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EFFECT ALGEBRAS AND RING-LIKE STRUCTURES

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Abstract

The dichotomic physical quantities, also called propositions, can be naturally associated to maps of the set of states into the real interval [0,1]. We show that the structure of effect algebra associated to such maps can be represented by quasiring structures, which are a generalization of Boolean rings, in such a way that the ring operation of addition can be non-associative and the ring multiplication non-distributive with respect to addition. By some natural assumption on the effect algebra, the associativity of the ring addition implies the distributivity of the lattice structure corresponding to the effect algebra. This can be interpreted as another characterization of the classicality of the logical systems of propositions, independent of the characterizations by Bell-like inequalities.

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1. Introduction

States and observables are typical ingredients of the description of a physical system. We shall adopt the approach in which the states are taken as primitive elements and we write S for the set they form. It is then physically natural to view an observable taking values in some outcome space Ξ as a map from the set S into the family of probability measures on Ξ. When the observable is dichotomic, i.e. has only two outcomes, then it becomes uniquely specified by the probability assigned by each state to one of the two outcomes: in other words a two-valued observable corresponds to a function from S into [0,1]. The requirements that S is convex and the maps associated to the observables are affine appear physically motivated: when these requirements are included the maps of S into [0,1] are generally called effects, when they are not explicitly included these maps will be called S-probabilities. The notion of S-probability, or state-supported probability, has been introduced and developed by the present authors to characterize classical and non-classical probabilities in the framework of Bell-type inequalities [1,2].

In this paper we would like to present some interrelations among different frames concerning models for observables and states: in particular, the frame of effect algebras and the frame of ring-like structures.

The theory of effect algebras has been used recently by many authors in studies on the mathematical foundations of quantum mechanics. The notion of effect algebra is sufficiently general to encompass the traditional order structure of the two-valued observables associated with a physical system (Boolean algebras in the classical case and orthomodular posets in the quantum one), but is sufficiently rich to endow the relation between the physically relevant notions of states and observables. Effect algebras have a large bibliography, let us mention [3,4] also for further quotations.

The theory of ring-like structures (or generalized Boolean quasirings, GBQRs) has been developed by D. Dorninger, H. Länger and M. Mączyński in a series of papers: reference [5] is the most recent one related to the present paper.

We will show that these frames can be unified in one approach which proves to be useful for characterization of classical and non-classical systems. In Section 2 we develop the theory of S-probabilities as effect algebras. We give examples of classical and nonclassical systems of S-probabilities and we investigate the notion of reflexivity related to Gleason's theorem. In Section 3 we recall the definitions of ring-like structures generalizing Boolean rings and we describe some relevant properties of the theory. Section 4 is devoted to the main result of this paper, namely the representation theorem for effect algebras of S-probabilities as ring-like structures. We characterize the classicality of the system by the associativity of the ring operation of addition, and we come to a logical operation which corresponds to the *exclusive or*. Notice that the traditional characterization of classicality by means of the distributivity involves two logical operations. In Section 5 we draw some conclusions from our characterization and we discuss physical interpretations.

2. S-probabilities as effect algebras

Let us recall that an *effect algebra* is a set $\mathcal{E}$ containing two special elements o, e, and equipped with a partial binary operation $\oplus$ satisfying the properties:

(i) $a \oplus b = b \oplus a$,

(ii) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,

(iii) for every $a \in \mathcal{E}$ there is in $\mathcal{E}$ a unique element, denoted $e \ominus a$, such that $a \oplus (e \ominus a)$ is defined and equals e,

(iv) if $a \oplus e$ is defined then $a = o$.

Let S be a nonempty set, to be interpreted in the sequel as the set of states of a physical system. We write $P(S)$ for the set of all functions from S into the real interval [0,1]; following the terminology of [1] the elements of $P(S)$ will be called *S-probabilities*. The zero and the unit functions will be denoted by 0 and 1. Let $f, g \in P(S)$: we can define a partial order relation in $P(S)$ by $f \leq g$ if and only if $f(\alpha) \leq g(\alpha)$ for all $\alpha \in S$. If the function $f + g$ defined as the sum of real functions belongs to $P(S)$, i.e. if $f + g \leq 1$, then we say that f and g are *orthogonal* and we write $f \perp g$. Hence the set $P(S)$ can be equipped with a partial binary operation $\oplus$ by defining, for $f, g, h \in P(S)$,

$$f \oplus g = h \text{ iff } f \perp g \text{ and } f + g = h.$$

It is easy to verify that $(P(S), \oplus)$ satisfies the properties (i)–(iv) listed above, with $o = 0$ and $e = 1$: thus $(P(S), \oplus)$ is an effect algebra. This fact ensures (see [3]) the possibility of getting probability measures on $P(S)$: let us recall that a map $m : P(S) \to [0,1]$ is a (σ-additive) probability measure if $m(0) = 0$, $m(1) = 1$ and, for every countable and orthogonal sequence $f_1, f_2, ... \in P(S)$ such that $\oplus_i f_i$ exists in $P(S)$, we have

