ON ABSOLUTE RETRACTS AND ABSOLUTE CONVEX RETRACTS IN SOME CLASSES OF $\ell$-GROUPS

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Abstract

By dealing with absolute retracts of $\ell$-groups we use a definition analogous to that applied by Halmos for the case of Boolean algebras. The main results of the present paper concern absolute convex retracts in the class of all archimedean $\ell$-groups and in the class of all complete $\ell$-groups.

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1. Introduction

Retracts of abelian $\ell$-groups and of abelian cyclically ordered groups were investigated in [6], [7], [8].

Suppose that $\mathcal{C}$ is a class of algebras. An algebra $A \in \mathcal{C}$ is called an absolute retract in $\mathcal{C}$ if, whenever $B \in \mathcal{C}$ and $A$ is a subalgebra of $B$, then $A$ is a retract of $B$ (i.e., there is a homomorphism $h$ of $B$ onto $A$ such that $h(a) = a$ for each $a \in A$). Cf., e.g., Halmos [3].

Further, let $\mathcal{C}$ be a class of $\ell$-groups. An element $A \in \mathcal{C}$ will be called an absolute convex retract in $\mathcal{C}$ if, whenever $B \in \mathcal{C}$ and $A$ is a convex $\ell$-subgroup of $B$, then $A$ is a retract of $B$. 

Let \( \mathcal{G} \) and \( \text{Arch} \) be the class of all \( \ell \)-groups, or the class of all archimedean \( \ell \)-groups, respectively.

It is easy to verify (cf. Section 2 below) that for \( A \in \mathcal{G} \) the following conditions are equivalent:

\[(i) \ A \text{ is an absolute retract in } \mathcal{G};\]
\[(ii) \ A \text{ is an absolute convex retract in } \mathcal{G};\]
\[(iii) \ A = \{0\}.
\]

In this note we prove

\((\alpha)\) Let \( A \) be an absolute retract in the class \( \text{Arch} \). Then the \( \ell \)-group \( A \) is divisible, complete and orthogonally complete.

By applying a result of [5] we obtain

\((\beta)\) Let \( A \in \text{Arch} \) and suppose that the \( \ell \)-group \( A \) is complete and orthogonally complete. Then \( A \) is an absolute convex retract in the class \( \text{Arch} \).

The question whether the implication in (\( \alpha \)) (or in (\( \beta \)), respectively) can be reversed remains open.

Let us denote by

\( \text{Compl} \) - the class of all complete \( \ell \)-groups;
\( \text{Compl}^* \) - the class of all \( \ell \)-groups which are complete and orthogonally complete.

\((\gamma)\) Let \( A \in \text{Compl} \). Then the following conditions are equivalent:

\[(i) \ A \text{ is orthogonally complete.}\]
\[(ii) \ A \text{ is an absolute convex retract in the class } \text{Compl}.\]

As a corollary we obtain that each \( \ell \)-group belonging to \( \text{Compl}^* \) is an absolute convex retract in the class \( \text{Compl}^* \).

We prove that if the class \( \mathcal{C} \subseteq \mathcal{G} \) is closed with respect to direct products and if \( A_i \ (i \in I) \) are absolute (convex) retracts in \( \mathcal{C} \), then their direct product \( \prod_{i \in I} A_i \) is also an absolute (convex) retract in \( \mathcal{C} \).
2. Preliminaries

For \( \ell \)-groups we apply the notation as in Conrad [1]. Hence, in particular, the group operation in an \( \ell \)-group is written additively.

We recall some relevant notions. Let \( G \) be an \( \ell \)-group. \( G \) is divisible if for each \( a \in G \) and each positive integer \( n \) there is \( x \in G \) with \( nx = a \). A system \( \emptyset \neq \{ x_i \}_{i \in I} \subseteq G^+ \) is called orthogonal (or disjoint) if \( x_{i(1)} \wedge x_{i(2)} = 0 \) whenever \( i(1) \) and \( i(2) \) are distinct elements of \( I \). If each orthogonal subset of \( G \) possesses the supremum in \( G \) then \( G \) is said to be orthogonally complete. \( G \) is complete if each nonempty bounded subset of \( G \) has the supremum and the infimum in \( G \). \( G \) is archimedean if, whenever \( 0 < x \in G \) and \( y \in G \), then there is a positive integer \( n \) such that \( nx \not\leq y \). For each archimedean \( \ell \)-group \( G \) there exists a complete \( \ell \)-group \( D(G) \) (the Dedekind completion of \( G \)) such that

(i) \( G \) is a closed \( \ell \)-subgroup of \( D(G) \);

(ii) for each \( x \in D(G) \) there are subsets \( \{ y_i \}_{i \in I} \) and \( \{ z_j \}_{j \in J} \) of \( G \) such that the relations

\[
\sup \{ x_i \}_{i \in I} = x = \inf \{ z_j \}_{j \in J}
\]

are valid in \( D(G) \).

