomorphisms $\tilde{F}(X,Y) : XF \otimes YF \to (X \otimes Y)F$ in $K'$ and an isomorphism $i_F$ between $I'$ and $IF$, where $I$ and $I'$ are the distinguished objects in $K$ and $K'$, respectively, ([5], [12], [14]). All symmetric monoidal categories with additional structures mentioned above are $ds$-categories. Of importance is the fact that a monoidal functor between at least $ds$-categories, which respects the diagonal morphisms except for isomorphisms, respects the canonical partial order relation and the distinguished terminal morphisms and the distinguished diagonal inversions, respectively, except for isomorphisms.

**Definition 3.3 ([14]).** A monoidal functor $F : K \to K'$ between $ds$-categories $K$ and $K'$ is called $d$-monoidal, if in addition the condition

$$(FD) \quad \forall A \in |K| \left( d_{AF}^{(K)}F = d_{AF'}^{(K')}\tilde{F}(A,A) \right)$$

holds with reference to the corresponding isomorphisms $\tilde{F}$ and $i_F$. A strictly monoidal functor $F$ fulfilling the condition (FD) is called strictly $d$-monoidal.

Obviously, the identical functor $1_K$ of $K^\bullet$ forms a strictly monoidal functor with respect to

$$\widetilde{1}_K = (\widetilde{1}_K(X,Y) = 1_{XF \otimes YF} | X, Y \in |K|), i_{1_K} = 1_I$$

and the constant functor $E : K^\bullet \to K'^\bullet (X \mapsto I', \varphi \mapsto 1_{I'})$ with reference to

$$\tilde{E} = (\tilde{E}(X,Y) = 1_{I'} | X, Y \in |K|), i_{\tilde{E}} = 1_{I'}$$

too, where $K^\bullet$ and $K'^\bullet$ are arbitrary symmetric monoidal categories.

Both functors are even $d$-monoidal functors, if $\underline{K} = (K^\bullet;d)$ and $\underline{K}' = (K'^\bullet;d')$ are $ds$-categories.

Moreover: Each $d$-monoidal functor $F : \underline{K} \to \underline{K}'$ between $dhts$-categories possesses the following properties with respect to the corresponding $F$, $i_F$ ([11], [14]):
(FT') \quad \ell_{tF}^{(K')} = i_F^{-1},

(Fmon) \quad \forall \varphi, \psi \in K \quad (\varphi \leq \psi \Rightarrow \varphi F \leq \psi F),

(FT) \quad \forall X \in |K| \quad \left( \ell_X^{(K)} F t_{tF}^{(K')} = t_{XF}^{(K')} \right),

(FP) \quad \forall X, Y \in |K| \quad \left( p_{\varphi, \psi}^{(K)} X Y F = (\tilde{F}(X, Y))^{-1} p_{\varphi, \psi}^{(K')} X Y F : \quad j = 1, 2 \right),

(FE) \quad \forall X \in |K| \quad \left( e \leq 1_X^{(K)} \Rightarrow e F \leq 1_X^{(K')} \right),

(FE\alpha) \quad \forall X, Y \in |K| \quad \forall \varphi \in K[X, Y] \quad \left( (\alpha^{(K)}(\varphi)) F = \alpha^{(K')}(\varphi F) \right).

Let $K, K'$ be $dhth\n-s$-categories and let $F : K \to K'$ be a $d$-monoidal functor. Then, in addition to the the properties above, the following holds ([14]):

(Finf) \quad \forall X, Y \in |K| \quad \forall \varphi, \psi \in K[X, Y] \quad \left( \left( d_X^{(K)} (\varphi \otimes \psi) \nabla_Y^{(K)} \right) F = d_X^{(K')} (\varphi F \otimes \psi F) \nabla_Y^{(K')} \right),

(Finf) \quad \forall X, Y \in |K| \quad \forall \varphi \in K[X, Y] \quad \left( (\varphi \otimes \varphi) \nabla_Y^{(K)} = \nabla_X^{(K)} \varphi \right)
\Rightarrow (\varphi F \otimes \varphi F) \nabla_Y^{(K')} = \nabla_{XF}^{(K')} (\varphi F) \right),

(F\n) \quad \forall X \in |K| \quad \left( \nabla_X^{(K')} = \tilde{F}(X, X) \nabla_X^{(K)} \right),

(F\n_1) \quad \forall X, Y, U \in |K| \quad \forall \varphi \in K[X, U] \quad \forall \psi \in K[Y, U] \quad \left( ((\varphi \otimes \psi) \nabla_U^{(K)}) F = \tilde{F}(X, Y) \left( (\varphi \otimes \psi) F \right) \nabla_U^{(K')} \right),

(F\n_2) \quad \forall X, Y \in |K| \quad \forall \varphi, \psi \in K[X, Y] \quad \left( (\varphi \otimes \psi) \nabla_Y^{(K)} = \nabla_X^{(K)} \varphi \right)
\Rightarrow (\varphi F \otimes \psi F) \nabla_Y^{(K')} = \nabla_{XF}^{(K')} (\varphi F) \right).

Obviously, property (Finf) is a special case of (F\n_2) and it expresses once more the monotony of the functor $F$, namely $\varphi \leq \psi \Rightarrow \varphi F \leq \psi F$.

The so-called zero functor $Z : K \to K'$ is defined by $X Z = O^{(K)}$ for all objects $X \in |K|$ and $\varphi Z = 1_{O^{(K')}}$ for all morphisms $\varphi \in K$. Trivially, this functor is a $d$-monoidal one.
Proposition 3.4 ([14]). Let $F : K \to K'$ be a $d$-monoidal functor between Hoehnke categories such that $F \neq Z$. Then one obtains:

\[
\forall X \in |K| \quad (\tilde{F}(X, O) = \tilde{F}(O, X) = 1^{(K')}_{O(K')}),
\]

\[
\forall X, Y \in |K| \quad (o^{(K)}_{X,Y}F = o^{(K')}_{X,F,YF}),
\]

\[
o^{(K)}F = t^{(K')}_{IF} o^{(K')} \quad (\Leftrightarrow o^{(K')} = i_F o^{(K)}) .
\]

By the structure of any Hoehnke categories $K$ and $K'$, each functor $F : K \to K'$ determines with respect to every pair of objects $X, Y \in |K|$ the morphism

\[
F^*(X, Y) := d^{(K')}_{(X \otimes Y)F} \left( p^{(K)}_{X,Y} F \otimes p^{(K)}_{X,Y} F \right) \in K'[ (X \otimes Y)F, XF \otimes YF]
\]

in the category $K'$.

Proposition 3.5 ([5]). In the case that $F : K \to K'$ is a $d$-monoidal functor with reference to $\tilde{F}$ and $i_F$, the morphisms $\tilde{F}(X, Y)$ are uniquely determined by

\[
(\tilde{F}(X, Y))^{-1} = d^{(K')}_{(X \otimes Y)F} \left( p^{(K)}_{X,Y} F \otimes p^{(K)}_{X,Y} F \right) = F^*(X, Y).
\]

Moreover:

Theorem 3.6 ([5], [14]). Assume that $F : K \to K'$ is any functor from a dhts-category $K$ into a dhts-category $K'$ satisfying the following conditions:

\[
(F^*) \quad \forall X, Y \in |K| (F^*(X, Y) \in Iso_{K'}),
\]

\[
(FI^*) \quad t^{(K')}_{IF} \in Iso_{K'},
\]

\[
(FM^*) \quad \forall \varphi, \psi \in K \quad ((\varphi \otimes \psi) F F^*(X', Y') = F^*(X, Y)(\varphi F \otimes \psi F)).
\]

Then $F : K \to K'$ is $d$-monoidal with reference to the morphisms

\[
\tilde{F}(X, Y) := (F^*(X, Y))^{-1}, \quad i_F := t^{(K')}_{IF}.
\]

The statements in 3.5 and 3.6 allow us to speak about $d$-monoidal functors between Hoehnke categories as such.

Hoehnke has shown in [5] that the composition of dht-symmetric functors $F : K \to K'$ and $G : K' \to K''$ between Hoehnke categories $K$, $K'$, $K''$.
respectively, yields a \(dht\)-symmetric functor \(FG : K \to K''\). The same is true for \(d\)-monoidal functors between Hoehnke categories. More precisely:

**Proposition 3.7.** Let \(F : K \to K'\) and \(G : K' \to K''\) be \(d\)-monoidal functors between Hoehnke categories \(K, K', K''\). Then the functor \(FG : K \to K''\) is a \(d\)-monoidal functor too.