$$m(\oplus_i f_i) = \sum_i m(f_i).$$

Every element α of S induces a probability measure m_α on $P(S)$ defined by

$$m_\alpha(f) := f(\alpha) \text{ for all } f \in P(S);$$

it is indeed immediate to show that m_α meets the properties of a probability measure. We denote by $\overline{S}$ the set of all probability measures on $P(S)$ induced by S, and by $\mathbf{S}(P(S))$ the set of all the probability measures on $P(S)$: of course, $\overline{S} \subseteq \mathbf{S}(P(S))$. Notice that $P(S)$ is separated by $\mathbf{S}(P(S))$ since it is obviously separated by $\overline{S}$.

Analogous notions can be extended to every subset $\mathcal{E}(S)$ of $P(S)$ which is an effect algebra with respect to the $\oplus$ operation, so that $(\mathcal{E}(S), \oplus)$ is a *sub-effect algebra* of $(P(S), \oplus)$. We denote by $\overline{S}_{\mathcal{E}(S)}$ the restriction of $\overline{S}$ to $\mathcal{E}(S)$, and by $\mathbf{S}(\mathcal{E}(S))$ the set of all the probability measures on $\mathcal{E}(S)$. We have $\overline{S}_{\mathcal{E}(S)} \subseteq \mathbf{S}(\mathcal{E}(S))$ and $\mathcal{E}(S)$ is obviously separated by $\overline{S}_{\mathcal{E}(S)}$, hence also by $\mathbf{S}(\mathcal{E}(S))$.

We will focus attention on the following definition:

Definition 2.1. The effect algebra $(\mathcal{E}(S), \oplus)$ is said to be *reflexive* if $\overline{S}_{\mathcal{E}(S)} = \mathbf{S}(\mathcal{E}(S))$.

Notice that in the context of standard quantum mechanics the property of reflexivity is related to Gleason's theorem. Indeed, let $\mathcal{H}$ be a complex separable Hilbert space and let S be the set of the quantum states, namely the set of all density operators (the positive trace-one operators) of $\mathcal{H}$. For every projector P of $\mathcal{H}$ we define a function $f_P : S \to$[0,1] by

$$f_P(D) := \mathrm{Tr}(DP) \text{ for all } D \in S$$

and we take the set $\mathcal{E}(S) := \{f_P \mid P \text{ a projector}\}$. Since $\mathcal{E}(S)$, considered as partially ordered orthocomplemented set, is isomorphic to the lattice of projections of $\mathcal{H}$, Gleason theorem ensures that every probability measure on $\mathcal{E}(S)$ comes from a density operator, that is from an element of S. Thus we get the reflexivity property $\overline{S}_{\mathcal{E}(S)} = \mathbf{S}(\mathcal{E}(S))$.

Also the context of standard classical mechanics fits with the reflexivity property. To the physical system under discussion a measurable space Ω (the "phase space") is attached whose elements are interpreted as the pure states. The set S of all states is now identified with the set $M_1^+(\Omega)$ of all the probability measures on the Boolean algebra $\mathcal{B}(\Omega)$ of the (measurable) subsets of Ω: notice that this structure of S mirrors the unique decomposition of non-pure states into pure ones, typical of the classical case. However, to obtain the reflexivity property for $\mathcal{E}(S)$, we have to include in $M_1^+(\Omega)$ also the finitely additive probability measures, which are not necessarily σ-additive.

This means that here $M_1^+(\Omega)$ includes also all homomorphisms from $\mathcal{B}(\Omega)$ onto the two-element Boolean algebra $\{0,1\}$: each $a \in \mathcal{B}(\Omega)$ determines a function $f_a : S \to [0,1]$ defined by $f_a(\alpha) := \alpha(a)$ for all $\alpha \in M_1^+(\Omega)$. Let