Let \( G_1 \) be a linearly ordered group and let \( G_2 \) be an \( \ell \)-group. The symbol \( G_1 \circ G_2 \) denotes the lexicographic product of \( G_1 \) and \( G_2 \). The elements of \( G_1 \circ G_2 \) are pairs \((g_1, g_2)\) with \( g_1 \in G_1 \) and \( g_2 \in G_2 \). For each \( g_2 \in G_2 \), the pair \((0, g_2)\) will be identified with the element \( g_2 \) of \( G_2 \). Then \( G_2 \) is a convex \( \ell \)-subgroup of \( G_1 \circ G_2 \).

**Lemma 2.1.** Let \( A \) be an \( \ell \)-group, \( A \neq \{0\} \), and let \( G_1 \) be a linearly ordered group, \( G_1 \neq \{0\} \). Put \( B = G_1 \circ A \). Then \( A \) fails to be a retract of \( B \).

**Proof.** By way of contradiction, suppose that \( A \) is a retract of \( B \). Let \( h \) be the corresponding retract homomorphism of \( B \) onto \( A \); i.e., \( h(a) = a \) for each \( a \in A \). There exists \( g_1 \in G_1 \) with \( g_1 > 0 \). Denote \((g_1, 0) = b, h(b) = a \). Further, there exists \( a_1 \in A \) with \( a_1 > a \). We have \( a_1 < b \), whence \( h(a_1) \leq h(b) \), thus \( a_1 \leq a \), which is a contradiction. \( \blacksquare \)

Let us denote by \( \mathcal{A} \) the class of all abelian lattice ordered groups. If \( A, G_1 \) and \( B \) are as in Lemma 2.1 and \( A, G_1 \in \mathcal{A} \), then also \( B \) belongs to \( \mathcal{A} \). Thus Lemma 2.1 yields
Proposition 2.2. Let $C \in \{G, A\}$ and let $A$ be an absolute retract (or an absolute convex retract, respectively) in the class $C$. Then $A = \{0\}$. ■

It is obvious that $\{0\}$ is an absolute (convex) retract in both the classes $G$ and $A$.

Let us remark that if $G_1, B \in G$ and if $G_1$ is a retract of $B$, then $G_1$ need not be a convex $\ell$-subgroup of $B$. This is verified by the following example:

Let $G_1$ be a linearly ordered group, $G_1 \neq \{0\}$. Further, let $G_2 \in G$, $G_2 \neq \{0\}$. Put $B = G_1 \circ G_2$. If $g_1 \in G_1$, then the element $(g_1, 0)$ of $B$ will be identified with the element $g_1$ of $G_1$. Thus $G_1$ turns out to be an $\ell$-subgroup of $B$ which is not a convex subset of $B$. For each $(g_1, g_2) \in B$ we put $h((g_1, g_2)) = g_1$. Then $h$ is a homomorphism of $B$ onto $G_1$ such that $h(g_1) = g_1$ for each $g_1 \in G_1$. Hence $G_1$ is a retract of $B$.

3. PROOFS OF ($\alpha$), ($\beta$) AND ($\gamma$)

In this section we assume that $A$ is an archimedean $\ell$-group. Hence $A$ is abelian.

It is well-known that there exists the divisible hull $A^d$ of $A$. Thus

(i) $A^d$ is a divisible $\ell$-group;

(ii) $A$ is an $\ell$-subgroup of $A^d$;

(iii) if $g \in A^d$, then there are $a \in A$, a positive integer $n$ and an integer $m$ such that $ng = ma$.

Lemma 3.1. Assume that $A$ is an absolute retract in the class Arch. Then the $\ell$-group $A$ is divisible.

Proof. By way of contradiction, suppose that $A$ fails to be divisible. Thus there are $a_1 \in A$ and $n \in \mathbb{N}$ such that there is no $x$ in $A$ with $nx = a_1$.

Put $B = A^d$. In view of the assumption, $A$ is a retract of $B$; let $h$ be the corresponding retract homomorphism.

There exists $b \in B$ with $nb = a_1$. Then $b \notin A$. Denote $h(b) = a$. We have

$$a_1 = h(a_1) = h(nb) = nh(b) = na,$$

which is a contradiction. ■
Lemma 3.2. Assume that $A$ is an absolute retract in the class $\text{Arch}$. Then $A$ is a complete $\ell$-group.

