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ON M-OPERATORS OF q-LATTICES

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Abstract

It is well known that every complete lattice can be considered as a complete lattice of closed sets with respect to appropriate closure operator. The theory of q-lattices as a natural generalization of lattices gives rise to a question whether a similar statement is true in the case of q-lattices. In the paper the so-called M-operators are introduced and it is shown that complete q-lattices are q-lattices of closed sets with respect to M-operators.

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1. Introduction

The idea of introducing lattice-like structure on a quasiordered set is due to I. Chajda in [1].

Having a quasiordered set $(A; Q)$ with a quasiorder relation Q (i.e. Q is both reflexive and transitive relation on A), denote by $E_Q = Q \cap Q^{-1}$ the equivalence on A induced by Q. The relation Q/E_Q on a factor set A/E_Q defined by

$$(B, C) \in Q/E_Q \text{ iff } (b, c) \in Q \text{ for some } b \in B, c \in C$$

is known to be a partial order relation on A/E_Q. To simplify notation we shall write $\leq$ instead of Q/E_Q.

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A mapping $\chi : A/E_Q \longrightarrow A$ with the property $\chi(B) \in B$ for each $B \in A/E_Q$ is called a *q-function* on A.

If for each $B, C \in A/E_Q$ there exist $sup_{\leq}(B, C)$ and $inf_{\leq}(B, C)$, then the triple (A, Q, χ) is called an *L-quasiordered set.* The equivalence class $[a]_{E_Q}$ will be denoted simply by $[a]$.

L-quasiordered sets give rise to lattice-like operations on A in the following manner [1]:

Lemma 1. *Let (A, Q, χ) be an L-quasiordered set. Let us define for $x, y \in A$ the operations*

$$x \vee y = \chi(sup_{\leq}([x], [y]),$$

$$x \wedge y = \chi(inf_{\leq}([x], [y]).$$

Then the algebra $(A; \vee, \wedge)$ satisfies the identities

$$\begin{array}{lll} x \vee y = y \vee x, & x \wedge y = y \wedge x & (\textit{commutativity}); \\ x \vee (y \vee z) = (x \vee y) \vee z, & x \wedge (y \wedge z) = (x \wedge y) \wedge z & (\textit{associativity}); \\ x \vee (x \wedge y) = x \vee x, & x \wedge (x \vee y) = x \wedge x & (\textit{weak-absorption}); \\ x \vee y = x \vee (y \vee y), & x \wedge y = x \wedge (y \wedge y) & (\textit{weak-idempotence}); \\ x \vee x = x \wedge x & & (\textit{equalization}). \end{array}$$

An algebra $\mathcal{A} = (A; \vee, \wedge)$ satisfying the axioms of Lemma 1 is called a *q-lattice.*

Conversely, having a q-lattice $\mathcal{A} = (A; \vee, \wedge)$, the relation Q on A defined by

$$(x, y) \in Q \text{ iff } x \vee y = y \vee y$$

is a quasiorder relation, the so-called *induced quasiorder on A*.

Let us note that $(x, y) \in Q$ iff $x \wedge y = x \wedge x$, see [1].

The set $Sk\mathcal{A} = \{x \in A : x \vee x = x\}$ of all idempotent elements of $\mathcal{A}$, the so-called *skeleton of $\mathcal{A}$*, forms a lattice with respect to the induced operations $\vee$ and $\wedge$; this lattice is called the *induced lattice of a q-lattice $\mathcal{A}$*.

Hence a q-lattice $\mathcal{A} = (A; \vee, \wedge)$ is a lattice if and only if $A = Sk\mathcal{A}$.

The set $C(a) = \{x \in A; a \vee a = x \vee x\}$ for $a \in A$ is called the *cell of* a. It is clear that every q-lattice is a disjoint union of cells and every cell contains exactly one element from the skeleton.