**Proof.** Ad (F*): Since every functor maps isomorphisms to isomorphism and

\[
(FG)^*(X, Y) = d^{(K'')}_{(X \otimes Y)(FG)}(p^{(K)_{X,Y}}_1 (FG) \otimes p^{(K)_{X,Y}}_2 (FG))
\]

\[
= d^{(K'')}_{(X \otimes Y)FG} \left( p^{(K)_{X,Y}}_1 F \otimes (p^{(K)_{X,Y}}_2 F)G \right)
\]

\[
= (d^{(K')}_{(X \otimes Y)F}) GG^*(X \otimes Y)F, (X \otimes Y)F \left( p^{(K)_{X,Y}}_1 F \otimes (p^{(K)_{X,Y}}_2 F)G \right)
\]

\[
= (d^{(K')}_{(X \otimes Y)F}) G \left( p^{(K)_{X,Y}}_1 F \otimes (p^{(K)_{X,Y}}_2 F)G \right) GG^*(XF, YF)
\]

\[
= (d^{(K')}_{(X \otimes Y)F}) \left( p^{(K)_{X,Y}}_1 F \otimes p^{(K)_{X,Y}}_2 F \right) GG^*(XF, YF)
\]

\[
= (F^*(X, Y)) GG^*(XF, YF)
\]

one obtains \((FG)^*(X, Y) \in Iso_{K''}\).

Ad (FI*): \(t^{(K'')}_{(FG)} = t^{(K'')}_{(FG)} = \left( t^{(K')}_{FG} \right) G t^{(K'')}_{(FG)} G) \in Iso_{K''}\)

since \(t^{(K'')}_{FG} \in Iso_{K''}\) and \(t^{(K')}_{FG} \in Iso_{K'}\).

Ad (FM*): \((\varphi \otimes \psi)(FG)(FG)^*(U, V) = ((\varphi \otimes \psi)F)G(F^*(U, V))GG^*(UF, VF)\)

\[
= ((\varphi \otimes \psi)FF^*(U, V))GG^*(UF, VF)
\]

\[
= (F^*(X, Y)(\varphi F \otimes \psi F))GG^*(UF, VF)
\]

\[
= (F^*(X, Y))(\varphi F \otimes \psi F)GG^*(UF, VF)
\]

\[
= (F^*(X, Y))GG^*(XF, YF)((\varphi F)G \otimes (\psi F)G)
\]

\[
= (FG)^*(X, Y)(\varphi (FG) \otimes (\psi FG)).
\]

\(\blacksquare\)
Lemma 3.8. Let $F : K \to K'$ be a functor from a Hoehnke category $K$ into a Hoehnke category $K'$ such that the conditions

\[(sFD)\] $\forall X \in |K| \left( d^{(K)}_X F = d^{(K')}_{XF} \right)$, \\
\[(sFT)\] $\forall X \in |K| \left( t^{(K)}_X F = t^{(K')}_{XF} \right)$, and \\
\[(sFM)\] $\forall \varphi, \psi \in K \ ((\varphi \otimes \psi) F = (\varphi F \otimes \psi F))$

are fulfilled.

Then $F$ has the properties

\[(sF^\ast)\] $\forall X, Y \in |K| \left( F^\ast \langle X, Y \rangle \in Un_{K'} \right)$ and \\
\[(sFI^\ast)\] $t^{(K')}_{1F} \in Un_{K'}$,

i.e. $F : K \to K'$ is a strictly $d$-monoidal functor.

Proof. Assuming (sFT) one has $1^{(K')}_{1F} = 1^{(K)}_{1F} = t^{(K)}_{1F} = t^{(K')}_{1F}$, hence $IF = I^{(K')}$ and (sFI^\ast) is fulfilled.

Moreover,

\[
\forall X, Y \in |K| \left( K'[X \otimes Y)F, (X \otimes Y)F] \ni 1^{(K)}_{X \otimes Y} F = \left( 1^{(K)}_X \otimes 1^{(K)}_Y \right) F \\
= 1^{(K)}_X F \otimes 1^{(K)}_Y F = 1^{(K')}_{XF \otimes YF} \in K'[X \otimes YF] \right),
\]

hence

\[
\forall X, Y \in |K| \ ((X \otimes Y) F = XF \otimes YF).
\]

Now let $X$ and $Y$ be any objects of $|K|$. Then

\[
F^\ast \langle X, Y \rangle = d^{(K')}_{(X \otimes Y)F} \left( p^{(K)}_{X,Y} 1_{1^{(K)}_1 F} \otimes p^{(K)}_{X,Y} 1_{1^{(K)}_2 F} \right) \quad \text{(by definition)}
\]

\[
= d^{(K)}_{X \otimes Y F} \left( p^{(K)}_{X,Y} 1_{1^{(K)}_1 F} \otimes p^{(K)}_{X,Y} 1_{1^{(K)}_2 F} \right) \quad \text{((sFD))}
\]

\[
= \left( d^{(K)}_{X \otimes Y} \left( p^{(K)}_{X,Y} 1_{1^{(K)}_1 F} \otimes p^{(K)}_{X,Y} 1_{1^{(K)}_2 F} \right) \right) F \quad \text{((sFM))}
\]

\[
= \left( 1^{(K)}_{X \otimes Y} \right) F = 1^{(K')}_{XF \otimes YF} \in Un_{K'}.
\]
Proposition 3.9. If functors $F : K \rightarrow K'$ and $G : K' \rightarrow K''$ between Hoehnke categories $K$, $K'$, $K''$ fulfil the conditions (sFD), (sFT), and (sFM), then the functor $FG : K \rightarrow K''$ satisfies the same conditions.

Corollary 3.10. If any functor $F : K \rightarrow K'$ as above fulfils (sFT) and (sFM), then $F$ is a $d$-monoidal functor satisfying (sFI$^*$$)$.

Proof. It remains to prove the validity of $(F^*)$.

$$F^*(X,Y) = d^{(K')}_{(X\otimes Y)} F \left( p^{(K)X,Y}_{1} F \otimes p^{(K)X,Y}_{2} F \right)$$

$$= d^{(K')}_{XF \otimes YF} \left( 1^{(K)}_X \otimes 1^{(K)}_Y \right) F \otimes \left( 1^{(K)}_X \otimes 1^{(K)}_Y \right) F$$

$$= d^{(K')}_{XF \otimes YF} \left( 1^{(K)}_X F \otimes t^{(K)}_Y F \right) \otimes \left( t^{(K)}_X F \otimes 1^{(K)}_Y F \right) \left( t^{(K)}_X F \otimes t^{(K)}_X F \right)$$

$$= \left( d^{(K')}_{XF} \left( 1^{(K)}_X \otimes t^{(K)}_X \right) F \otimes d^{(K')}_{YF} \left( t^{(K)}_Y \otimes 1^{(K)}_Y \right) F \right) b^{(K')}_{XF,F,YF}$$

$$= \left( \left( r^{(K')}_{XF} \right)^{-1} \otimes \left( l^{(K')}_{YF} \right)^{-1} \right) \left( \langle X \otimes IF, IF \otimes YF \rangle \right) \left( t^{(K)}_X F \otimes t^{(K)}_X F \right)$$

$$= \left( r^{(K')}_{XF} \right)^{-1} t^{(K)}_X F \otimes \left( l^{(K')}_{YF} \right)^{-1} l^{(K)}_X F \in Iso_{K'}.$$

4. Functors between theories, theory morphisms

The following considerations are confined to $dhts$-theories, but it is easily to see that all results are transferable to $dhth\n\n$-theories and $dts$-theories, respectively.

Lemma 4.1. Let $F$ be a $d$-monoidal functor from a Hoehnke theory $T$ into a Hoehnke theory $T'$ such that all morphisms $F(A,B)$ and $i_F$ are central morphisms only. Then $F$ maps every central morphism $c \in C_T$ to a central morphism $c_F \in C_{T'}$.

Proof. Every functor maps unit morphisms to unit morphism. Any $d$-monoidal functor fulfils the conditions (FA), (FR), and (FL) and since $i_F$
and every $\tilde{F}(A, B)$ are central morphisms, all images $a_{A, B, C}F$, $r_A F$, $l_A F$, $(a_{A, B, C})^{-1}F$, $(r_A)^{-1}F$, $(l_A)^{-1}F$ are central morphisms in $T'$.

Therefore, the images of all morphisms of $C_T^{(0)}$ are central morphisms in $T'$.

Assuming that all morphisms of $C_T^{(n)}$ for any $n \in \mathbb{N}$ are mapped by $F$ to central morphisms in $T'$ one has

$$\forall \varphi \in C_T^{(n+1)} \setminus C_T^{(n)} \exists \varphi_1, \varphi_2 \in C_T^{(n)} \quad (\varphi F = (\varphi_1 \varphi_2) F =$$

$$(\varphi_1 F)(\varphi_2 F) \in C_{T'} \wedge \varphi F = (\varphi_1 \otimes \varphi_2) F =$$

$$(\tilde{F}(\text{dom } \varphi_1, \text{dom } \varphi_2))^{-1}(\varphi_1 F \otimes \varphi_2 F) \tilde{F}(\text{cod } \varphi_1, \text{cod } \varphi_2) \in C_{T'}),$$

hence $\forall \varphi \in C_T \quad (\varphi F \in C_{T'})$, 

Observe that especially strict $d$-monoidal functors map central morphisms to central morphisms.