**Proof.** By way of contradiction, suppose that $A$ fails to be complete. Put $B = D(A)$. Then $A$ is an $\ell$-subgroup of $B$ and $A \neq B$. Thus there is $b \in B$ such that $b$ does not belong to $A$.

In view of the assumption, $A$ is a retract of $B$; let $h$ be the corresponding retract homomorphism. Put $h(b) = a$.

There exists a subset $\{a_i\}_{i \in I}$ of $A$ such that the relation

$$b = \bigvee_{i \in I} a_i$$

is valid in $B$. Hence $a_i \leq b$ for each $i \in I$. This yields

$$a_i = h(a_i) \leq h(b) = a$$

for each $i \in I$. Thus $b \leq a$.

At the same time, there exists a subset $\{a'_j\}_{j \in J}$ of $A$ such that the relation

$$b = \bigwedge_{j \in J} a'_j$$

holds in $B$. Hence $b \leq a'_j$ for each $j \in J$, thus by applying the homomorphism $h$ we obtain that $a \leq a'_j$ for each $j \in J$. Therefore $a \leq b$. Summarizing, $a = b$ and we arrived at a contradiction.

Lemma 3.3. Suppose that $H$ is a complete $\ell$-group. Then there exists an $\ell$-group $K$ such that

(i) $H$ is a convex $\ell$-subgroup of $K$;

(ii) $K$ is complete and orthogonally complete;

(iii) for each $0 < k \in K$ there exists a disjoint subset $\{x_i\}_{i \in I}$ of $H$ such that the relation

$$k = \bigvee_{i \in I} x_i$$

is valid in $K$.  


Proof. This is a consequence of results of [5]. □

Lemma 3.4. Assume that $A$ is an absolute retract in the class $\text{Arch}$. Then the $\ell$-group $A$ is orthogonally complete.

Proof. In view of Lemma 3.2, $A$ is complete. Put $A = H$ and let $K$ be as in Lemma 3.3. According to the assumption, $A$ is a retract of $K$. Let $h$ be the corresponding retract homomorphism.

Let $0 < k \in K$ and let $\{x_i\}_{i \in I}$ be as in Lemma 3.3. Put $h(k) = a$. Then $a \geq h(x_i) = x_i$ for each $i \in I$, whence $k \leq a$. Thus the condition (i) of Lemma 3.3 yields that $k \in A$. Hence $K^+ \subseteq A$ and then $K \subseteq A$. Therefore $K = A$ and so $A$ is orthogonally complete. □

From Lemmas 3.1, 3.2 and 3.4 we conclude that (α) is valid.

Let $G_1, G_2 \in G$; their direct product is denoted by $G_1 \times G_2$. If $g_1 \in G_1$, then the element $(g_1, 0)$ of $G_1 \times G_2$ will be identified with $g_1$. Similarly, for $g_2 \in G_2$, the element $(0, g_2)$ of $G_1 \times G_2$ will be identified with $g_2$. Under this identification, both $G_1$ and $G_2$ are convex $\ell$-subgroups of $G_1 \times G_2$.

Definition 3.5. (Cf. [2].) Let $G_1 \in \text{Arch}$. We say that $G_1$ has the splitting property if, whenever $H \in \text{Arch}$ and $G_1$ is a convex $\ell$-subgroup of $H$, then $G_1$ is a direct factor of $H$.

Proposition 3.6. (Cf. [4].) Let $G_1 \in \text{Arch}$. Then the following conditions are equivalent:

(i) $G_1$ has the splitting property.

(ii) The $\ell$-group $G_1$ is complete and orthogonally complete.

Lemma 3.7. Let $H \in G$ and let $G_1$ be a direct factor of $H$. Then $G_1$ is a retract of $H$.

Proof. There exists $G_2 \in G$ such that $H = G_1 \times G_2$. For $(g_1, g_2) \in H$ we put $h((g_1, g_2)) = g_1$. Then $h$ is a retract homomorphism of $H$ onto $G_1$. □

Proof of (β). Let $A, B \in \text{Arch}$ and suppose that $A$ is a convex $\ell$-subgroup of $B$. Further, suppose that $A$ is complete and orthogonally complete. In view of Proposition 3.6, $A$ is a direct factor of $B$. Hence according to Lemma 3.7, $A$ is a retract of $B$. Therefore $A$ is an absolute convex retract in the class $\text{Arch}$. □
**Lemma 3.8.** Let $A \in \text{Compl}$. Suppose that $A$ is an absolute convex retract in the class $\text{Compl}$. Then $A$ is orthogonally complete.

