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ON THE CHARACTERISATION OF MAL'TSEV AND JÓNSSON-TARSKI ALGEBRAS

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Abstract

There are very strong parallels between the properties of Mal'tsev and Jónsson-Tarski algebras, for example in the good behaviour of centrality and in the factorization of direct products. Moreover, the two classes between them include the majority of algebras that actually arise "in nature". As a contribution to the research programme building a unified theory capable of covering the two classes, along with other instances of good centrality and factorization, the paper presents a common framework for the characterisation of Mal'tsev and Jónsson-Tarski algebras. Mal'tsev algebras are characterized by simplicial identities in the product complex of an algebra. In the dual of a pointed variety, a simplicial object known as the pointed complex is then constructed. The basic simplicial Mal'tsev identity in the pointed complex characterises Jónsson-Tarski algebras. Higher-dimensional simplicial Mal'tsev identities in the pointed complex are characteristic of a class of algebras lying properly between Goldie and Jónsson-Tarski algebras.

Keywords: Mal'tsev variety, Mal'tsev algebra, Jónsson-Tarski variety, Jónsson-Tarski algebra, Goldie variety, Goldie algebra, congruence permutability, simplicial object.

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1. Introduction

A variety of universal algebras is a *Mal'tsev variety* if there is a derived ternary operation P (a so-called *Mal'tsev operation* or *Mal'tsev parallelogram*) such that the identities

$$P(y, y, x) = x = P(x, y, y) \tag{1.1}$$

hold (see: [9], [10], [11])*. Members of a Mal'tsev variety are called *Mal'tsev algebras*. A variety of universal algebras is a *Jónsson-Tarski* variety if there is a derived binary operation + and a nullary operation (constant) selecting a subalgebra $\{0\}$ such that the identities

$$0 + x = x = x + 0 \tag{1.2}$$

hold ([7]). Members of a Jónsson-Tarski variety are called *Jónsson-Tarski algebras*. A Mal'tsev variety having a nullary operation selecting a subalgebra $\{0\}$ is called a *Goldie variety*, and its members are called *Goldie algebras* ([2]). Goldie algebras become Jónsson-Tarski algebras on defining

$$x + y = P(x, 0, y). \tag{1.3}$$

Thus the class of Goldie algebras represents the intersection of the classes of Mal'tsev and Jónsson-Tarski algebras. Goldie algebras include loops, and, in particular, groups, as well as many kinds of algebras having group reducts (*groups with operators* in Emmy Noether's sense), such as rings, Lie algebras, Jordan algebras, etc. Equationally defined quasigroups and Heyting algebras provide examples of Mal'tsev algebras that are not Goldie algebras. Monoids provide examples of Jónsson-Tarski algebras that are not Goldie algebras.

Jónsson-Tarski algebras and Mal'tsev algebras share many desirable properties, in particular the good behaviour of centrality and direct products. For example, under mild finiteness assumptions (which include non-emptiness in the case of Mal'tsev algebras), one may cancel the factor A from an isomorphism

$$A \times B \cong A \times C$$

to obtain a central isotopy ([11], p. 70)

$$B \simeq C$$

*The various spellings of Mal'tsev's name are the result of changing conventions for transliteration from the Cyrillic to the Latin alphabet.

(compare Theorem 3.11(ii) of [7] and Theorem 424 of [11], recalling that central isotopy reduces to isomorphism for Jónsson-Tarski algebras). Given the very close parallels between the behaviour of Jónsson-Tarski and Mal'tsev algebras, it is natural to seek a common framework for characterizing the two classes. The purpose of this paper is to propose such a framework, as a contribution to the development of a unified treatment of Jónsson-Tarski and Mal'tsev algebras, along with other classes of algebras that share their good behaviour.