When visualizing a q-lattice $\mathcal{A} = (A; \vee, \wedge)$, we firstly draw the lattice skeleton $Sk\mathcal{A}$ and then we add the corresponding cells. For example, the diagram

$$\begin{array}{ccc} c & \qquad\qquad & d \\ & & \\ & & \\ a & & b \end{array}$$

represents a q-lattice with a skeleton $Sk\mathcal{A} = \{a, c\}$ and with two cells $C(a) = C(b) = \{a, b\}, C(c) = C(d) = \{c, d\}$.

2. M-OPERATORS

A q-lattice $\mathcal{A} = (A; \vee, \wedge)$ is called *complete* if $Sk\mathcal{A}$ is a complete lattice. Since the join (the meet) of two (not necessarily distinct) elements of a q-lattice $\mathcal{A}$ is always a skeletal element, $\mathcal{A}$ is complete iff $\bigvee\{a; a \in X\}$ (or $\bigwedge\{a; a \in X\}$) exists for an arbitrary subset X of A.

By an operator on A we mean a mapping $C : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ of all subsets $\mathcal{P}(A)$ of A into itself. A subset $X \subseteq A$ is called *closed with respect to* C (or C-*closed*) if $C(X) = X$. The set of all C-closed sets will be denoted by $\mathcal{L}(C)$.

The set $\mathcal{P}(A)$ can be quasiordered in a natural way as follows:

Lemma 2. *Let A be a set, $M \in \mathcal{P}(A)$. Let us define the relation $\leq$ on $\mathcal{P}(A)$ for $X, Y \in \mathcal{P}(A)$ by*

$$X \leq Y \ \textit{ iff } \ X \cap M \subseteq Y \cap M.$$

Then $\leq$ is a quasiorder relation on $\mathcal{P}(A)$ and, moreover, $\mathcal{P}(A)$ is a q-lattice with respect to the operations

$$X \wedge Y = X \cap Y \cap M,$$
$$X \vee Y = (X \cup Y) \cap M$$

with $Sk\mathcal{P}(A) = \mathcal{P}(M)$.

Proof. Easy. ■

The q-lattice from Lemma 2 will be called a *set-M-q-lattice* on A. It is easy to see that set-A-q-lattice on A is just a set-lattice on A. (i.e. lettice of all subsets of A)

We are ready to formulate our natural problem:

Given a complete q-lattice $\mathcal{A}$, does there exist an operator C on A and $M \subseteq A$ such that the set $\mathcal{L}(C)$ of all C-closed sets on A is closed under the operations $\wedge$ and $\vee$ (as introduced in Lemma 2) and the set-M-q-lattice $\mathcal{L}(C)$ is isomorphic to $\mathcal{A}$?

In the following we give a positive answer to the above problem.

Remember that an operator $C : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ is called a *closure operator* on A if for each $X, Y \in \mathcal{P}(A)$:

(C1) $X \subseteq C(X)$,

(C2) $X \subseteq Y \Rightarrow C(X) \subseteq C(Y)$,

(C3) $C(C(X)) = C(X)$.

For a singleton $a \in A$, we shall write $C(a)$ instead of $C(\{a\})$.

We start from the following definition:

Definition 1. Let C be a closure operator on A, $M \subseteq A$ and $M' = A \setminus M$. Let us define a C_M-closure of $X \subseteq A$ as follows:

$$C_M(X) =$$

$$\begin{cases} (C(X) \cap M) \cup \{m'\}, & \text{if } X \cap M' = \{m'\} \text{ and } C(X) \cap M = C(m') \cap M; \\ C(X) \cap M, & \text{otherwise.} \end{cases}$$

The C_M-closure does not have the properties (C1)–(C3) of a closure operator. Its properties are listed in the following proposition.

Proposition 1. *C_M-closure operator on A has the following properties for $X, Y \subseteq A$:*

(1) $X \subseteq M \Rightarrow\ C_M(X) \subseteq M$,

(2) $X \cap M \subseteq\ C_M(X)$,

(3) $X \subseteq Y \Rightarrow\ C_M(X) \cap M \subseteq C_M(Y) \cap M$,

(4) $C_M(C_M(X \cap M)) = C_M(X \cap M)$.