$$\mathcal{E}(S) := \{f_a \mid a \in \mathcal{B}(\Omega)\}.$$

The set $\mathcal{E}(S)$ equipped with the real function ordering and with the orthocomplementation $'$ defined by $f_a' := 1 - f_a$ is isomorphic to the Boolean algebra $\mathcal{B}(\Omega)$. Indeed, the correspondence $a \mapsto f_a$ is clearly a homomorphism of $\mathcal{B}(\Omega)$ onto $\mathcal{E}(S)$: to show that it is one-to-one we have only to observe that if $a \neq b$, $a, b \in \mathcal{B}(\Omega)$, then there exists $\alpha \in M_1^+(\Omega)$ such that $\alpha(a) \neq \alpha(b)$, hence $f_a(\alpha) \neq f_b(\alpha)$ so that $f_a \neq f_b$. This follows from the Stone representation theorem for Boolean algebras, which states that for every $a \neq b$, with $a \neq 0$, there is a two-valued homomorphism $\alpha \in M_1^+(\Omega)$ such that $\alpha(a) = 1$, $\alpha(b) = 0$, hence $f_a(\alpha) \neq f_b(\alpha)$ (see, e.g., [6]). Due to the isomorphism between $\mathcal{E}(S)$ and $\mathcal{B}(\Omega)$ every probability measure on $\mathcal{E}(S)$ can be identified with a probability measure on $\mathcal{B}(\Omega)$, i.e. with an element of $S = M_1^+(\Omega)$. This implies the reflexivity property $\overline{S}_{\mathcal{E}(S)} = \mathbf{S}(\mathcal{E}(S))$.

We see from the above discussion that in standard classical mechanics we do not have a counterpart of Gleason's theorem for σ-additive probability measures. For finitely additive probability measures the counterpart of Gleason's theorem is the Stone representation theorem as said above. This agrees with the fact that the logic of classical mechanics can be completely determined by the two-valued (classical) logic, whereas in quantum mechanics the logic is not classical and not two-valued.

Let us remark that the reflexivity property is related to the convexity of the set S of states, as specified by the following theorem.

Theorem 2.1. *If S admits an effect algebra $\mathcal{E}(S)$ of S-probabilities which separates S and is reflexive then S is convex.*

Proof. Since the set $\mathbf{S}(\mathcal{E}(S))$ of all the probability measures on $\mathcal{E}(S)$ is obviously convex, the reflexivity property $\overline{S}_{\mathcal{E}(S)} = \mathbf{S}(\mathcal{E}(S))$ implies the convex structure of $\overline{S}_{\mathcal{E}(S)}$. If $\mathcal{E}(S)$ separates S then the correspondence between $\overline{S}_{\mathcal{E}(S)}$ and S is one-to-one because $\alpha \neq \beta$, $\alpha, \beta \in S$, implies the existence of $f \in \mathcal{E}(S)$ such that $f(\alpha) \neq f(\beta)$, hence $m_\alpha(f) \neq m_\beta(f)$, i.e. $m_\alpha \neq m_\beta$. Hence the convex structure of $\overline{S}_{\mathcal{E}(S)}$ induces the convex structure of S. ∎

As a consequence of the above theorem a set S which is not convex does not admit an effect algebra $\mathcal{E}(S)$ of S-probabilities which has the reflexivity property and separates S.

Notice that the convexity of S is in general not sufficient to make the effect algebra $\mathcal{E}(S)$ reflexive. The next theorem provides an example of additional conditions, suggested by the frame of quantum mechanics, that allow the recovering of the reflexivity property.

Theorem 2.2. *Let $\mathcal{E}(S)$ be an effect algebra of S-probabilities, which is assumed to be a partially ordered orthocomplemented set with respect to the natural order of real functions and the orthocomplementation $f' = 1 - f$. Assume that $\mathcal{E}(S)$ separates the elements of S, and assume that S is convex with respect to the convex structure given by the isomorphism $S \cong \overline{S}$. Let $V(S)$ be the affine space generated by S with respect to this convex structure. Assume next that $\mathcal{E}(S)$ is the restriction to S of the family of functions*

$$\overline{f} : V(S) \to R^+$$

satisfying the condition

$$\overline{f}(x+y) + \overline{f}(x-y) = 2\overline{f}(x) + 2\overline{f}(y) \quad \text{for all } x, y \in V(S) \tag{1}$$

(we will write f for the restriction to S of $\overline{f}$). Then $\mathcal{E}(S)$ is reflexive, i.e. we have $\mathbf{S}(\mathcal{E}(S)) = \overline{S}$.

Proof. By a theorem of von Neumann, condition (1) implies that there exists an inner product on $V(S)$, say $(.,.)$, and a family of projectors such that for every $f \in \mathcal{E}(S)$ we have $\overline{f}(x) = (Px, x)$ for some projector P and for all $x \in V(S)$. By taking the completion of $V(S)$ with respect to the norm defined by $(.,.)$, i.e. $\|x\| = (x,x)^{1/2}$, we obtain a Hilbert space H such that every projector P is positive and hence self-adjoint. The partially ordered orthocomplemented set $\mathcal{E}(S)$ is now isomorphic to $L(H)$, the lattice of projections of H. As mentioned previously, by Gleason's theorem the set $\mathcal{E}(S) \cong L(H)$ is reflexive, so we have $\mathbf{S}(\mathcal{E}(S)) = \overline{S}$. ∎

We see from Theorem 2.2 that the essential properties which are to be attributed to S-probabilities arising from a quantum mechanical system are:

1° The convex structure of the set of states S;

2° The fact that every S-probability is in fact a quadratic functional on the affine space $V(S)$ generated by the convex structure of S (quadratic functionals are defined as positive-valued functionals on $V(S)$ with the property (1), they are clearly non-additive on $V(S)$).