Following the publication of [11] and the extension of its methods from Mal'tsev algebras to more general algebras having modular congruence lattices (by Hagemann, Herrmann [5], and many others later), the question of developing a unified theory for Mal'tsev and Jónsson-Tarski algebras was raised by the author in an invited paper presented at the Special Session on Lattice Theory and General Algebra during the American Mathematical Society meeting in Boulder, Colorado in March, 1980 [12] (cf. p. 68 of [11]). Except for two results at the end of Gumm's Habilitationsschrift ([4], Th. 11.11 and Cor. 12.3), however, there was little response to this challenge. In retrospect, the reason has become clear. Researchers of that period were focussed almost exclusively on the lattice of congruences as the primary object of interest. On the other hand, although it is natural to work with congruences in Mal'tsev algebras, the theory of Jónsson-Tarski algebras expounded in [7] works with subalgebras instead. There is thus an evident duality between the two theories: subobjects in Jónsson-Tarski algebras, quotients in Mal'tsev algebras. The formulation proposed here recognises this duality. The fundamental object associated with a Mal'tsev algebra A is its *product complex* (Section 2 below, cf. (0.13.2.2) of [1] or p. 117 of [11]), the 0-coskeleton of the truncated complex consisting of the unique arrow from A to the terminal object. The defining identities (1.1) for Mal'tsev algebras are formulated in terms of this complex. In fact, the formulation yields single identities, the *simplicial Mal'tsev identities* of Definition 2.3. The main task undertaken in this paper is to exhibit the corresponding simplicial object associated with each Jónsson-Tarski algebra, the so-called *pointed complex* of Section 3. This is not itself a simplicial object in a category of Jónsson-Tarski algebras and homomorphisms (there are too many degeneracies and too few face operators), but it does become a simplicial object when one passes to the opposite category (Theorem 3.1). Once the pointed complex is established, one may translate back and forth between the theories of Mal'tsev and Jónsson-Tarski by transposing the

product and pointed complexes. Thus Theorem 4.1 shows that Jónsson-Tarski varieties are characterized by satisfaction of the basic simplicial Mal'tsev identity in each pointed complex. Theorems 4.3 and 4.4 show that satisfaction of the higher-dimensional Mal'tsev identities in each pointed complex of a pointed variety is characteristic of a class of varieties that lie properly between Jónsson-Tarski and Goldie varieties.

In general, this paper will follow the algebraic and categorical conventions of [13]. In particular, mappings will normally be placed to the right of their arguments, so that in passing from text to mathematics one may continue to read from left to right, avoiding a profusion of parentheses. On the other hand, mappings that are images of morphisms under contravariant functors (such as the identity functor from a category of Jónsson-Tarski algebras to its opposite) will usually be placed to the left of their arguments, and composition of such mappings will be denoted by $\circ$ rather than by simple juxtaposition. The description of the product complex in Section 2 is designed to act as a quick introduction to the calculus of simplicial objects for readers who might not be familiar with its intricacies.

2. Product complexes and Mal'tsev algebras

Let A be a set. For each positive integer n and natural number i less than n, define the *face map* ε^i or

$$\varepsilon^i_n : A^n \to A^{n-1}; (x_0, \ldots, x_{n-1}) \mapsto (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1}) \tag{2.1}$$

(mnemonic: ε^i_n for "excise x_i from the n-tuple"). Define the *degeneracy* δ^i or

$$\delta^i_n : A^n \to A^{n+1}; (x_0, \ldots, x_{n-1}) \mapsto (x_0, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_{n-1}) \tag{2.2}$$

(mnemonic: δ^i_n for "duplicate x_i within the n-tuple".) The set of direct powers of A, together with the face maps and degeneracies, forms the *product complex* (cf. (0.13.2.2) of [1] or p. 117 of [11]). The face maps and degeneracies satisfy the following *simplicial identities*: the *face identities*

$$\varepsilon^i_n \varepsilon^{j-1}_{n-1} = \varepsilon^j_n \varepsilon^i_{n-1} \tag{2.3}$$

for $0 \leq i < j < n$ (i.e. consecutively excise x_i and x_j in either order), the *degeneracy identities*

$$\delta^i_n \delta^{j+1}_{n+1} = \delta^j_n \delta^i_{n+1} \tag{2.4}$$

for $0 \leq i \leq j \leq n$ (i.e. consecutively duplicate x_i and x_j in either order), and the *mixed identities*

$$\delta_n^i \varepsilon_{n+1}^j = \begin{cases} \varepsilon_n^j \delta_{n-1}^{i-1}, & \text{if } j < i, \\ \varepsilon_n^{j-1} \delta_{n+1}^i, & \text{if } j > i+1, \\ 1, & \text{otherwise} \end{cases} \tag{2.5}$$

for $0 \leq i, j \leq n$ (cf. (0.1.2) of [1] and VII(11)–(13) of [8]). Note the duality between the face identities (2.3) and the degeneracy identities (2.4), as well as the self-duality of the mixed identities (2.5) except in the final cases $j \in \{i, i+1\}$.