**Theorem 4.2.** Let $T$ be a $J$-sorted Hoehnke theory. Then the function $\Phi$ as defined in 2.5 induces a $d$-monoidal functor $\Phi : T \to T$ relative to $\tilde{\Phi}$ and $i_\Phi$ such that

$$\forall X, Y \in |T| \quad (\tilde{\Phi}(X, Y) := (c_X^{-1} \otimes c_Y^{-1}) c_{X \otimes Y}) \quad \text{and} \quad i_\Phi := 1_I.$$ 

**Proof.** The object mapping is given by the function $\Phi : |T| \to |T|$, namely

$$X \Phi = \begin{cases} 
X, & \text{for all } X \in J \cup \{I, O\}, 
I, & \text{for all } X \in \langle I \rangle, 
\otimes A_j & \text{for all } X \in |T| \setminus \langle I \rangle^o,
\end{cases}
$$

where $A_1, \ldots, A_n \in J$ are exactly the factors appearing in $X$ in this sequence independently of brackets.

Using the uniquely determined central morphisms $c_X \in C_T[X, X \Phi]$ define a morphism mapping by

$$T[X, Y] \ni \varphi \mapsto \varphi \Phi := c_X^{-1} \varphi c_Y \in T[X \Phi, Y \Phi].$$

Then the functor conditions are fulfilled, since
∀φ ∈ T ((dom φ)Φ = dom(φ Φ), (cod φ)Φ = cod(φ Φ)) by definition,

∀X ∈ |T| (1_X Φ = c^{-1}_X 1_X c_X = 1_X Φ),

∀X, Y, P ∈ |T| ∀ψ ∈ T[X, Y] ∀ψ ∈ T[Y, P]

((ψ ψ)Φ = c^{-1}_X ψ c_P = c^{-1}_X ψ c_Y c^{-1}_Y ψ c_P = (ψ Φ)(ψ Φ)).

By Theorem 3.6, it is sufficient to prove the conditions (F*), (FI*), and

(FA*) for the functor Φ.

Ad (F*): Let X and Y be arbitrary objects of T. Then, by definition,

\[ \Phi^*(X, Y) = d_{(X \otimes Y) \Phi}(p_1^{X, Y} \Phi \otimes p_1^{X, Y} \Phi) = d_{(X \otimes Y) \Phi}(c^{-1}_{X, Y} p_1^{X, Y} \Phi \otimes p_1^{X, Y} \Phi c_{X, Y}^{-1}) \]

\[ = c_{X, Y}^{-1} d_{(X \otimes Y)}(p_1^{X, Y} \otimes p_1^{X, Y})(c_X \otimes c_Y) = c_{X, Y}^{-1} (c_X \otimes c_Y) \in CT \subseteq Iso_T. \]

Ad (FI*): Because of IΦ = I, t_1Φ = t_I = 1_I ∈ Iso_T.

Ad (FA*): For all objects X_1, X_2, Y_1, Y_2 and all morphisms \( \varphi_i \in T[X_i, Y_i], i \in \{1, 2\} \), the equation

\[ (\varphi_1 \otimes \varphi_2)\Phi^*(Y_1, Y_2) = c^{-1}_{X_1 \otimes X_2} (\varphi_1 \otimes \varphi_2) c_{Y_1 \otimes Y_2} c^{-1}_{Y_1 \otimes Y_2} (c_{Y_1} \otimes c_{Y_2}) \]

\[ = c^{-1}_{X_1 \otimes X_2} (\varphi_1 c_{Y_1} \otimes \varphi_2 c_{Y_2}) \]

\[ = c^{-1}_{X_1 \otimes X_2} (c_{X_1} \otimes c_{X_2}) \left( c^{-1}_{X_1} \varphi_1 c_{Y_1} \otimes c^{-1}_{X_2} \varphi_2 c_{Y_2} \right) \]

\[ = \Phi^*(X_1, X_2) (\varphi_1 \Phi \otimes \varphi_2 \Phi) \]

is valid. Therefore, (Φ, \tilde{Φ}, i_Φ) with \( \tilde{Φ} := (Φ^*)^{-1} \) and \( i_Φ := 1_I \) is a d-monoidal functor from T into T.

The functor Φ shall be called the canonical functor of T.

**Corollary 4.3.** Let T be a J-sorted dhts-theory. Then the canonical functor of T possesses the following properties:
Adjointness between theories and strict theories

(1) \( \forall X \in |T| \ ((X\Phi)\Phi = X\Phi) \),

(2) \( \forall X \in |T| \ ((t_X)\Phi = t_X\Phi) \),

(3) \( \forall X \in |T| \ ((r_X)\Phi = 1_X\Phi = (l_X)\Phi) \),

(4) \( \forall X \in |T| \ (d_X\Phi\Phi^*(X, X) = d_X\Phi) \),

(5) \( \forall X \in |T| \ (\nabla_X\Phi = \Phi^*(X, X)\nabla_X\Phi) \),

(6) \( \forall X \in |T| \ (\Phi^*(X, I) = (r_X\Phi)^{-1}, \Phi^*(I, X) = (l_X\Phi)^{-1}) \),

(7) \( \forall X \in |T| \ ((c_X)\Phi = 1_X\Phi = (1_X)\Phi = c_X\Phi) \),

(8) \( \forall \varphi \in T (\text{dom}\varphi = \bigotimes_{j=1}^n A_j \land \text{cod}\varphi = \bigotimes_{k=1}^m B_k \land A_j, B_k \in J \Rightarrow \varphi\Phi = \varphi) \),

(9) \( \forall \varphi \in T ((\varphi\Phi)\Phi = \varphi\Phi) \).

**Proof.** Ad (1): \((X\Phi)\Phi = \left( \bigotimes_{j=1}^n A_j \right) \Phi = \bigotimes_{j=1}^n A_j = X\Phi\).

Ad (2): \((t_X)\Phi = c_X^{-1}t_Xc_I = t_X\Phi\) since \(c_X \in Iso_T \land c_I = 1_I\).

Ad (3): The assertion is a special case of (7).

Ad (4): \(d_X\Phi = c_X^{-1}d_Xc_{X\otimes X} = d_X\Phi \left( c_X^{-1} \otimes c_X^{-1} \right) c_{X\otimes X} = d_X\Phi(\Phi^*(X, X))^{-1} = (\Phi^*(X, X))^{-1} \)

\[ \Rightarrow d_X\Phi\Phi^*(X, X) = d_X\Phi. \]

Ad (5): \(\nabla_X\Phi = (c_{X\otimes X})^{-1}\nabla_Xc_X = (c_{X\otimes X})^{-1}(c_X \otimes c_X)\nabla_X\Phi = \Phi^*(X, X)\nabla_X\Phi\).

Ad (6): \(\Phi^*(X, I) \in C_T[\Phi^*(X, X) \otimes I] \) and \(r_X\Phi \in C_T[X\Phi \otimes I, X\Phi]\),

hence \(\Phi^*(X, I) = (r_X\Phi)^{-1}\) by the coherence principle.
Ad (7): $c_X \in C_T[X, X\Phi] \Rightarrow (c_X)\Phi \in C_T[X\Phi, (X\Phi)\Phi] = C_T[X\Phi, X\Phi] \ni 1_{X\Phi}$

\[ \Rightarrow (c_X)\Phi = 1_{X\Phi} = 1_{X\Phi}. \]

$c_{X\Phi} \in C_T[X\Phi, X\Phi] = C_T[X\Phi, X\Phi]$ shows $c_{X\Phi} = 1_{X\Phi}$.

Ad (8): $\varphi\Phi = c_{X\Phi}^{-1} c_\varphi c_{Y\Phi} = \varphi$, where $X = \bigotimes_{j=1}^n A_j = X\Phi \land Y = \bigotimes_{k=1}^m B_k = Y\Phi$.

Ad (9): $(\varphi\Phi)\Phi = (c_{X\Phi}^{-1} c_\varphi c_{Y\Phi}) \Phi = (c_{X\Phi}^{-1}) \Phi(\varphi)\Phi(c_{Y\Phi})\Phi = \varphi\Phi.$

\[ \square \]

**Definition 4.4.** Let $\mathbf{T}$ be a $J$-sorted Hoehnke theory and let $\Phi$ be the canonical $d$-monoidal functor of $\mathbf{T}$. Then define a binary relation $\kappa$ for objects and morphisms of $\mathbf{T}$ as follows:

\[(X, Y) \in \kappa :\Leftrightarrow X\Phi = Y\Phi,\]

\[(\varphi_1, \varphi_2) \in \kappa :\Leftrightarrow \varphi_1\Phi = \varphi_2\Phi.\]

**Theorem 4.5.** The relation $\kappa$ defined by the canonical $d$-monoidal functor $\Phi$ of a $J$-sorted Hoehnke theory $\mathbf{T}$ as above is a “generalized” congruence on $\mathbf{T}$.

**Proof.** Considering small categories as many-sorted total algebras, a congruence $\rho$ is defined as a family of equivalence relations on the isolated morphism sets, i.e. $(\varphi, \psi) \in \rho \Rightarrow \text{dom}\varphi = \text{dom}\psi \land \text{cod}\varphi = \text{cod}\psi$.