**Proof.** Put $H = A$ and let $K$ be as in Lemma 3.3. In view of Lemma 3.3 (i), $A$ is a convex $\ell$-subgroup of $K$. Hence according to the assumption, $A$ is a retract of $K$. Now it suffices to apply the same method as in the proof of Lemma 3.4.

**Lemma 3.9.** Let $A \in \text{Compl}$. Suppose that $A$ is orthogonally complete. Then $A$ is an absolute convex retract in the class $\text{Compl}$.

**Proof.** In view of $(\beta)$, $A$ is an absolute convex retract in the class $\text{Arch}$. It is well-known that the class $\text{Compl}$ is a subclass of $\text{Arch}$. Hence $A$ is an absolute convex retract in the class $\text{Compl}$.

From Lemmas 3.8 and 3.9 we conclude that $(\gamma)$ holds.

**Corollary 3.10.** Let $A \in \text{Compl}^*$. Then $A$ is an absolute convex retract in the class $\text{Compl}^*$.

4. **Direct products**

Let $A_i (i \in I)$ be $\ell$-groups; consider their direct product

$$A = \prod_{i \in I} A_i.$$  

Without loss of generality we can suppose that $A_i(\mathfrak{1}) \cap A_{i(2)} = \{0\}$ whenever $i(\mathfrak{1})$ and $i(\mathfrak{2})$ are distinct elements of $I$. For $a \in A$ and $i \in I$, we denote by $a_i$ or by $a(A_i)$ the component of $a$ in the direct factor $A_i$.

Let $i \in I$. Put

$$A'_i = \{a \in A : a_i = 0\}.$$  

Then we have

$$A = A_i \times A'_i,$$

$$A'_i = \prod_{j \in I \setminus \{i\}} A_j.$$

Let $i(0) \in I$ and $a_i^{(0)} \in A_i^{(1)}$. There exists $a \in A$ such that

$$a_i = \begin{cases} 
  a_i^{(0)} & \text{if } i = i(0), \\
  0 & \text{otherwise}.
\end{cases}$$

Then the element $a$ of $A$ will be identified with the element $a_i^{(0)}$ of $A_i^{(0)}$. Under this identification, each $A_i$ turns out to be a convex $\ell$-subgroup of $A$.

**Lemma 4.1.** Let $B$ be an $\ell$-group and let $A$ be an $\ell$-subgroup of $B$. Suppose that (1) is valid. Let $i$ be a fixed element of $I$ and assume that $A_i$ is a retract of $B$; the corresponding retract homomorphism will be denoted by $h_i$. Then for each $a \in A$ the relation

$$h_i(a) = a_i$$

is valid.

**Proof.** a) At first let $0 \leq a' \in A'_i$ and $0 \leq a^i \in A_i$. Then $a' \land a^i = 0$, thus

$$0 = h_i(a') \land h_i(a^i) = h_i(a') \land a^i.$$ 

Since this is valid for each $a^i \in A_i$ and $h_i(a') \in A_i$ we conclude that $h_i(a') = 0$. Then $h_i(-a') = 0$ as well and this yields that $h_i(a'') = 0$ for each $a'' \in A'_i$.

b) Let $a \in A$. In view of (2) we have

$$a = a_i + a(A'_i).$$

Thus

$$h_i(a) = h_i(a_i) + h_i(a(A'_i)).$$

According to a), $h_i(a(A'_i)) = 0$. Thus $h_i(a) = a_i$. 

**Lemma 4.2.** Let $B$ be an $\ell$-group and let $A$ be an $\ell$-subgroup of $B$. Suppose that (1) is valid and that for each $i \in I$, $A_i$ is a retract of $B$; the corresponding retract homomorphism will be denoted by $h_i$. For $b \in B$ we put

$$h(b) = b^1 \in A,$$

where $b^1_i = h_i(b)$ for each $i \in I$. Then
(i) \( h \) is a homomorphism of \( B \) into \( A \);
(ii) \( h(a) = a \) for each \( a \in A \).

**Proof.** The definition of \( h \) and the relation (1) immediately yield that (i) is valid. Let \( a \in A \) and \( i \in I \). Put \( h(a) = a^1 \). We have

\[
a = a_i + a(A'_i),
\]

thus by applying (i),

\[
h(a) = h(a_i) + h(a(A'_i)),
\]

\[
a^1_i = h_i(a_i) + h_i(a(A'_i)).
\]

Since \( h_i(a_i) = a_i \) and because \( (a(A'_i))_i = 0 \), according to Lemma 4.1, we obtain

\[
a^1_i = a_i \quad \text{for each } i \in I,
\]

thus \( a^1 = a \).