Proof. (1) is easily seen from the definition of C.

Since C is a closure operator on A, (2) follows from the fact that $X \subseteq C(X)$ for each $X \subseteq A$.

Further we have $C_M(X) \cap M = C(X) \cap M$ for each $X \subseteq A$, hence $X \subseteq Y$ yields $C(X) \subseteq C(Y)$ and

$$C_M(X) \cap M = C(X) \cap M \subseteq C(Y) \cap M = C_M(Y) \cap M;$$

this proves the property (3).

Let us verify the property (4). Since $X \cap M \subseteq M$, we have by (1) $C_M(X \cap M) \subseteq M$, and, moreover

$$C_M(C_M(X \cap M)) = C_M(C(X \cap M) \cap M) = C(C(X \cap M) \cap M) \cap M =$$

$$= C(C(X \cap M)) \cap M = C(X \cap M) \cap M = C_M(X \cap M)$$

by (C3) and (C2) of the operator C. ∎

Proposition 1 leads us to the following definition:

Definition 2. Let $M \subseteq A$. An operator C^* on A with properties

(MC1) $X \subseteq M \Rightarrow\ C^*(X) \subseteq M$,

(MC2) $X \cap M \subseteq\ C^*(X)$,

(MC3) $X \subseteq Y \Rightarrow\ C^*(X) \cap M \subseteq C^*(Y) \cap M$,

(MC4) $C^*(C^*(X \cap M)) = C^*(X \cap M)$,

for each $X, Y \subseteq A$, is called an *M-operator* on A.

Let us note that for an M-operator C^* on A, the set $\mathcal{L}(C^*)$ of all C^*-closed sets is non-empty. Indeed, by (MC4) we have $\{C^*(X) : X \subseteq M\} \subseteq \mathcal{L}(C^*)$, and, by (MC1) and (MC2) $M \in \mathcal{L}(C^*)$.

Next we will show that $\mathcal{L}(C^*)$ can be endowed by a set-M-q-lattice structure:

Proposition 2. *Let C^* be an M-operator on A, let $X_\alpha \in \mathcal{L}(C^*), \alpha \in \Lambda$. Then $\mathcal{L}(C^*)$ is a complete q-lattice w.r.t. operations*

$$\bigwedge X_\alpha \ = \bigcap X_\alpha \cap M,$$

$$\bigvee X_\alpha \ = \bigwedge \{Y \in \mathcal{L}(C^*) : \quad X_\alpha \leq Y \quad \text{for each} \quad \alpha \in \Lambda\},$$

where $\leq$ is the quasiorder on A induced by $\wedge$. Moreover, $Sk\mathcal{L}(C^*) = \{X \in \mathcal{L}(C^*) : X \subseteq M\}$.

Proof. Firstly we have to prove that the operations are well-defined, i.e. that $\bigcap X_\alpha \cap M \in \mathcal{L}(C^*)$ whenever $X_\alpha \in \mathcal{L}(C^*)$ for each $\alpha \in \Lambda$. By (MC2) we have $\bigcap X_\alpha \cap M \subseteq C^*(\bigcap X_\alpha \cap M)$. Conversely, $\bigcap X_\alpha \cap M \subseteq X_\alpha$ for each $\alpha \in \Lambda$, hence using (MC1) and (MC3) one gets

$$C^*\left(\bigcap X_\alpha \cap M\right) = C^*\left(\bigcap X_\alpha \cap M\right) \cap M \subseteq \ C^*(X_\alpha) \cap M = X_\alpha \cap M$$

for each $\alpha \in \Lambda$. But this yields also

$$C^*\left(\bigcap X_\alpha \cap M\right) \subseteq \bigcap X_\alpha \cap M$$

verifying the closedness of the set $\bigcap X_\alpha \cap M$.