In terms of the product complex, the Mal'tsev parallelogram identities (1.1) reduce to a single equation.

Proposition 2.1. *A set A is a Mal'tsev algebra if and only if it is endowed with a ternary operation P such that*

$$P\left(\varepsilon_3^0, \varepsilon_3^2 \varepsilon_2^0 \delta_1^0, \varepsilon_3^2\right) = \varepsilon_3^1. \tag{2.6}$$

Proof. Applied to the general element (a_0, a_1, a_2) of A^3, the equality (2.6) of the proposition becomes

$$\begin{aligned} (P(a_1, a_1, a_0), P(a_2, a_1, a_1)) &= P((a_1, a_2), (a_1, a_1), (a_0, a_1)) \\ &= P\left((a_0, a_1, a_2)\varepsilon_3^0, (a_0, a_1, a_2)\varepsilon_3^2 \varepsilon_2^0 \delta_1^0, (a_0, a_1, a_2)\varepsilon_3^2\right) \\ &= (a_0, a_1, a_2) P\left(\varepsilon_3^0, \varepsilon_3^2 \varepsilon_2^0 \delta_1^0, \varepsilon_3^2\right) \\ &= (a_0, a_1, a_2)\varepsilon_3^1 \\ &= (a_0, a_2), \end{aligned}$$

the two components of which are equivalent to (1.1). ∎

In fact, the identity of Proposition 2.1 works in all dimensions for which it is defined.

Corollary 2.2. *A set A is a Mal'tsev algebra if and only if it is endowed with a ternary operation P such that*

$$P\left(\varepsilon^0, \varepsilon^2 \varepsilon^0 \delta^0, \varepsilon^2\right) = \varepsilon^1 \tag{2.7}$$

at any object of the product complex for which the equation is defined.

Proof. Applied to the general element $(a_0, \ldots, a_{n-1})$ of A^n for $n \geq 3$, the left hand side of the equality (2.7) of the corollary becomes

$$\begin{aligned}(a_0,&a_1, a_2, a_3, \ldots, a_{n-1})P\left(\varepsilon^0, \varepsilon^2\varepsilon^0\delta^0, \varepsilon^2\right)\\ &= P((a_1, a_2, a_3, \ldots, a_{n-1}), (a_1, a_1, a_3, \ldots, a_{n-1}), (a_0, a_1, a_3, \ldots, a_{n-1}))\\ &= (P(a_1, a_1, a_0), P(a_2, a_1, a_1), P(a_3, a_3, a_3), \ldots, P(a_{n-1}, a_{n-1}, a_{n-1})),\end{aligned}$$

while the right hand side becomes

$$(a_0, a_2, a_3, \ldots, a_{n-1}).$$

These two expressions certainly agree in a Mal'tsev algebra. On the other hand, suppose that they agree. Then the equality between their first two components yields (1.1). ∎

Definition 2.3. The equation (2.6) of Theorem 2.1 is called the (2-*dimensional*) *simplicial Mal'tsev identity.* The equation

$$P\left(\varepsilon_n^0, \varepsilon_n^2\varepsilon_{n-1}^0\delta_{n-2}^0, \varepsilon_n^2\right) = \varepsilon_n^1 \tag{2.8}$$

of Corollary 2.2, for $n > 3$, is called the *simplicial Mal'tsev identity of dimension* $n-1$. Collectively, the identities (2.8) for any $n > 3$ are called the *higher-dimensional simplicial Mal'tsev identities.*