This is in contradiction to classical mechanics, where S-probabilities may be additive in the affine space generated by the states.

Let us finally recall the conditions assuring that a set of S-probabilities takes the classical structure of a Boolean algebra: they are expressed by the following theorem proved in [1]:

A set $\mathcal{E}(S)$ of S-probabilities is a Boolean algebra (with respect to the natural ordering of real functions and the complementation $f \mapsto f' = 1 - f$) if and only if the following conditions hold:

(i) $0 \in \mathcal{E}(S)$,

(ii) $f \in \mathcal{E}(S)$ implies $1 - f \in \mathcal{E}(S)$,

(iii) if $f_1, f_2, f_3 \in \mathcal{E}(S)$ and $f_i + f_j \leq 1$ for $i \neq j$, then $f_1 + f_2 + f_3 \in \mathcal{E}(S)$ (we express this fact by saying that the triple (f_1, f_2, f_3) has the triangular property),

(iv) for every $f_1, f_2 \in \mathcal{E}(S)$ there are $g_1, g_2, g_3 \in \mathcal{E}(S)$ such that they form a triple having the triangular property and $f_1 = g_1 + g_2$ and $f_2 = g_2 + g_3$.

Clearly, if the conditions of this theorem are met, then $(\mathcal{E}(S), \oplus)$ is also an effect algebra with $f_1 \vee f_2 = g_1 \oplus g_2 \oplus g_3$, $f_1 \wedge f_2 = g_2$.

3. Ring-like structures generalizing Boolean rings (GBQRs)

First we recall some definitions and properties of ring-like structures introduced in [5].

Definition 3.1. An algebra $(R, +, \cdot)$ of type (2,2) (with two binary operations $+$ and $\cdot$) is called a *generalized Boolean quasiring* (GBQR) if there are two elements $0, 1 \in R$ such that the following axioms hold:

(1) $x + y = y + x$,

(2) $0 + x = x$,

(3) $(xy)z = x(yz)$,

(4) $xy = yx$,

(5) $xx = x$,

(6) $x0 = 0$,

(7) $x1 = x$,

(8) $1 + (1 + xy)(1 + x) = x$

for all $x, y, z \in R$.

A weaker structure can be obtained if the binary operation $+$ is assumed to become a partial operation denoted by $\oplus$: omitting then the axiom (1) we come to the following definition:

Definition 3.2. A partial algebra $(R, \oplus, \cdot)$ of type (2, 2) (with a partial binary operation $\oplus$ and a total binary operation $\cdot$) is called a *partial generalized Boolean quarising* (pGBQR) if there are $0, 1 \in R$ such that $\oplus : \{0, 1\} \times R \to R$, $\cdot : R \times R \to R$ and the following axioms hold:

(2′) $0 \oplus x = x$,

(3′) $(xy)z = x(yz)$,

(4′) $xy = yx$,

(5′) $xx = x$,

(6′) $x0 = 0$,

(7′) $x1 = x$,

(8′) $1 \oplus (1 \oplus xy)(1 \oplus x) = x$

for all $x, y, z \in R$.

If in a pGBQR $(R, \oplus, \cdot)$ we define

$$x \vee y := 1 \oplus (1 \oplus x)(1 \oplus y), \quad x \wedge y := xy, \quad x' := 1 \oplus x$$

for all $x, y \in R$, then $(R, \vee, \wedge, ', 0, 1)$ is a bounded lattice with an involutory antiautomorphism . Conversely, if we have a bounded lattice $(R, \vee, \wedge, ', 0, 1)$ with an involutory antiautomorphism and we define

$$0 \oplus x := x, \quad 1 \oplus x := x', \quad xy := x \wedge y$$

for all $x, y \in R$, then we obtain a pGBQR $(R, \oplus, \cdot)$. We denote the lattice $(R, \vee, \wedge, ', 0, 1)$ associated with the $(R, \oplus, \cdot)$ by $L(R)$.

A Boolean ring $(R, +, \cdot)$ is clearly also a GBQR. Namely, the axiom (8) follows from the following axioms holding in a Boolean ring

(i) $x + (y + z) = (x + y) + z$,

(ii) $x(y + z) = xy + xz$,

(iii) $x + x = 0$.