That is not true for the relation $\kappa$, since only $\forall \varphi, \psi \in \mathbf{T}((\varphi, \psi) \in \kappa \Rightarrow (\text{dom}\varphi)\Phi = (\text{dom}\psi)\Phi \land (\text{cod}\varphi)\Phi = (\text{cod}\psi)\Phi)$, because of

\[ (\varphi, \psi) \in \kappa \Rightarrow \varphi\Phi = \psi\Phi \Rightarrow c_{\text{dom}\varphi}^{-1} \varphi c_{\text{cod}\varphi} = c_{\text{dom}\psi}^{-1} \psi c_{\text{cod}\psi} \]

\[ \Rightarrow (\text{dom}\varphi)\Phi = (\text{dom}\psi)\Phi \land (\text{cod}\varphi)\Phi = (\text{cod}\psi)\Phi. \]

Moreover, the relation $\kappa$ is not compatible with the morphism composition in the strong sense.
By definition, the relation $\kappa$ is reflexive, symmetric, and transitive for objects and morphisms, respectively.

The relation is compatible with $\otimes$-operation of morphisms and objects, respectively, because of the following argumentation.

Using (FM*) and Corollary 4.3 (5) one has for morphisms:

$$(\varphi_1, \varphi_2), (\psi_1, \psi_2) \in \kappa \Rightarrow (\varphi_1 \otimes \psi_1) \Phi = \Phi^* (X_1, P_1)(\varphi_1 \Phi \otimes \psi_1 \Phi)(\Phi^* (Y_1, Q_1))^{-1}$$

$$= \Phi^* (X_1, P_1)(\varphi_2 \Phi \otimes \psi_2 \Phi)(\Phi^* (Y_1, Q_1))^{-1}$$

$$= c_{X_1 \otimes P_1}^{-1} (c_{X_1} c_{X_2}^{-1} \otimes c_{P_1} c_{P_2}^{-1}) (\varphi_2 \otimes \psi_2) (c_{Y_2} c_{Y_1}^{-1} \otimes c_{Q_2} c_{Q_1}^{-1}) c_{Y_1 \otimes Q_1}$$

$$\Rightarrow (\varphi_1 \otimes \psi_1) \Phi = (\varphi_2 \otimes \psi_2) \Phi$$

$$= \left( c_{X_1 \otimes P_1}^{-1} (c_{X_1} c_{X_2}^{-1} \otimes c_{P_1} c_{P_2}^{-1}) (\varphi_2 \otimes \psi_2) (c_{Y_2} c_{Y_1}^{-1} \otimes c_{Q_2} c_{Q_1}^{-1}) c_{Y_1 \otimes Q_1} \right) \Phi$$

$$= (\varphi_2 \otimes \psi_2) \Phi$$

$$\Rightarrow (\varphi_1 \otimes \psi_1, \varphi_2 \otimes \psi_2) \in \kappa.$$ 

Concerning the object relation one obtains

$$(X_1, X_2) \in \kappa \land (Y_1, Y_2) \in \kappa \Rightarrow X_1 \Phi = X_2 \Phi \land Y_1 \Phi = Y_2 \Phi$$

$$\Rightarrow 1_{X_1} \Phi = 1_{X_2} \Phi \land 1_{Y_1} \Phi = 1_{Y_2} \Phi$$

$$\Rightarrow (1_{X_1}, 1_{X_2}) \in \kappa \land (1_{Y_1}, 1_{Y_2}) \in \kappa$$

$$\Rightarrow (1_{X_1} \otimes 1_{Y_1}, 1_{X_2} \otimes 1_{Y_2}) \in \kappa$$

$$\Rightarrow (1_{X_1} \otimes 1_{Y_1}, 1_{X_2} \otimes 1_{Y_2}) \in \kappa$$

$$\Rightarrow 1_{X_1} \otimes 1_{Y_1} \Phi = 1_{X_2} \otimes 1_{Y_2} \Phi$$

$$\Rightarrow 1_{(X_1 \otimes 1_{Y_1})} \Phi = 1_{(X_2 \otimes 1_{Y_2})} \Phi$$

$$\Rightarrow (X_1 \otimes Y_1) \Phi = (X_2 \otimes Y_2) \Phi$$

$$\Rightarrow (X_1 \otimes Y_1, X_2 \otimes Y_2) \in \kappa.$$
The relation $\kappa$ is, as already mentioned, reflexive, therefore it preserves all morphisms which are determined by constant operation symbols.

For the morphism composition:

Let $\varphi_i \in T[X_i, Y_i], \psi_i \in T[P_i, Q_i]$ for $i \in \{1, 2\}$ be arbitrary morphisms of $T$. Then, for $Y_1 = P_1$, i.e. $\varphi_1$ is composable with $\psi_1$,

$$(\varphi_1, \varphi_2), (\psi_1, \psi_2) \in \kappa \Rightarrow (\varphi_1 \psi_1)\Phi = (\varphi_1 \Phi)(\psi_1 \Phi)$$

$$= (\varphi_2 \Phi)(\psi_2 \Phi) = c_{X_1}^{-1} \varphi_2 c_{Y_2}^{-1} \psi_2 c_{Q_2},$$

therefore, by Corollary 4.3 (7) and (5),

$$(\varphi_1 \psi_1)\Phi = ((\varphi_1 \psi_1)\Phi)\Phi = \left(c_{X_2}^{-1} \varphi_2 c_{Y_2}^{-1} \psi_2 c_{Q_2}\right)\Phi = (\varphi_2 c_{Y_2} c_{P_2} \psi_2)\Phi,$$

hence $(\varphi_1 \psi_1, \varphi_2 c_{Y_2} c_{P_2} \psi_2) \in \kappa$.

Observe that especially $\varphi_2$ and $\psi_2$ have not to be composable in general, but there is a central morphism $c$ such that there exists the compositum $\varphi_2 c \psi_2$.

**Remark.** It is easy to verify that the generating central morphisms $1, a, a^{-1}, r, r^{-1}, l, l^{-1}$ of any $J$-sorted theory $T$ fulfil even the following conditions:

$$\forall X, Y, P \in |T| \ ((1_{X \otimes (Y \otimes P)}, 1_{(X \otimes Y) \otimes P}) \in \kappa),$$

$$\forall X, Y, P \in |T| \ ((a_{X,Y,P}, 1_{X \otimes (Y \otimes P)} \in \kappa \land ((a_{X,Y,P})^{-1}, 1_{(X \otimes Y) \otimes P}) \in \kappa),$$

$$\forall X \in |T| \ ((1_{X \otimes I}, 1_X), (1_{I \otimes X}, 1_X) \in \kappa),$$

$$\forall X \in |T| \ ((r_X, 1_{X \otimes I}), ((r_X)^{-1}, 1_X), ((l_X), 1_{I \otimes X}), ((l_X)^{-1}, 1_X) \in \kappa).$$

**Theorem 4.6.** To every $J$-sorted Hoehnke theory

$T \in |Th_{dht}(J)|$
there exists in a natural manner a $J$-sorted strict Hoehnke theory

\[ T_s \in |sTh_{\text{sh}}^J(J)|. \]

**Proof.** The canonical $d$-monoidal functor \( \Phi : T \to T \) related to any $J$-sorted Hoehnke theory $T$ induces the “generalized” congruence $\kappa$.

Construct a new category $T_s$ by using the knowledge about $H^\circ, S^\circ$ and the functions $W$ and $W^*$.

$|T_s| := S^\circ$ (:= $S$),

$T_s := \{ [\varphi]_\kappa \mid \varphi \in T \}$, where $[\varphi]_\kappa = \{ \varphi' \in T \mid \varphi \Phi = \varphi' \Phi \}$,

$\text{dom}^{(T_s)}[\varphi]_\kappa := \left( \text{dom}^{(T)} \varphi \right) W^*$, $\text{cod}^{(T_s)}[\varphi]_\kappa := \left( \text{cod}^{(T)} \varphi \right) W^*$,

$1_A^{(T_s)} := \left[ 1_{AW} \right]_\kappa$,

$[\varphi]_\kappa \cdot^{(T_s)} [\psi]_\kappa := [\varphi \otimes^{(T)} \psi]_\kappa$, where $Y \Phi = (\text{cod} \varphi) \Phi = (\text{dom} \psi) \Phi = P \Phi$

$\leftrightarrow Y W^* = (\text{cod} \varphi) W^* = (\text{dom} \psi) W^*) = P W^*$,

$A \otimes^{(T_s)} B = (AW \otimes^{(T)} BW) W^*$ (by (W4)),

$[\varphi]_\kappa \otimes^{(T_s)} [\psi]_\kappa := [\varphi \otimes^{(T)} \psi]_\kappa$,

$d_{A,B,C}^{(T_s)} := \left[ d_{AW,BW,CW}^{(T)} \right]_\kappa = \left[ 1_{AW \otimes^{(T)} BW \otimes^{(T)} CW} \right]_\kappa$,

$r_A^{(T_s)} := \left[ r_{AW}^{(T)} \right]_\kappa = \left[ 1_{AW}^{(T)} \right]_\kappa = \left[ l_{AW}^{(T)} \right]_\kappa = \left[ l_A^{(T_s)} \right]_\kappa$,

$s_{A,B}^{(T_s)} := \left[ s_{AW,BW}^{(T)} \right]_\kappa$, $d_A^{(T_s)} := \left[ d_{AW}^{(T)} \right]_\kappa$, $t_A^{(T_s)} := \left[ t_{AW}^{(T)} \right]_\kappa$, $\nabla_A^{(T_s)} := \left[ \nabla_{AW}^{(T)} \right]_\kappa$,

$\circ^{(T_s)} := \left[ \circ^{(T)} \right]_\kappa$.