3. Pointed complexes

Let $\mathcal{C}$ be a category. Recall that a *simplicial object* in $\mathcal{C}$ consists of an object A_n of $\mathcal{C}$ for each positive integer n (in which case the *dimension* of A_n is defined to be the natural number $n-1$), *face* morphisms ε^i or $\varepsilon_n^i : A_{n-1} \to A_{n-2}$ for $0 \leq i < n > 1$, and *degeneracy* morphisms δ^i or $\delta_n^i : A_{n-1} \to A_n$ for $0 \leq i < n \geq 1$, such that the simplicial identities are satisfied (cf. (0.1) of [1], §VII.4 of [8], p. 114 of [11]). For example, the product complexes of Section 2 are simplicial objects in the category of sets, and in each variety of algebras (construed as a category with homomorphisms as morphisms). In the product complex determined by a set or algebra A, the object at dimension n, for each natural number n, is the direct power A^{n+1}.

A variety $\mathfrak{V}$ of algebras is said to be *pointed* if there is a nullary operation (constant) selecting a subalgebra $\{0\}$. Thus $\{0\}$ becomes a zero object when $\mathfrak{V}$ is construed as a category. For each algebra A in a pointed variety $\mathfrak{V}$,

a simplicial object in the opposite category $\mathfrak{V}^{op}$ will be constructed, having the direct power A^n as the object at dimension n. Note that the object at dimension zero is the zero object $\{0\}$. The complex, the so-called *pointed complex*, will play the same role amongst Jónsson-Tarski algebras that the product complex plays amongst Mal'tsev algebras.

For an integer $n > 1$ and $0 \leq i < n$, the degeneracy morphism $\delta_n^i : A_{n-1} \to A_n$ in $\mathfrak{V}^{op}$ is defined to be the projection homomorphism

$$\delta^i : A^n \to A^{n-1}; (a_0, \ldots, a_{n-1}) \mapsto (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n-1}). \tag{3.1}$$

Note that these homomorphisms appear as face maps (2.1) in the product complex of A. The degeneracy morphism $\delta_1^0 : A_0 \to A_1$ is determined uniquely (as the constant homomorphism from A to $\{0\}$) by the fact that A_0 is the zero object of $\mathfrak{V}^{op}$. In similar fashion, the face morphisms $\varepsilon_2^i : A_1 \to A_0$, for $0 \leq i < 2$, are determined uniquely as the homomorphisms $\varepsilon_2^i : A^0 \to A^1$ inserting $\{0\}$ into A. For an integer $n > 2$ and $0 < i < n-1$, the face morphism $\varepsilon_n^i : A_{n-1} \to A_{n-2}$ in $\mathfrak{V}^{op}$ is defined to be the diagonal homomorphism

$$\varepsilon^i : A^{n-2} \to A^{n-1}; (a_0, \ldots, a_{n-3}) \mapsto (a_0, \ldots, a_{i-1}, a_{i-1}, \ldots, a_{n-3}). \tag{3.2}$$

Note that these homomorphisms appear as degeneracies (2.2) in the product complex of A. The face morphism $\varepsilon_n^0 : A_{n-1} \to A_{n-2}$ in $\mathfrak{V}^{op}$ is defined to be the homomorphism

$$\varepsilon^0 : A^{n-2} \to A^{n-1}; (a_0, \ldots, a_{n-3}) \mapsto (0, a_0, \ldots, a_{n-3}). \tag{3.3}$$

Finally, the face morphism $\varepsilon_n^{n-1} : A_{n-1} \to A_{n-2}$ in $\mathfrak{V}^{op}$ is defined to be the homomorphism

$$\varepsilon^{n-1} : A^{n-2} \to A^{n-1}; (a_0, \ldots, a_{n-3}) \mapsto (a_0, \ldots, a_{n-3}, 0). \tag{3.4}$$

Theorem 3.1. *If $\mathfrak{V}$ is a pointed variety, then the pointed complex is a simplicial object in $\mathfrak{V}^{op}$.*