The operation $\wedge$ on $\mathcal{L}(C^*)$ is then well defined and induces a quasiorder relation $\leq$ on $\mathcal{L}(C^*)$ as follows:

$$X \leq Y \ \text{ iff } \ X \cap M \subseteq Y \cap M.$$

We show that $\bigcap X_\alpha \cap M$ is the greatest lower bound of X_α's w.r.t. induced quasiorder. Indeed, let $X \in \mathcal{L}(C^*)$ and suppose that $X \leq X_\alpha$ for each $\alpha \in \Lambda$. Then $X \cap M \subseteq X_\alpha \cap M$, hence also $X \cap M \subseteq \bigcap X_\alpha \cap M$ verifying $X \leq \bigcap X_\alpha \cap M$.

It is immediately seen that $\bigvee X_\alpha$ is the least upper bound of X_α's w.r.t. $\leq$ and, altogether, $\mathcal{L}(C^*)$ is a complete q-lattice. ∎

Now we are ready to show that complete q-lattices can be viewed as q-lattices of closed sets w.r.t. appropriate M-operators.

Theorem. *Let $\mathcal{L} = (L, \vee, \wedge)$ be a complete q-lattice and let $\leq$ be the induced quasiorder on L. Then the operator C on L defined by*

$$C(X) = \{y \in Sk\mathcal{L} : \ y \leq \bigvee X\} \cup X$$

is a closure operator and for $M = Sk\mathcal{L}$ we have $\mathcal{L}(C_M) \cong \mathcal{L}$.

Proof. According to Proposition 1 and Definition 2, the operator C_M is an M-operator on L, and by Proposition 2, $\mathcal{L}(C_M)$ is a complete q-lattice. It is easily seen that C is a closure operator on L. Hence it is enough to prove that the q-lattices $\mathcal{L}(C_M)$ and $\mathcal{L}$ are isomorphic. Denote for $a \in L$ by $L_{Sk}(a)$ the set of all skeletal elements lying below a.

Let us describe all C_M-closed sets:

- by (MC4) all the sets $C_M(X)$ for $X \subseteq M$ are C_M-closed, i.e. the sets $C_M(X) = \{y \in Sk\mathcal{L} : \ y \leq \bigvee X\} = L_{Sk}(\bigvee X)$;
- let us consider the sets $X \subseteq L$ with $|X \cap M'| \geq 2$.

Then $C_M(X) = C(X) \cap M \subseteq M$, so $C_M(X) \neq X$ and X is not C_M-closed:

- suppose that $X \subseteq L$ with $X \cap M' = \{m'\}$ and $M \cap C(m') \neq M \cap C(X)$. Then again

$$C_M(X) = C(X) \cap M \subseteq M \text{ and since } m' \notin M, \ X \text{ is not } C_M\text{- closed;}$$

- finally, let $X \cap M' = \{m'\}$ and $M \cap C(m') = M \cap C(X)$ for $X \subseteq L$. This gives

$$C_M(X) = \{y \in Sk\mathcal{L} : \ y \leq \vee X\} \cup \{m'\} = \{y \in Sk\mathcal{L} : \ y \leq m' \vee m'\} \cup \{m'\},$$

and the sets

$$\{y \in Sk\mathcal{L} : \ y \leq m' \vee m'\} \cup \{m'\} \text{ for } m' \notin Sk\mathcal{L}$$

are C_M-closed.

Let us verify that the mapping $\phi : L \longrightarrow \mathcal{L}(C_M)$ defined by

$$\phi(x) = L_{Sk}(x) \text{ for } x \in Sk\mathcal{L},$$

$$\phi(y) = L_{Sk}(y \vee y) \cup \{y\} \text{ for } y \notin Sk\mathcal{L}$$

is the desired isomorphism.