Obviously, $(S^\circ; \otimes, I, O)$ is an algebra of type $(2, 0, 0)$ with an associative binary operation, a unit element $I$, and a zero element $O$.

Moreover, $(|T_s|, T_s, \cdot, \text{dom}, \text{cod}, 1)$ is a small category, since $|T_s|$ is a set and
\[ \varphi \in T_s[A, B] \Rightarrow \varphi \in T[X, Y] \land A = XW^*, B = YW^* \Rightarrow 1_A[\varphi]_x \]

\[ = [1X]_x[\varphi]_x = [1XcX]_x[\varphi]_x = [\varphi]_x = [\varphi]_x[1Y]_x = [\varphi]_x[1B]_x \]

\[ [\varphi]_x \in T_s[A, B], [\psi]_x \in T_s[B, C], [\chi]_x \in T_s[C, D] \]

\[ \Rightarrow [\varphi]_x([\psi]_x[\chi]_x) = [\varphi]_x[\psi c_{P, Q} \chi]_x = [\varphi c_X Y \psi c_{P, Q} \chi]_x \]

\[ = [\varphi c_X Y \psi]_x[\chi]_x = ([\varphi]_x[\psi]_x)[\chi]_x. \]

\((T_s; \otimes, I, 1, 1, 1, s)\) is a symmetric strictly monoidal category since the defining conditions are fulfilled. Observe that to every morphism \( \rho \in T_s[A, B] \) there is a morphism \( \varphi \in T[X, Y] \) such that \( A = XW^*, B = YW^*, \rho = [\varphi]_x. \)

Ad (F1): \( \forall \rho, \rho' \in T_s \) (dom \((\rho \otimes \rho') = \) dom \(([\varphi]_x \otimes [\varphi']_x)\)

\[ = \text{dom} [\varphi \otimes \varphi']_x = (\text{dom} (\varphi \otimes \varphi'))W^* \]

\[ = ((\text{dom} \varphi) \otimes (\text{dom} \varphi'))W^* = (\text{dom} \varphi)W^* \otimes (\text{dom} \varphi')W^* \]

\[ = (\text{dom} [\varphi]_x) \otimes (\text{dom} [\varphi']_x) = \text{dom} \rho \otimes \text{dom} \rho'. \]

Ad (F2): The assertion \( \forall \rho, \rho' \in T_s \) (cod \((\rho \otimes \rho') = \) cod \(\rho \otimes \text{cod} \rho') \) will be proved in the same manner.

Ad (F3): \( \forall A, B \in T_s \) (1\(A \otimes B\) = 1\((A \otimes B)W\)\) = 1\(A W \otimes B W\)\)\)

\[ = [1AW \otimes 1BW]_x = [1AW]_x \otimes [1BW]_x = 1A \otimes 1B, \]

since \( T \) is a symmetric monoidal category and for all \( A, B \in S^\circ \) one has \( (A \otimes B)W \Phi = (A \otimes B)WW^*W = (A \otimes B)W = (AWW^* \otimes BW)W = (AW \otimes BW) \Phi. \)

Ad (F4): \( \forall A, B, C, A', B', C' \in T_s \) \( \forall \rho \in T_s[A, B] \)

\[ \forall \sigma \in T_s[B, C] \forall \rho' \in T_s[A', B'] \forall \sigma' \in T_s[B', C'] \]
\((\rho \otimes \rho')(\sigma \otimes \sigma') = ([\varphi]_\mathcal{X} \otimes [\varphi']_\mathcal{X})([\psi]_\mathcal{X} \otimes [\psi']_\mathcal{X})\)

\[= [\varphi \otimes \varphi']_\mathcal{X} [\psi \otimes \psi']_\mathcal{X}\]

\[= [(\varphi \otimes \varphi')c_{Y \otimes Y', P \otimes P'}(\psi \otimes \psi')]_\mathcal{X}\]

\[= [(\varphi \otimes \varphi')(c_{Y, P} \otimes c_{Y', P'})(\psi \otimes \psi')]_\mathcal{X}\]

\[= [\varphi_{c_{Y, P}}(\varphi \otimes \varphi')c_{Y', P'}(\psi \otimes \psi')]_\mathcal{X}\]

\[= \rho_\sigma \otimes \rho'_\sigma').\]

Ad (M1), (M2), (M3): The conditions are trivially fulfilled since \(a\) and \(r\) consist of unit morphisms only.

Ad (M4): \(\forall A, B \in |\mathcal{T}_s| \quad (s_{A,B}^{(T_s)} s_{B,A}^{(T_s)}) = \left[ s_{AW,BW}^{(T)} \right]_\mathcal{X} \left[ s_{BW,AW}^{(T)} \right]_\mathcal{X}\]

\[= \left[ s_{AW,BW}^{(T)} s_{BW,AW}^{(T)} \right]_\mathcal{X} = \left[ 1_{AW \otimes BW}^{(T)} \right]_\mathcal{X} = \left[ 1_{AW \otimes BW}^{(T_s)} \right]_\mathcal{X} = \left[ 1_{A \otimes B}^{(T_s)} \right].\]

Ad (M5): \(\forall A \in |\mathcal{T}_s| \quad (s_{A,I}^{(T_s)} l_{A}^{(T_s)}) = \left[ l_{AW}^{(T)} \right]_\mathcal{X} = \left[ l_{AW}^{(T)} \right]_\mathcal{X}

\[= \left[ r_{AW}^{(T_s)} \right]_\mathcal{X} = l_{A}^{(T_s)} = \left[ l_{A}^{(T_s)} \right].\]

Ad (M6): \(\forall A, B, C, A', B', C' \in |\mathcal{T}_s| \quad \forall \rho \in |\mathcal{T}_s[A, A']|\)

\[\forall \sigma \in |\mathcal{T}_s[B, B']| \forall \tau \in |\mathcal{T}_s[C, C']|\]

\(\left( a_{A,B,C}^{(T_s)}((\rho \otimes \sigma) \otimes \tau) = \left[ a_{X,Y,F}^{(T)} (([\varphi]_\mathcal{X} \otimes [\psi]_\mathcal{X}) \otimes [\chi]_\mathcal{X})\right)\right]_\mathcal{X}\)
\[
\begin{align*}
&H.-J. \text{ Vogel} \\
&\kappa = \left[ a^{(T)}_{X,Y,\psi}(X \otimes Y) \otimes P \right]_\times \\
&\kappa = \left[ (\psi \otimes (\psi \otimes \chi))a^{(T)}_{X',Y',\psi} \right]_\times \\
&\kappa = \left[ (\psi \otimes (\psi \otimes \chi))c^{(X' \otimes (Y' \otimes P))}_{X,Y,P} \right]_\times \\
&\kappa = \left( [\psi]_\times \otimes [\psi]_\times \otimes [\chi]_\times \right) \left[ a^{(T)}_{X,Y,\psi} \right]_\times \\
&\kappa = \left( (\rho \otimes (\sigma \otimes \tau))a^{(T)}_{A',B',C'} \right).
\end{align*}
\]

**Ad (M7):** \(\forall A, A' \in |T_s| \ \forall \rho \in T_s [A, A'] \left( r^{(T)}_{A'} \rho = [r^{(T)}_{A'}]_\times \kappa \right) \) (by \(XW^* = AW^* = A\))

\[
\begin{align*}
&\kappa = \left[ [c_{AW,\psi} \otimes 1^{(T)}_I] r^{(T)}_{X'} \right]_\times \\
&\kappa = \left( [c_{AW,\psi} \otimes 1^{(T)}_I] c_{X,Y}^{(X',Y')} \right) \left[ r^{(T)}_{X'} \right]_\times \\
&\kappa = \left( [c_{AW,\psi}]_\times \otimes [1^{(T)}_I]_\times \right) \left[ r^{(T)}_{X'} \right]_\times \\
&\kappa = \left( [\psi]_\times \otimes [1^{(T)}_I]_\times \right) \left[ r^{(T)}_{X'} \right]_\times \\
&\kappa = \left( \rho \otimes 1^{(T)}_{A'} \right) r^{(T)}_{A'} \right) \) (by \(XW^* = A'W^* = A'\)).
\end{align*}
\]

**Ad (M8):** \(\forall A, B \in |T_s| \ \forall \rho \in T_s [A, A'], \sigma \in T_s [B, B'] \left( s^{(T)}_{A,B} (\sigma \otimes \rho) = [s^{(T)}_{A,B}]_\times (\psi \otimes \varphi) \right) \)

\[
\begin{align*}
&\kappa = \left[ s^{(T)}_{A,B} \right]_\times (\psi \otimes \varphi) \times \\
&\kappa = \left[ s^{(T)}_{A,B} \right]_\times [\psi \otimes \varphi]_\times \\
&\kappa = \left[ s^{(T)}_{A,B} c_{B,Y \otimes X}^{(X,\psi,\varphi)} \right]_\times \\
&\kappa = \left[ c_{A,B,Y \otimes X}^{(X,\psi,\varphi)} \right]_\times
\end{align*}
\]
Adjointness between theories and strict theories

\[ e_{AW \otimes BW, X \otimes Y}(\varphi \otimes \psi)s^{(T)}_{X', Y'} \]

\[ = \left( \varphi \otimes \psi \right) e_{X' \otimes Y', X' \otimes Y'}s^{(T)}_{X', Y'} \]

\[ = [\varphi \otimes \psi]s^{(T)}_{X', Y'} \]

\[ = (\varphi \otimes \psi)[s^{(T)}_{X', Y'}] \]

\[ = (\rho \otimes \sigma)s^{(T_s)}_{X', Y'} \]

where

\[ XW^* = AWW^* = A, \quad X'W^* = A'WW^* = A' \]

\[ YW^* = BWW^* = B, \quad Y'W^* = B'WW^* = B' \]

**Theorem 4.7.** Let \( T \in |Th_{\text{dht}}^J| \) be a \( J \)-sorted Hoehnke theory. Then there exists in a natural manner a strictly \( d \)-monoidal functor \( \Psi \) into the corresponding strict Hoehnke theory \( T_s \in |sth_{\text{dht}}^J| \).