Proof. The simplicial identities in the pointed complex must be verified. By the duality between the face identities (2.3) and the degeneracy identities (2.4), the degeneracy identities

$$\delta_n^i \circ \delta_{n+1}^{j+1} = \delta_n^j \circ \delta_{n+1}^i \tag{3.5}$$

for the pointed complex of A reduce to the face identities (2.3) for the product complex of A. Similarly, most of the face identities

$$\varepsilon_n^i \circ \varepsilon_{n-1}^{j-1} = \varepsilon_n^j \circ \varepsilon_{n-1}^i \tag{3.6}$$

for the pointed complex of A (those not involving factors of the form ε_m^0 or ε_m^{m-1}) reduce to the degeneracy identities (2.4) for the product complex of A. If $j = 1$ in (3.6), then necessarily $i = 0$, and using (3.2), (3.3) one verifies

$$\begin{aligned}\varepsilon_n^0 \circ \varepsilon_{n-1}^0(a_0, \ldots, a_{n-3}) &= \varepsilon_n^0(0, a_0, \ldots, a_{n-3}) = (0, 0, a_0, \ldots, a_{n-3}) \\ &= \varepsilon_n^1(0, a_0, \ldots, a_{n-3}) = \varepsilon_n^1 \circ \varepsilon_{n-1}^0(a_0, \ldots, a_{n-3})\end{aligned}$$

as required for satisfaction of (3.6) in this case. The verifications of the other exceptional cases of (3.6) are similar.

It remains to check the mixed identities

$$\delta_n^i \circ \varepsilon_{n+1}^j = \begin{cases} \varepsilon_n^j \circ \delta_{n-1}^{i-1}, & \text{if } j < i, \\ \varepsilon_n^{j-1} \circ \delta_{n+1}^i, & \text{if } j > i+1, \\ 1, & \text{otherwise} \end{cases} \tag{3.7}$$

in the pointed complex. As for the face identities, most of the cases reduce to the corresponding identities (2.5) for the product complex. If $j = n$ and $i < n - 1$, one verifies using (3.1) and (3.4) that

$$\begin{aligned}\delta_n^i \circ \varepsilon_{n+1}^n(a_0, \ldots, a_{n-2}) &= \delta_n^i(a_0, \ldots, a_{n-2}, 0) \\ &= (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n-2}, 0) \\ &= \varepsilon_n^{n-1}(a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n-2}) \\ &= \varepsilon_n^{n-1} \circ \delta_{n+1}^i(a_0, \ldots, a_{n-2}).\end{aligned}$$

For $i = n - 1$, one obtains

$$\delta_n^{n-1} \circ \varepsilon_{n+1}^n(a_0, \ldots, a_{n-2}) = \delta_n^{n-1}(a_0, \ldots, a_{n-2}, 0) = (a_0, \ldots, a_{n-2}),$$

as required for satisfaction of (3.7) in this case. Treatment of the other exceptional cases of (3.7) is similar. ∎

4. Mal'tsev identities for pointed algebras

Following the establishment in Theorem 3.1 of the pointed complex belonging to each algebra in a pointed variety, one may now study the satisfaction of the simplicial Mal'tsev identities in these complexes. The first result shows that the original 2-dimensional simplicial Mal'tsev identity (2.6) in pointed varieties serves to characterize Jónsson-Tarski varieties.

Theorem 4.1. *A pointed variety $\mathfrak{V}$ is a Jónsson-Tarski variety if and only if there is a ternary derived operation P such that the simplicial Mal'tsev identity* (2.6) *holds in the pointed complex of each member A of $\mathfrak{V}$.*

Proof. Applied to an element x of A, the simplicial Mal'tsev identity

$$P\left(\varepsilon_3^0, \varepsilon_3^2 \circ \varepsilon_2^0 \circ \delta_1^0, \varepsilon_3^2\right) = \varepsilon_3^1$$

becomes

$$\begin{aligned}(P(0,0,x), P(x,0,0)) &= P((0,x),(0,0),(x,0)) \\ &= P\left(\varepsilon_3^0(x), \varepsilon_3^2 \circ \varepsilon_2^0 \circ \delta_1^0(x), \varepsilon_3^2(x)\right) \\ &= P\left(\varepsilon_3^0, \varepsilon_3^2 \circ \varepsilon_2^0 \circ \delta_1^0, \varepsilon_3^2\right)(x) \\ &= \varepsilon_3^1(x) = (x,x),\end{aligned}$$