We can verify this by a direct algebraic computation. Hence axiom (8) provides a weakening of the axioms of a Boolean ring, this is why we call the system a quasiring. Because we still have the axiom (5), the product idempotency characteristic for a Boolean ring, we call our quasiring Boolean. Axiom (8) was selected in such a way that it still allows us to preserve the lattice structure leading to the lattice $L(R)$, but the system is not associative with respect to $+$ and not distributive like in Boolean rings.

Every Boolean algebra becomes a Boolean ring if we define $+$ to be the symmetric difference

$$x + y = x \dot{-} y = x \bigtriangleup y := (x \wedge y') \vee (x' \wedge y)$$

and

$$xy := x \wedge y.$$

Note that in a Boolean algebra the symmetric difference can also be defined equivalently by

$$x \dot{-} y = x \bigtriangledown y := (x \vee y) \wedge (x \wedge y)'.$$

These two definitions of the symmetric difference are equivalent in a Boolean ring but not in a partial generalized Boolean quasiring. In fact, if we have a pGBQR $(R, \oplus, \cdot)$, then we can extend it to a GBQR by putting, for all $x \in R$,

$$0 + x = x + 0 := 0 \oplus x,$$

$$1 + x = x + 1 := 1 \oplus x,$$

and, for all $x, y \in R \backslash \{0, 1\}$,

$$x + y = y + x := z$$

arbitrarily with any $z \in R$. Hence, we see that an extension of a pGBQR to a GBQR is not unique because we can define $+$ to be completely arbitrary in the domain $R \backslash \{0, 1\}$, only preserving the commutativity of $+$.

There are two canonical extensions of $\oplus$ to a total operation:

$$x +_1 y := 1 \oplus (1 \oplus x(1 \oplus y))(1 \oplus (1 \oplus x)y),$$

$$x +_2 y := (1 \oplus (1 \oplus x)(1 \oplus y))(1 \oplus xy).$$

In a Boolean ring they take the form

$$x +_1 y = x \bigtriangleup y, \quad x +_2 y = x \bigtriangledown y$$

so that $+_1$ and $+_2$ become identical while in a pGBQR we only have the relation

$$x +_1 y \leq x +_2 y,$$

where the partial order in $(R, \oplus, \cdot)$ is defined by

$$x \leq y \Leftrightarrow xy = x$$

and coincides with the partial order in the lattice $L(R)$. Observe also that in a Boolean ring we have

$$x' \bigtriangleup y' = x \bigtriangleup y$$

and

$$x \wedge y' \leq x \bigtriangleup y \leq x \vee y = (x' \wedge y')'.$$

In [5] it was shown that if we preserve the above properties for an extended $+$ assuming that

(i) $x + y = x' + y'$

and

(ii) $xy' \leq x + y \leq (x'y')'$,

then we get

$$x +_1 y \leq x + y \leq x +_2 y.$$

Having in mind that in a Boolean ring $+_1$ and $+_2$ are equivalent expressions of the logical operation of "exclusive or", the above property suggests to interprete $+$, also in the generalized case, as such a logical operation. Hence, $p + q$ will mean that "p or q but not both". Thus, the operator $+$ appears, from a logical point of view, to be more natural than the traditional lattice operators $\wedge$ and $\vee$ used in the quantum logic approach. We come back to this interpretation in the last section.

4. Representation of effect algebras of S-probabilities as GBQR's

In this section we will show that an effect algebra of S-probabilities can be interpreted as a pGBQR, and consequently as a GBQR, if we assume an additional requirement whose formulation needs some auxiliary definitions. First let us observe that every effect algebra of S-probabilities $(\mathcal{E}(S), \oplus)$ is in a natural way a partially ordered set $(\mathcal{E}(S), \leq)$ if we define the partial order by

$$f \leq g \Leftrightarrow f(\alpha) \leq g(\alpha) \ \forall \alpha \in S, \quad f, g \in \mathcal{E}(S),$$

i.e. $\leq$ is defined pointwise. We now have the following definitions.

Definition 4.1. Let $(L, \leq)$ be a partially ordered set. We define a partial commutative binary operation $\sqcap$ on L by

$$a \sqcap b = a = b \sqcap a \text{ whenever } a \leq b.$$

Definition 4.2. Let $\cdot$ be a total commutative binary operation on $(L, \leq)$ which is an extension of $\sqcap$. We say that $\cdot$ is a *perfect extension* of $\sqcap$ if

$$a \cdot b = a \Rightarrow a \sqcap b \text{ exists and } a \sqcap b = a.$$

Note that $a \sqcap b = a$ implies $ab = a$ so in fact we have for a perfect extension

$$a \cdot b = a \Leftrightarrow a \sqcap b = a.$$

To simplify the notation we omit, as usual, the $\cdot$ in the next formulas.