**Proof.** Defining \( X\Psi := XW^* \), \( \varphi\Psi := [\varphi]_\kappa \) one obtains for arbitrary objects \( X, Y, P \) and morphisms \( \varphi \in T[X, Y], \psi \in T[Y, P] \)

\[ \left( \text{dom}^{(T)}\varphi \right) \Psi = X\Psi = XW^* = \text{dom}^{(T_s)}[\varphi]_\kappa = \text{dom}^{(T_s)}(\varphi\Psi) \]

\[ \left( \text{cod}^{(T)}\varphi \right) \Psi = Y\Psi = YW^* = \text{cod}^{(T_s)}[\varphi]_\kappa = \text{cod}^{(T_s)}(\varphi\Psi) \]

\[ 1^{(T)}_X \Psi = 1^{(T)}_{XW^*} = 1^{(T_s)}_{X\Psi} \]

\[ (\varphi 
abla (T) \psi)\Psi = [\varphi \nabla (T) \psi]_\kappa = [\varphi]_\kappa \nabla (T_s) [\psi]_\kappa = (\varphi\Psi) \nabla (T_s) \psi\Psi \]

hence \( \Psi \) is a functor.

By Lemma 3.8, it is sufficient to show (sFD), (sFT), and (sFM).

Ad (sFD):

\[ d^{(T)}_X \Psi = \left[ d^{(T)}_X \right]_\kappa = \left[ d^{(T)}_{XW^*} \right]_\kappa = d^{(T_s)}_{XW^*} = d^{(T_s)}_{X\Psi} \]
Ad (sFT): \[ t_X^{(T)} \Psi = [t_X^{(T)}]_{\kappa} = [t_{XW^*W}]_{\kappa} = t_{XW^*} \Psi = t_X^{(T)} \Psi. \]

Ad (sFM): \((\varphi \otimes \psi)\Psi = [\varphi \otimes \psi]_{\kappa} = [\varphi]_{\kappa} \otimes [\psi]_{\kappa} = \varphi \Psi \otimes \psi \Psi.\)

Therefore, \(\Psi : T \rightarrow T_s\) is a strictly \(d\)-monoidal functor.

The converse question is also positively answered by the following theorem:

**Theorem 4.8.** Let \(T_s \in \vert sT_{dht}^0(J)\vert\) be a strict \(J\)-sorted Hoehnke theory. Then there corresponds to \(T_s\) in a natural way a \(J\)-sorted Hoehnke theory \(T \in \vert T_{dht}^0(J)\vert\).

**Proof.** Take \(\vert T \vert = H^* \ (\vert T \vert = H)\), where \((H^*; \otimes, I, O) ((H; \otimes, I))\) is the free \(G^*-\text{algebra (free } G\text{-algebra) freely generated by } J.\)

Defining \(T[X,Y] := \{([X,\varphi, Y] \mid \varphi \in T_s[XW^*, YW^*]\})\) for arbitrary \(X,Y \in H^* \ (X, Y \in H)\) one obtains obviously \(T[X,Y] \cup T[X',Y'] = \emptyset\) if \(X \neq X'\) or \(Y \neq Y'\) and, by definition, \(\text{dom}^{(T)}(X,\varphi,Y) = X, \text{cod}^{(T)}(X,\varphi,Y) = Y\) and \(1_X^{(T)} = (X,1_{XW^*},X).\)

Morphisms \((X,\varphi,Y)\) and \((P,\psi,Q)\) are composable for \(Y = P\) defined by

\[(X,\varphi,Y) \circ^{(T)} (Y,\psi,Q) := (X,\varphi \circ^{(T_s)} \psi, Q).\]

Then
\[
1_X^{(T)} \circ^{(T)} (X,\varphi,Y) = (X,1_{XW^*}^{(T_s)},X) \circ^{(T)} (X,\varphi,Y) = (X,1_{XW^*}^{(T_s)} \varphi,Y) = (X,\varphi,Y),
\]
\[
(X,\varphi,Y) \circ^{(T)} 1_Y^{(T)} = (X,\varphi,Y) \circ^{(T)} (Y,1_{YW^*}^{(T_s)},Y) = (X,\varphi 1_{YW^*}^{(T_s)},Y) = (X,\varphi,Y),
\]
\[
(X,\varphi,Y) \circ^{(T)} ((Y,\psi,P) \circ^{(T)} (P,\chi,Q)) = (X,\varphi(\psi \chi),Q)
\]
\[
= (X,\varphi(\psi \chi),Q) = ((X,\varphi,Y) \circ^{(T)} (Y,\psi,P)) \circ^{(T)} (P,\chi,Q),
\]

hence one has a category.

By the agreements
\[(X_1,\varphi_1,Y_1) \otimes^{(T)} (X_2,\varphi_2,Y_2) := (X_1 \otimes^{(T_s)} X_2,\varphi_1 \otimes^{(T_s)} \varphi_2,Y_1 \otimes^{(T_s)} Y_2),\]
Adjointness between theories and strict theories

\( a_{X,Y,P}^{(T)} := (X \otimes (Y \otimes P), 1_{XW^* \otimes YW^* \otimes PW^*}, (X \otimes Y) \otimes P) \),

\( r_X^{(T)} := \left( X \otimes I, 1_{XW^*}, X \right) \),

\( l_X^{(T)} := \left( I \otimes X, 1_{XW^*}, X \right) \),

\( s_{X,Y}^{(T)} := \left( X \otimes Y, s_{XW^* \otimes YW^*}, Y \otimes X \right) \),

\( d_X^{(T)} := \left( X, d_{XW^*}, X \otimes X \right) \),

\( t_X^{(T)} := \left( X, t_{XW^*}, I \right) \),

\( o^{(T)} := (I, o^{(T)}) \)

one obtains a \( \text{dhts}-\text{category} \ (T, \otimes^{(T)}, I, a^{(T)}, r^{(T)}, l^{(T)}, s^{(T)}, t^{(T)}, o^{(T)}) \), i.e. a Hoehnke theory in \( |T_{\text{dht}}(J)| \), since the validity of the defining axioms obviously carries over from \( T_s \) into \( T \).

**Remark.** If \( T_s \in |sT_{\text{dht}}(J)| \) is even any strict \( J \)-sorted Hoehnke theory with halfdiagonalinversions, then one obtains by the additional agreement

\[ \nabla_X^{(T)} := \left( X \otimes X, \nabla_{XW^*}^{(T_s)}, X \right) \]

a \( \text{dhth}s\)-category \( (T, \otimes^{(T)}, I, a^{(T)}, r^{(T)}, l^{(T)}, s^{(T)}, t^{(T)}, \nabla^{(T)}, o^{(T)}) \), i.e. a Hoehnke theory in \( |T_{\text{dhth}}(J)| \).

**Definition 4.9.** Let \( T \) and \( T' \) be \( J \)-sorted Hoehnke theories in \( |Th_{\text{dht}}(J)| \) and \( |sTh_{\text{dht}}(J)| \), respectively.

Then a \( d \)-monoidal functor \( F : T \to T' \) is called \textit{theory morphism}, if, in addition, the conditions

\[ (\text{Th1}) \quad \forall X \in |T| \ (XF = X), \]

\[ (sF^*) \quad \forall X, Y \in |T| \ (F(X, Y) \in \text{Un}_K) \]

are fulfilled.
Lemma 4.10. Every theory morphism $F : T \rightarrow T'$ has the properties (sFD), (sFT), (sFM), (sFI$^*$).

Conversely, any functor $F : T \rightarrow T'$ is a theory morphism between $J$-sorted Hoehnke theories $T$ and $T'$, whenever $F$ satisfies (Th1), (sFD), (sFT), and (sFM).

Proof. The assertion is an immediate consequence of Lemma 3.8 and Corollary 3.10.