whose two components are

$$P(0,0,x) = x \text{ and } P(x,0,0) = x. \tag{4.1}$$

If $\mathfrak{V}$ is a Jónsson-Tarski variety, then (4.1) holds with

$$P(x,y,z) = x + (y + z).$$

Conversely, suppose that (4.1) holds. By analogy with (1.3), define

$$x + y = P(x,0,y).$$

Then (4.1) reduces to the Jónsson-Tarski identities (1.2). ∎

In contrast with the direct characterization of Jónsson-Tarski algebras given by Theorem 4.1, the issue of satisfaction of the higher-dimensional simplicial Mal'tsev identities in the pointed complex of a member of a pointed variety is more involved. On the one hand, these identities are sufficient, but not

necessary, for the variety to be a Jónsson-Tarski variety. On the other hand, they are necessary, but not sufficient, for the variety to be a Goldie variety.

Lemma 4.2. *A derived ternary operation P of a pointed variety $\mathfrak{V}$ satisfies each higher-dimensional simplicial Mal'tsev identity* (2.8) *in the pointed complex of each algebra A of $\mathfrak{V}$ if and only if the identities*

$$P(0,0,x) = x \tag{4.2}$$

and

$$P(x,y,y) = x \tag{4.3}$$

are satisfied in $\mathfrak{V}$.

Proof. Applied to an element $(a_0, \ldots, a_{n-3})$ of A^{n-2}, the higher-dimensional simplicial Mal'tsev identity

$$P\left(\varepsilon^0_n, \varepsilon^2_n \circ \varepsilon^0_{n-1} \circ \delta^0_{n-2}, \varepsilon^2_n\right) = \varepsilon^1_n$$

becomes

$$\begin{aligned}
&(P\ (0,0,a_0), P(a_0,a_1,a_1), P(a_1,a_1,a_1), \ldots, P(a_{n-3},a_{n-3},a_{n-3}))\\
&\quad = P((0,a_0,a_1,\ldots,a_{n-3}), (0,a_1,a_1,\ldots,a_{n-3}), (a_0,a_1,a_1,\ldots,a_{n-3}))\\
&\quad = P\left(\varepsilon^0_n(a_0,\ldots,a_{n-3}), \varepsilon^2_n \circ \varepsilon^0_{n-1} \circ \delta^0_{n-2}(a_0,\ldots,a_{n-3}), \varepsilon^2_n(a_0,\ldots,a_{n-3})\right)\\
&\quad = P\left(\varepsilon^0_n, \varepsilon^2_n \circ \varepsilon^0_{n-1} \circ \delta^0_{n-2}, \varepsilon^2_n\right)(a_0,\ldots,a_{n-3})\\
&\quad = \varepsilon^1_n(a_0,\ldots,a_{n-3}) = (a_0,a_0,a_1,\ldots,a_{n-3}),
\end{aligned}$$

whose components are

$$P(0,0,a_0) = a_0,\ P(a_0,a_1,a_1) = a_0,\ P(a_i,a_i,a_i) = a_i \text{ for } 1 \leq i \leq n-3. \tag{4.4}$$

Note that the first two identities of (4.4) are (4.2) and (4.3). Conversely, if (4.2) and (4.3) are satisfied, then (4.3) yields the idempotence of P which completes the list (4.4) of identities in $\mathfrak{V}$. ∎

Theorem 4.3. *In a pointed variety $\mathfrak{V}$, the existence of a ternary derived operation P such that the higher-dimensional simplicial Mal'tsev identities* (2.8) *are satisfied in the pointed complex of each member A of $\mathfrak{V}$ is sufficient, but not necessary, for $\mathfrak{V}$ to be a Jónsson-Tarski variety.*

Proof. If (4.2) and (4.3) hold, then (4.1) follows, so that $\mathfrak{V}$ is a Jónsson-Tarski algebra.