Definition 4.3. The extension $\cdot$ of $\sqcap$ is said to be *absorbing* if for all $a, b \in L$ we have

$$(ab)a = a(ab) = ab.$$

Let now $(\mathcal{E}, \oplus)$ be an effect algebra of S-probabilities (to simplify the notation we write $\mathcal{E}$ instead of $\mathcal{E}(S)$ in the sequel of this section).

Definition 4.4. The effect algebra $(\mathcal{E}, \oplus)$ is said to have the *extension property* if there exists a perfect extension $\cdot$ of the partial operation $\sqcap$ which is commutative, associative and absorbing.

First we have the following lemma analogous to a lemma of [5]:

Lemma 4.1. *If the extension property holds in $(\mathcal{E}, \oplus)$, then the extension $\cdot$ introduced by $\sqcap$ is unique and we have $ab = \inf_{\leq}(a, b)$.*

Proof. Let $a, b, \in \mathcal{E}$. Since $\cdot$ is absorbing we have $ab \leq a$. Using the commutativity of $\cdot$ we obtain $ab = ba \leq b$, i.e. $ab \leq a, b$. Let $c \in \mathcal{E}$ with $c \leq a$ and $c \leq b$. Hence $c \sqcap a = c = c \sqcap b$. Since $\cdot$ is an extension of $\sqcap$, we have further $ca = cb = c$. This implies $(ca)(cb) = cc$. By commutativity and associativity of $\sqcap$, it follows that $(cc)(ab) = cc$. Since $cc = c \sqcap c = c$, we infer that $c(ab) = c$. Because $\sqcap$ is perfect, this implies $c \sqcap (ab) = c$, i.e. $c \leq ab$. Hence ab is uniquely defined as $ab = \inf_{\leq}(a, b)$. ∎

We can now state our main theorems.

Theorem 4.1. *Let $(\mathcal{E}, \oplus)$ be an effect algebra of S-probabilities having the extension property. We extend $\oplus$ to a partial binary operation $\uplus$ on $\mathcal{E}$ by defining*

$$a \uplus b = c \text{ if } a \oplus b = c, \quad 1 \uplus a = 1 - a \text{ for all } a \in \mathcal{E}$$

(note that $1 \oplus a$ exists only if $a = 0$, we then have $1 \oplus 0 = 1$ which agrees with $1 \uplus 0 = 1 - 0 = 1$). Then $(\mathcal{E}, \uplus, \cdot)$ is a pGBQR.

Proof. If suffices to show the validity of the axioms (2')–(8') of Definition 3.2 with $\oplus$ replaced by $\uplus$. Obviously (2')–(7') hold. If $x, y \in \mathcal{E}$, then

$$xy \leq x \Rightarrow x' \leq (xy)' \Rightarrow (xy)'x' = x' \Rightarrow [(xy)'x']' = x \Rightarrow 1 \uplus (1 \uplus xy)(1 \uplus x) = x$$

by use of Lemma 4.1 and of de Morgan's law with the involutory antiautomorphism $x' = 1 \uplus x$. ∎

We can extend $\uplus$ to a total operation $+$ on $\mathcal{E}$ by defining, for $a, b \in \mathcal{E}$,

$$a + b = \begin{cases} c, & \text{if } a \uplus b = c, \\ d, & \text{with an arbitrary } d, \text{ if } a \uplus b \text{ does not exist,} \end{cases}$$

in such a way that $+$ is commutative.

Then we have

Theorem 4.2. *$(\mathcal{E}, +, \cdot)$ is a GBQR extending the effect algebra $(\mathcal{E}, \oplus)$ of S-probabilities.*

The proof is obvious, because the extension of a pGBQR to a GBQR can be defined arbitrarilouly respecting the commutativity of $+$. ∎

However, there arises the question under what condition put on $+$ the lattice $L(\mathcal{E})$ associated with $(\mathcal{E}, +, \cdot)$ is distributive. First we have the following theorem characterizing the extension of $\oplus$ to $+$.

Theorem 4.3. *Let $(\mathcal{E}, \oplus)$ be an effect algebra of S-probabilities with the extension property. Assume that the extension of $(\mathcal{E}, \oplus)$ to a GBQR $(\mathcal{E}, +, \cdot)$ satisfies the following conditions*

(i) $x + y = (1 + x) + (1 + y)$,

(ii) $x(1 + y) \leq x + y \leq 1 + (1 + x)(1 + y)$

for all $x, y \in \mathcal{E}$.