Theorem 4.11. All $J$-sorted Hoehnke theories together with the corresponding theory morphisms form a category $\text{Th}_{\text{dht}}^J(J)$ and $s\text{Th}_{\text{dht}}^J(J)$, respectively, where the composition of theory morphisms is defined by the usual composition of functors.

Proof. Obviously, $\text{dom}(F : T \rightarrow T') = T$, $\text{cod}(F : T \rightarrow T') = T'$.

The identical functor $1_T : T \rightarrow T$ is a theory morphism with respect to

$$1_T = (1_T(X,Y) = 1_{X \otimes Y} | X, Y \in H^o), \quad i_1_T = 1_I.$$

Let $F : T \rightarrow T'$ and $G : T' \rightarrow T''$ be theory morphisms. Then, by definition, $FG$ is a functor fulfilling the condition (Th1).

Moreover, because of Lemma 4.10 and Proposition 3.9, $FG$ is a theory morphism.

Trivially, $F 1_T = F = F 1_{T'}$ and $F(GH) = (FG)H$ for every theory morphism $F$ and all composable theory morphisms $F$, $G$ and $H$.

Theorem 4.12. Let $\text{Th}_{\text{dht}}^J(J)$ and $s\text{Th}_{\text{dht}}^J(J)$ be the categories introduced above. Then there are the functors

$$\Sigma : \text{Th}_{\text{dht}}^J(J) \rightarrow s\text{Th}_{\text{dht}}^J(J)$$

$$T \mapsto T \Sigma := T_s$$

defined by

$$XW^* \mapsto XW^*, \; [\varphi]_x \mapsto [\varphi F]_{x'}$$
and

$$\Pi : sTh^0_{\text{dht}}(J) \to Th^0_{\text{dht}}(J)$$

$$T_s \mapsto T_s \Pi := T_s \quad \text{(see 4.7),}$$

$$(F : T_s \to T_s') \mapsto (F \Pi : T \to T') \quad \text{defined by}$$

$$X \mapsto X, \quad (X, \varphi, Y) \mapsto (X, \varphi F, Y)$$

such that $\Sigma$ is a left-adjoint functor of the functor $\Pi$.

**Proof. a)** The functor property of $\Sigma$:

The mapping on objects is well defined by Theorem 4.5. Let $F$ be a theory morphism from a $J$-sorted theory $T$ into a $J$-sorted theory $T'$, i.e. $F \in Th^0_{\text{dht}}(J)[T, T']$. Then $F \Sigma$, defined as above, is a theory morphism too, more precisely,

$$F \Sigma \in sTh^0(J)[T \Sigma, T' \Sigma].$$

By definition, the mapping $F \Sigma$ respects “dom” and “cod” and one obtains

$$1_{(XW^\ast)}(F \Sigma) = 1_{(X^T)}(F) = 1_{(X^T)}(F) = 1_{(XW^\ast)}(F \Sigma)$$

for all objects $X \in |T|$.

Now let $[\varphi]_\Sigma \in T \Sigma[XW^\ast, YW^\ast]$, $[\psi]_\Sigma \in T \Sigma[UU^\ast, VW^\ast]$ be arbitrary morphisms such that $YW^* = UW^*$. Then

$$([\varphi]_\Sigma[\psi]_\Sigma)(F \Sigma) = [\varphi \circ_{Y, U} \psi]_\Sigma(F \Sigma) = [\varphi F]_\Sigma[\psi F]_\Sigma = [\varphi F]_\Sigma[\psi F]_\Sigma = [\varphi F]_\Sigma[\psi F]_\Sigma = [\varphi F]_\Sigma(F \Sigma).$$

Furthermore, the functor $F \Sigma$ satisfies (Th1) by definition, (sFD) and (sFT) since for all $A \in S^0$ one has
\[ d_A^{(T \Sigma)}(F \Sigma) = \left[ d_A^{(T)} \right]_{\chi}(F \Sigma) = \left[ d_A^{(T)} \right]_{\lambda} = \left[ d_A^{(T')} \right]_{\lambda} = \left[ d_A^{(T')} \right]_{\lambda} = d_A^{(T \Sigma)} \]

and

\[ t_A^{(T \Sigma)}(F \Sigma) = \left[ t_A^{(T)} \right]_{\chi}(F \Sigma) = \left[ t_A^{(T)} \right]_{\lambda} = \left[ t_A^{(T')} \right]_{\lambda} = \left[ t_A^{(T')} \right]_{\lambda} = t_A^{(T \Sigma)} \]

and (sFM) since for all \( \varphi \in T[X, U], \psi \in T[Y, V] \) the equation

\[
\left( [\varphi]_{\chi} \otimes [\psi]_{\lambda} \right)(F \Sigma) = \left( [\varphi \otimes \psi]_{\lambda} \right) = \left( [\varphi \otimes \psi]_{\lambda} \right) = \left( [\varphi \otimes \psi]_{\lambda} \right) = \left( [\varphi \otimes \psi]_{\lambda} \right) = \left( [\varphi \otimes \psi]_{\lambda} \right) = \left( [\varphi \otimes \psi]_{\lambda} \right) = \left( [\varphi \otimes \psi]_{\lambda} \right) = \left( [\varphi \otimes \psi]_{\lambda} \right) = \left( [\varphi \otimes \psi]_{\lambda} \right) = \left( [\varphi \otimes \psi]_{\lambda} \right) = \left( [\varphi \otimes \psi]_{\lambda} \right)
\]

is valid.

b) The functor property of \( \Pi \):

The mapping on objects \( T_s \) is well defined by Theorem 4.7.

Let \( (F : T_s \rightarrow T'_s) \) be a theory morphism. Then \( (F \Pi : T \rightarrow T') \) defined by

\[
X \mapsto X, (X, \varphi, Y) \mapsto (X, \varphi F, Y)
\]

is a theory morphism too, since the conditions (Th1), (sFD), (sFT), and (sFM) are satisfied.

Ad (Th1): \( \forall X \in H^\circ \ (X(F \Pi) = X) \) by definition.

Ad (sFD):

\[
\forall X \in H^\circ \left( d_X^{(T)} (F \Pi) = \left( X, d_X^{(T_s)} \right) (F \Pi) = \left( X, d_X^{(T')} \right) = d_X^{(T')} \right) = \left( X, d_X^{(T')} \right)
\]

Ad (sFT):

\[
\forall X \in H^\circ \left( t_X^{(T)} (F \Pi) = \left( X, t_X^{(T_s)} \right) (F \Pi) = \left( X, t_X^{(T')} \right) = t_X^{(T')} \right)
\]
Ad (sFM): \( \forall \rho \in T[X, U], \sigma \in T[Y, V] \left( (\rho \otimes \sigma)(F\Pi) \right) \)

\[
= ((X, \varphi, U) \otimes (Y, \psi, V))(F\Pi)
\]

\[
= (X \otimes Y, \varphi \otimes \psi, U \otimes V)(F\Pi)
\]

\[
= (X \otimes Y, (\varphi \otimes \psi)F, U \otimes V)
\]

\[
= (X, \varphi F, U \otimes V)
\]

\[
= \rho(F\Pi) \otimes \sigma(F\Pi).
\]

c) It remains to show that \( \Sigma \) is a left-adjoint of \( \Pi \). We will prove in several steps that for every \( T \in |Th_{dht}(J)| \) and every \( T_s \in |sTh_{dht}(J)| \) there is an isomorphism between the sets \( sTh_{dht}(J)[T, T_s\Pi] \).

1. A functor from a theory \( T \) into \( T(\Sigma \Pi) \):

Define a mapping \( \Theta_T \) on objects and morphisms of any Hoehnke theory by \( X^{\Theta_T} := X \) and \( \varphi^{\Theta_T} := (X, [\varphi]^r, Y) \) for \( \varphi \in T[X, Y] \). This mappings are well defined and the values are objects and morphisms of \( T(\Sigma \Pi) \).

\( \Theta_T : T \rightarrow T(\Sigma \Pi) \) is a functor, since the object mapping is compatible with “dom” and “cod” and

\[
1_X^{(T)} \Theta_T = \left( X, \left[ 1_X^{(T)} \right]^r, X \right) = \left( X, 1_X^{(T(\Sigma))}, X \right) = 1_X^{(T(\Sigma))} = 1_X^{(T(\Sigma \Pi))},
\]

\[
(\varphi \psi)^{\Theta_T} = \left( X, [\varphi \psi]^r, U \right) = \left( X, [\varphi]^r[\psi]^r, U \right) = (\varphi^{\Theta_T})(\psi^{\Theta_T}).
\]
Moreover, $\Theta_T : T \to T(\Sigma \Pi)$ is even a theory morphism because of the validity of (Th1), (sFD), (sFT), and (sFM) as follows:

\[
\forall X \in |T| \ (X \Theta_T = X) \text{ by definition.}
\]

\[
\forall X \in |T| \left( d_X^{(T)} \Theta_T = \left( X, \left[ d_X^{(T)} \right]_X, X \otimes X \right) = \left( X, d_X^{(T \Sigma)}, X \otimes X \right) \right.
\]
\[
= d_{X}^{(T(\Sigma)\Pi)} = d_{X}^{(T(\Sigma)\Pi)} \right).
\]

\[
\forall X \in |T| \left( t_X^{(T)} \Theta_T = \left( X, \left[ t_X^{(T)} \right]_X, I \right) = \left( X, t_X^{(T \Sigma)}, I \right) = t_{X}^{(T(\Sigma)\Pi)} \right).
\]

\[
\forall \varphi \in T[X, U], \ \psi \in T[Y, V] \ ((\varphi \otimes \psi) \Theta_T = (X \otimes Y, [\varphi \otimes \psi]_X, U \otimes V)
\]
\[
= (X, [\varphi]_X, U) \otimes (Y, [\psi]_Y, V) = \varphi \Theta_T \otimes \psi \Theta_T).
\]

In such a way, every theory morphism $G' \in |sTh_{dht}(J)|$ determines uniquely a theory morphism $G := \Theta_T(G' \Pi) \in Th_{dht}(J)[T, T_s \Pi]$.