Conversely, consider the variety $\mathfrak{V}$ of commutative monoids, certainly a Jónsson-Tarski variety. Each derived ternary operation P of $\mathfrak{V}$, as an element of the free $\mathfrak{V}$-algebra on the three-element generating set $\{x, y, z\}$, has the normal form

$$ax + by + cz,$$

with natural numbers a, b, c, inside the free abelian group on $\{x, y, z\}$. Then (4.2) would force $c = 1$, contradicting the equation $b = c = 0$ given by (4.3). ■

Theorem 4.4. *In a pointed variety $\mathfrak{V}$, the existence of a ternary derived operation P such that the higher-dimensional simplicial Mal'tsev identities* (2.8) *are satisfied in the pointed complex of each member A of $\mathfrak{V}$ is necessary, but not sufficient, for $\mathfrak{V}$ to be a Goldie variety.*

Proof. If $\mathfrak{V}$ is a Goldie variety with Mal'tsev operation P, then the identity (4.3) is just the right hand side of (1.1), while in a pointed variety (4.2) is a consequence of the left hand side of (1.1).

Conversely, consider the set $\mathbb{N}$ of natural numbers, equipped with a nullary operation selecting 0, the binary operation $+$ of ordinary addition, and a binary operation

$$x \smallsetminus y = \begin{cases} y, & \text{if } x < y, \\ x - y, & \text{otherwise} \end{cases}$$

called *pseudosubtraction*. Consider the derived ternary operation

$$P(x, y, z) = (x + y) \smallsetminus z.$$

Then P satisfies (4.2) and (4.3), so the higher-dimensional simplicial Mal'tsev identities (2.8) are satisfied in the pointed complex of each member A of the variety $\mathfrak{V}$ generated by the algebra $(\mathbb{N}, 0, +, \smallsetminus)$.

It will now be shown that $\mathfrak{V}$ is not a Goldie variety. Consider the identical embedding of the (monoid reduct of the) fourth power $\mathbb{N}^4$ of the algebra

$(\mathbb{N}, 0, +, \smallsetminus)$ into the abelian group $\mathbb{Z}^4$. Let V be the kernel congruence of the operation of integer subtraction, considered as a subgroup of $\mathbb{Z}^4$. Then the intersection W of V with $\mathbb{N}^4$ is a subalgebra of $(\mathbb{N}, 0, +, \smallsetminus)^4$, since the values of the basic operations of $\mathfrak{V}$ are obtained either by group operations or by projections performed on their vector of arguments. Moreover, W is a reflexive subalgebra of $(\mathbb{N}^2)^2$, since V is a congruence on $\mathbb{Z}^2$. If $\mathfrak{V}$ were a Goldie variety, then W would be a congruence on $\mathbb{N}^2$ ([11], Proposition 143), having the diagonal $\widehat{\mathbb{N}}$ as a congruence class. The algebra $\mathbb{N}$ would then be central ([11], p. 43), and so there would be an isomorphism

$$\mathbb{N} \to \mathbb{N}^2/\widehat{\mathbb{N}} \; ; \; n \mapsto (0, n)^W \tag{4.5}$$

([11], 414). But (4.5) does not surject, since $(1, 0)^W$ does not lie in its image. ∎

5. Concluding remarks

The Mal'tsev identities (1.1) are equivalent to simplicial identities in the product complex (Proposition 2.1, Corollary 2.2). For a pointed variety, the Jónsson-Tarski identities (1.2) are equivalent to the 2-dimensional simplicial Mal'tsev identity in each pointed complex (Theorem 4.1). The higher-dimensional Mal'tsev identities in each pointed complex characterize a class of pointed varieties lying properly between the classes of Jónsson-Tarski and Goldie varieties (Theorems 4.3, 4.4).

Given the common framework for Mal'tsev and Jónsson-Tarski algebras, one may begin the programme of unifying the two theories. For example, the extension theory for Mal'tsev varieties ([11], Ch. 6) should translate readily to Jónsson-Tarski varieties. It may also prove fruitful to take other Mal'tsev conditions (such as those for modularity of the congruence lattice as summarised nicely in [14], or those for higher-order permutability of congruences [3] [6]), translate them to simplicial form, and then interpret them in pointed complexes of pointed varieties.

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