Then $+$ is bounded by the canonical extensions $+_1$ and $+_2$ defined in Section 3, i.e. we have

(iii) $x +_1 y \leq x + y \leq x +_2 y$ *for all $x, y \in \mathcal{E}$.*

The proof of this theorem follows from Lemma 3.2 of [5] and therefore we omit it. Of course the canonical extensions $+_1$ and $+_2$ satisfy the conditions (i) and (ii). ∎

It turns out that the associativity of one of the canonical extensions makes the lattice $(L(\mathcal{E}), \vee, \wedge)$ distributive. Namely we have the following theorem:

Theorem 4.4. *Let $(\mathcal{E}, \oplus)$ be a non-degenerate algebra of S-probabilities with the extension property. Assume that a canonical extension $+ \in \{+_1, +_2\}$ is associative, i.e.*

$$x + (y + z) = (x + y) + z \text{ for all } x, y, z \in \mathcal{E}.$$

Then the lattice $(L(\mathcal{E}), \vee, \wedge)$ associated with $(\mathcal{E}, +, \cdot)$ is distributive.

The proof of this theorem follows from Theorem 4.3 of [5]. ∎

Let us recall that an effect algebra $(\mathcal{E}, \oplus)$ of S-probabilities is said to be *non-degenerate* if $f \perp f$ implies $f = 0$, the orthogonality relation in $(\mathcal{E}, \oplus)$ being defined as usual by $f + g \Leftrightarrow f + g \leq 1$. If $(\mathcal{E}, \oplus)$ is non-degenerate, then the orthogonal kernel $OK(\mathcal{E})$ defined by

$$OK(\mathcal{E}) = \{x \in \mathcal{E} \mid x \perp x\}$$

consists only of the 0 element, and hence it is distributive as required in Theorem 4.3 of [5]. This implies, by that theorem, that $L(\mathcal{E})$ is distributive. Hence Theorem 4.4 holds.

Notice that in general the distributivity of the lattice $(L(\mathcal{E}), \vee, \wedge)$ does not imply the ring distributivity of $(\mathcal{E}, +, \cdot)$. However, if $(\mathcal{E}, +)$ is a quasigroup then this implication holds. We have the following theorem which follows from Theorem 5.2 of [5]:

Theorem 4.5. *Let $(\mathcal{E}, \oplus)$ be an effect algebra of S-probabilities. Let $+$ be a canonical extension of $\oplus$ to a GBQR $(\mathcal{E}, +, \cdot)$. Assume that $(\mathcal{E}, +)$ is a quasigroup, i.e. for all $a, b \in \mathcal{E}$, the equation $a + x = b$ has a unique solution in $\mathcal{E}$. Then $(L(\mathcal{E}), \vee, \wedge)$ is a Boolean algebra and the quasiring $(\mathcal{E}, +, \cdot)$ is distributive.* ∎

5. Conclusions and physical interpretation

As we have mentioned in Section 2, we can refer to two standard examples of effect algebras of S-probabilities. First the effect algebra of S-probabilities defined on a Boolean algebra A where S is defined to be the set of all two-valued (also finitely additive) probability measures on A. In this case the effects are defined as maps $f : S \to [0, 1]$ such that for each $a \in A$ we have $f_a(\alpha) = \alpha(a)$ for every $\alpha \in S$. By the Stone representation theorem it is clear that the correspondence $a \to f_a$ establishes the isomorphism between A and $\mathcal{E}(S)$. Hence the operation $\oplus$ defined by $(f \oplus g)(\alpha) = f(\alpha) + g(\alpha)$ for all $\alpha \in S$ can be extended to the operation $+$ defined by $(f + g)(\alpha) = f(\alpha) + g(\alpha) - 2(f \wedge g)(\alpha)$, $\alpha \in S$, which is the symmetric difference on the Boolean algebra $(\mathcal{E}(S), \wedge, \vee)$. This extension is associative, hence $(\mathcal{E}(S), +, \cdot)$ is a Boolean ring. This corresponds to the classical frame.

We obtain another example of an effect algebra of S-probabilities by taking S to be the unit sphere of a Hilbert space H of dimension > 1 $S = S^1(H)$, and defining effects as the maps $f : S \to [0, 1]$, namely for each positive operator $0 \leq A \leq I$ we define f_A as

$$f_A(\alpha) = (A\alpha, \alpha) \text{ for all } \alpha \in S.$$

The partial operation $\oplus$ is defined by

$$(f \oplus g)(\alpha) = f(\alpha) + g(\alpha), \ \alpha \in S$$

for $f + g \leq 1$.