2. A construction of a strictly $d$-monoidal functor $G : T \to T_s$:

To every theory morphism $G \in Th_{dht}(J)[T, T_s \Pi]$ there is assigned in a natural manner a strictly $d$-monoidal functor $G : T \to T_s$ as follows:

Let be given any $G \in Th_{dht}(J)[T, T_s \Pi]$. Then

\[
XG = X \ (X \in |T|) \text{ and}
\]

\[
T[X, U] \ni \varphi \mapsto \varphi G = (X, \varphi G, U) \in T_s \Pi[X, U],
\]

where $\varphi G \in T_s[XW^*, UW^*]$.

The agreements

\[
H^0 \ni X \mapsto X \Xi := XW^* \in S^0
\]

and
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$$\mathcal{T}_s \Pi [X, U] \ni (X, \psi, U) \mapsto (X, \psi, U) \Xi := \psi \in \mathcal{T}_s [XW^*, UW^*]$$

define a functor $\Xi : \mathcal{T}_s \Pi \to \mathcal{T}_s$ because of:

$$\text{dom}^{(\mathcal{T}_s)}((X, \psi, U) \Xi) = \text{dom}^{(\mathcal{T}_s)}(\psi) = X \Xi = (\text{dom}^{(\mathcal{T}_s \Pi)}(X, \psi, U)) \Xi,$$

$$\text{cod}^{(\mathcal{T}_s)}((X, \psi, U) \Xi) = \text{cod}^{(\mathcal{T}_s)}(\psi) = U \Xi = (\text{cod}^{(\mathcal{T}_s \Pi)}(X, \psi, U)) \Xi,$$

$$1^{(\mathcal{T}_s \Pi)} \Xi = \left( X, 1^{(\mathcal{T}_s)} XW^*, X \right) \Xi = 1^{(\mathcal{T}_s)} X \Xi,$$

$$(X, \psi_1, U_1) \otimes (X, \psi_2, U_2) \Xi = (X_1 \otimes X_2, \psi_1 \otimes \psi_2, U_1 \otimes U_2) \Xi$$

$$= \psi_1 \otimes \psi_2 = (X_1, \psi_1, U_1) \Xi \otimes (X_2, \psi_2, U_2) \Xi.$$

The compositum $\mathcal{G} := G \Xi$ is strictly $d$-monoidal functor from $\mathcal{T}$ into $\mathcal{T}_s$.

3. The induced theory morphism $G' \in sTh_{dh}^c(J)$:

Let $G$, $\Xi$, and $\mathcal{G}$ be given as above. Then define a mapping $G'$ by $\mathcal{A}G' := A$ for all $A \in S^s$ and $[\varphi], G' := \varphi \mathcal{G} = (\varphi G) \Xi = (X, \varphi_G, U) \Xi = \varphi_G \in \mathcal{T}_s [XW^*, UW^*]$ for all $\varphi \in \mathcal{T}[X, U]$, where $\varphi_G$ is a well-defined morphism of $\mathcal{T}_s$.

Because of

$$\varphi_1 \in \mathcal{T}[X_1, U_1] \land \varphi_2 \in \mathcal{T}[X_2, U_2] \land [\varphi_1] \ast = [\varphi_2] \ast \Rightarrow$$

$$\Rightarrow X_1W^* = X_2W^* := A \land U_1W^* = U_2W^* := B$$

$$\land c_{X_1}^{-1} \varphi_1 c_{U_1} = c_{X_2}^{-1} \varphi_2 c_{U_2} \in \mathcal{T}_s [AW, BW] \Rightarrow$$
since $G$ is a theory morphism in $sT_h\cup dht(J)$ because of the validity of (Th1) by definition and the validity of (sFD), (sFT), and (sFM) as follows:

\[ d_A^{(T_\Sigma)} G' = [d_A^{(T)}] \cong d_A^{(T)} G \cong d_A^{(T_\Pi)} G = d_A^{(T_\Pi)} \Xi = d_A^{(T_A)} = d_A^{(T_S)}. \]

\[ t_A^{(T_\Sigma)} G' = [t_A^{(T)}] \cong t_A^{(T)} G = t_A^{(T_\Pi)} \Xi = t_A^{(T_A)} = t_A^{(T_S)}. \]
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\[(\varphi_\times \otimes [\psi]_\times)G' = ([\varphi \otimes \psi]_\times)G' = (\varphi \otimes \psi \overline{G}) = (\varphi \overline{G}) \otimes (\psi \overline{G}) = [\varphi]_\times G' \otimes [\psi]_\times G'.\]

By the functor \(\Pi : sTh^\circ - dht(J) \to Th^\circ_{dht}(J)\), \(G' \Pi : T(\Sigma \Pi) \to T_s \Pi\) is a theory morphism.

Moreover, this theory morphism has the property \(G = \Theta_T(G' \Pi)\). This is a consequence of

\[H^\circ \ni X \mapsto X(\Theta_T(G' \Pi)) = (X\Theta_T)(G' \Pi) = X(G' \Pi) = X = XG\]

and

\[T[X, U] \ni \varphi \mapsto \varphi(\Theta_T(G' \Pi)) = (\varphi \Theta_T)(G' \Pi) = (X, [\varphi]_\times, U)(G' \Pi) = (X, [\varphi]_\times G', U) = (X, \varphi G, U) = \varphi G.\]

Finally, let \(L : T\Sigma \to T_s\) be a theory morphism such that \(\Theta_T(L \Pi) = G\). Then

\[\forall X \in H^\circ \ ((XW^*)G' = XW^* = (XW^*)G)\]

and

\[\forall X, U \in H^\circ \forall \varphi \in T[X, U] \ ((X, [\varphi]_\times G', U) = (X, \varphi \overline{G}, U) = \varphi G = (\varphi(\Theta_T)(L \Pi)) = (\varphi \Theta_T)(L \Pi) = (X, [\varphi]_\times, U)(L \Pi) = (X, [\varphi]_\times L, U)\]

\[\Rightarrow [\varphi]_\times G' = [\varphi]_\times L,\]

thus \(L = G'\), i.e. \(G'\) is the only theory morphism in \(sTh^\circ_{dht}(J)\) with the property \(G = \Theta_T(G' \Pi)\).
The diagram illustrates the individual \(d\)-monoidal functors and theory morphisms, respectively, which are considered in the proof of the last theorem. This diagram is commutative in all of its parts, namely \(G = \Theta_T(G'\Pi)\) was shown above, \(\overline{G} = G\Xi\) by definition, and \(\overline{G} = \Psi G'\) follows by

\[
X(\Psi G') = (X\Psi)G' = (XW^*)G' = XW^* = X\overline{G}
\]

and

\[
\varphi(\Psi G') = (\varphi \Psi)G' = [\varphi]_XG' = \varphi_G = \varphi\overline{G}.
\]

**Corollary 4.13.** The theory morphisms \(\Theta_T, \overline{T} \in |Th_{\text{dht}}^\circ(J)|\) form a natural transformation \(\Theta : Id_{Th_{\text{dht}}^\circ(J)} \rightarrow \Sigma\Pi\).

**Proof.** \(\Theta = (\Theta_T \mid \overline{T} \in |Th_{\text{dht}}^\circ(J)|)\) is a natural transformation \(\Theta : Id_{Th_{\text{dht}}^\circ(J)} \rightarrow \Sigma\Pi\) because of the commutativity of the following diagram for arbitrary theories and theory morphisms of \(Th_{\text{dht}}^\circ(J)\):
Let $X$ be any object of $\mathcal{T}$. Then

$$X(F\Theta T') = (XF)\Theta T' = X\Theta T' = X$$

and

$$X(\Theta TF(\Sigma\Pi)) = (X\Theta T)((F\Sigma)\Pi) = X.$$

For every morphism $\varphi \in \mathcal{T}[X,U]$ one has

$$\varphi(F\Theta T') = (\varphi F)\Theta T' = (X, [\varphi F]_{\lambda'}, U)$$

and

$$\varphi(\Theta TF(\Sigma\Pi)) = (\varphi \Theta T)((F\Sigma)\Pi) =$$

$$= (X, [\varphi]_{\lambda'}(F\Sigma), U) = (X, [\varphi F]_{\lambda'}, U),$$

hence

$$\Theta TF(\Sigma\Pi) = F\Theta T'.$$

\[\square\]
References


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