**Corollary 4.3.** Let the assumptions of Lemma 4.2 be valid. Then \( A \) is a retract of \( B \).

From Corollary 4.3 we immediately conclude

**Proposition 4.4.** Assume that \( \mathcal{C} \) is a class of \( \ell \)-groups which is closed with respect to direct products. Let \( A_i \) \( (i \in I) \) be absolute retracts in \( \mathcal{C} \) and let (1) be valid. Then \( A \) is an absolute retract in \( \mathcal{C} \).

**Proposition 4.5.** Assume that \( \mathcal{C} \) is a class of \( \ell \)-groups which is closed with respect to direct products. Let \( A_i \) \( (i \in I) \) be absolute convex retracts in \( \mathcal{C} \) and let (1) be valid. Then \( A \) is an absolute convex retract in \( \mathcal{C} \).

**Proof.** Let \( B \in \mathcal{C} \) and suppose that \( A \) is a convex \( \ell \)-subgroup of \( B \). Then all \( A_i \) are convex \( \ell \)-subgroups of \( B \). Hence in view of the assumption, all \( A_i \) are retracts of \( B \). Thus according to Corollary 4.3, \( A \) is a retract of \( B \). Therefore \( A \) is an absolute convex retract in the class \( \mathcal{C} \).
5. An example

The assertions of the following two lemmas are easy to verify; the proofs will be omitted.

**Lemma 5.1.** Let $A$ be an $\ell$-group which is complete and divisible. Then

(i) we can define (in a unique way) a multiplication of elements of $A$ with reals such that $A$ turns out to be a vector lattice;

(ii) if $r > 0$ is a real, $0 < a \in A$, $X = \{ q_1 \in Q : 0 < q_1 \leq r \}$, $Y = \{ q_2 \in R : r \leq q_2 \}$, then the relations

$$\sup(q_1 a) = ra = \inf(q_2 a)$$

are valid in $A$;

(iii) if $A_1$ is an $\ell$-subgroup of $A$ such that $A_1$ is complete and divisible, and $a_1 \in A$, then for each real $r$ the multiplication $ra_1$ in $A_1$ gives the same result as the multiplication $ra_1$ in $A$.

**Lemma 5.2.** Let $A$ be as in Lemma 5.1 and suppose that $A = \prod_{i \in I} A_i$. Then all $A_i$ are complete and divisible; moreover, for each real $r$, each $a \in A$ and each $i \in I$ we have

$$(ra)_i = ra_i.$$
$A = \varphi(R)$.

$A$ is an $\ell$-subgroup of $B$; if $I$ has more than one element, then $A$ fails to be convex in $B$.

Let $B$ be as above; suppose that $A$ is an $\ell$-group isomorphic to $R$ and that $A$ is an $\ell$-subgroup of $B$. Let $0 < a \in A$. Then $a_i = a(B_i) \geq 0$ for each $i \in I$ and there exists $i(0) \in I$ with $a_{i(0)} > 0$. Thus, in view of Lemma 5.1, we have $(ra)_{i(0)} > 0$ for each $r \in R$ with $r \neq 0$. Further, for each $a_1 \in A$ there exists a uniquely determined element $r \in R$ with $a_1 = ra$. This yields that the mapping

$$\varphi_{i(0)} : a_1 \mapsto (a_1)_{i(0)}$$

is an isomorphism of $A$ into $B_{i(0)}$.

Let $b \in B_{i(0)}$. There exists a unique $r \in R$ such that

$$b = ra_{i(0)}.$$  

Then, in view of Lemma 5.2, $b = (ra)_{i(0)}$ and hence the mapping $\varphi_{i(0)}$ is an isomorphism of $A$ onto $B_{i(0)}$.

For each $b \in B$ we put

$$h(b) = \varphi^{-1}_{i(0)}(b_{i(0)}).$$

Then $h$ is a homomorphism of $B$ into $A$. For $a_1 \in A$ the definition of $\varphi_{i(0)}$ yields

$$h(a_1) = a_1.$$  

Thus we obtain

**Lemma 5.3.** Let $B \in \mathcal{C}_R$ and let $A$ be an $\ell$-subgroup of $B$ such that $A$ is isomorphic to $R$. Then $A$ is a retract of $B$.  

**Corollary 5.4.** Let $A$ be an $\ell$-group isomorphic to $R$. Then $A$ is an absolute retract in the class $\mathcal{C}_R$.  

From Lemma 5.4 and Corollary 4.5 we conclude

**Proposition 5.5.** Each element of $\mathcal{C}_R$ is an absolute retract in the class $\mathcal{C}_R$.  

References


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