It is clear that this operation does not admit an extension to an associative canonical operation $+ \in \{+_1, +_2\}$. Indeed both extensions $+_1$ and $+_2$ are not associative. This follows from the fact that if we restrict $\mathcal{E}(H)$ to the effects induced by projections, then this restriction is isomorphic to the lattice of projections on H, which is known to be non-distributive, and consequently $+$ is not associative by Theorem 4.4. Observe that if we extend $\oplus$ to $+$, this extension does not correspond to the addition of functions. In particular, in the property

$$f + g = (1 + f) + (1 + g)$$

even in the case, where $f + g \leq 1$ (and hence when on the left-hand side $+$ corresponds to the addition of functions), $+$ on the right-hand side is not addition of functions, it is a binary operation extending $\oplus$. This is clear when we write

$$1 + f = 1 - f;$$

on the left-hand side we have $+$ as a binary operation, whereas on the right-hand side $-$ is the subtraction of functions.

Let us go back to our assumption in Theorem 4.1. The partial operation $\sqcap$ in $(\mathcal{E}, \oplus)$ means that if $a \leq b$ (i.e. if $a(\alpha) \leq b(\alpha)$ $\forall \alpha \in S$) then $a \sqcap b$ is defined by $a \sqcap b = a$ and corresponds to the classical "and". In this case classical and non-classical "and" coincide. The extension property means that we assume the possibility of extending this partial "and" to a total binary operation $\cdot$ which coincides with $\sqcap$ if $ab = a$ and has the absorbing property $(ab)a = ab$ of the classical "and". This in particular means that $\cdot$ is idempotent ($aa = a$). Lemma 4.1 shows that such an extension is unique: This means that the classical properties of "and" mentioned above determine uniquely a non-classical extension of it. On the other hand, the extension of the classical operation of "or" for orthogonal effects ($a \oplus b$ corresponds to "a or b") to a total binary operation $+$ of "exclusive or" is not unique, but we can always obtain the structure of a GBQR, as shown in Theorem 4.2. In general, this extension $+$ is not associative. The classicality of $+$ is characterized by properties (i) and (ii) of Theorem 4.3. Namely, (i) means that $+$ is invariant with respect to the negation (the measurement of "p exclusive or q" means the same as the measurement of "$(\neg p)$ exclusive or $(\neg q)$"). The condition (ii) means that for our general "exclusive or" we want to preserve the classical property that $p+q$ is between $p \wedge \neg q$ and

$\neg(\neg p \wedge \neg q)$. This means that $p \wedge \neg q$ should imply $p+q$, and $p+q$ should imply $\neg(\neg p \wedge \neg q)$ (i.e. $p \vee q$). Without these restrictions $+$ could be completely arbitrary. Theorem 4.3 shows that with these assumptions $+$ is bounded by the canonical extensions $+_1$ and $+_2$, which means that $p +_1 q \to p + q$ and $p + q \to p +_2 q$. Theorem 4.4 characterizes the distributivity (i.e. the classicality) of the lattice $(L(\mathcal{E}), \wedge, \vee)$.

This shows that the distributivity is implied by the associativity (with some natural additional assumption put on $\mathcal{E}$). Hence to show that we deal with a classical system, it is sufficient to show that $p+(q+r) = (p+q)+r$ for all $p, q, r \in \mathcal{E}$. The verification of this equality requires only 4 measurements $q+r$, $p+(q+r)$, $p+q$, $(p+q)+r$, whereas the verification of distributivity $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ requires 5 measurements $(q \vee r)$, $p \wedge (q \vee r)$, $p \wedge q$, $p \wedge r$, $(p \wedge q) \vee (p \wedge r)$.

Notice also that the associativity involves only one operation $+$, whereas the distributivity involves two operations $\wedge$ and $\vee$. Theorem 4.5 shows that the ring distributivity of $(\mathcal{E}, +, \cdot)$ is implied by the unique solvability of the equation $a + x = b$ when $+ = +_1$ or $+ = +_2$. In this case, this is equivalent to the fact that $(L(\mathcal{E}), \wedge, \vee)$ is a Boolean algebra, i.e. to the fact that we deal with a classical system.

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REFERENCES

[1] E.G. Beltrametti and M.J. Mączyński, *On a characterization of classical and non-classical probabilities*, J. Math. Phys. **32** (1991), 1280–1286.

[2] E.G. Beltrametti and M.J. Mączyński, *On the range of non-classical probability*, Rep. Math. Phys. **36** (1995), 195–213.

[3] E.G. Beltrametti and S. Bugajski, *Effect algebras and statistical theories*, J. Math. Phys. **38** (1997), 3020–3030.

[4] E.G. Beltrametti, S. Bugajski and V.S. Varadarajan, *Extensions of convexity models*, J. Math. Phys. **41** (2000), 1415–1429.

[5] D. Dorninger, H. Länger and M. Mączyński, *On ring-like structures induced by Mackey's probability function*, Rep. Math. Phys. **43** (1999), 499–515.

[6] R. Sikorski, *Boolean Algebras*, 2nd ed., Springer-Verlag, Berlin-Heidelberg-New York 1964.

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