Injectivity of ϕ is easily seen from its definition, surjectivity then yields from the fact that the elements of $\mathcal{L}(C_M)$ are of the form $L_{Sk}(x)$ for $x \in Sk\mathcal{L}$ or $L_{Sk}(y \vee y) \cup \{y\}$ for $y \notin Sk\mathcal{L}$.

Now let $x, y \in L$. To verify that ϕ is a homomorphism, we distinguish three cases:

Case 1. Assume $x, y \in Sk\mathcal{L}$. Then $x \wedge y \in Sk\mathcal{L}$ and

$$\phi(x) \wedge \phi(y) = L_{Sk}(x) \wedge L_{Sk}(y) = (L_{Sk}(x) \cap L_{Sk}(y)) \cap Sk\mathcal{L} =$$

$$= L_{Sk}(x) \cap L_{Sk}(y) = L_{Sk}(x \wedge y) = \phi(x \wedge y)$$

By the definition of join in $\mathcal{L}(C_M)$ we have

$$\phi(x) \vee \phi(y) = \bigwedge\{Y \in \mathcal{L}(C_M) :\ \phi(x) \leq Y, \phi(y) \leq Y\} =$$

$$= \bigcap\{Y \in \mathcal{L}(C_M) :\ L_{Sk}(x) \subseteq Y \cap Sk\mathcal{L}, L_{Sk}(y) \subseteq Y \cap Sk\mathcal{L}\} \cap Sk\mathcal{L}.$$

Evidently, $x \vee y \in Sk\mathcal{L}$, $L_{Sk}(x \vee y) \in \mathcal{L}(C_M)$ and

$$L_{Sk}(x) \cup L_{Sk}(y) \subseteq L_{Sk}(x \vee y) = L_{Sk}(x \vee y) \cap Sk\mathcal{L} = \phi(x \vee y).$$

To prove the converse inclusion, we have to show that

$$\phi(x \vee y) = L_{Sk}(x \vee y) \subseteq Y \cap Sk\mathcal{L}$$

for each $Y \in \mathcal{L}(C_M)$ with $L_{Sk}(x) \cup L_{Sk}(y) \subseteq Y \cap Sk\mathcal{L}$.

If $Y = L_{Sk}(z)$ for some $z \in Sk\mathcal{L}$, we get

$$L_{Sk}(x) \cup L_{Sk}(y) \subseteq L_{Sk}(z) \cap Sk\mathcal{L} = L_{Sk}(z),$$

i.e. $x \leq z,\ y \leq z$, and since $Sk\mathcal{L}$ is the lattice, $x \vee y \leq z$. But then

$$L_{Sk}(x \vee y) \subseteq L_{Sk}(z) = Y = Y \cap Sk\mathcal{L}.$$

In the remaining case, we have $Y = L_{Sk}(u \vee u) \cup \{u\}$ for some $u \notin Sk\mathcal{L}$. This yields $x \leq u \vee u,\ y \leq u \vee u$ and hence $x \vee y \leq u \vee u$. Finally, we get

$$L_{Sk}(x \vee y) \subseteq L_{Sk}(u \vee u) = Y \cap Sk\mathcal{L},$$

finishing the Case 1.

Case 2. Assume that $x, y \notin Sk\mathcal{L}$. Then

$$\begin{aligned} &\phi(x) \wedge \phi(y) = \\ &= (L_{Sk}(x \vee x) \cup \{x\}) \cap (L_{Sk}(y \vee y) \cup \{y\}) \cap Sk\mathcal{L} = \\ &= L_{Sk}(x \vee x) \cap L_{Sk}(y \vee y) = L_{Sk}((x \vee x) \wedge (y \vee y)). \end{aligned}$$

By Lemma 1, $(x \vee x) \wedge (y \vee y) = x \wedge y$, hence

$$L_{Sk}((x \vee x) \wedge (y \vee y)) = L_{Sk}(x \wedge y) = \phi(x \wedge y),$$

veryfying that, in the Case 2, ϕ is $\wedge$-preserving.

The join of $\phi(x)$ and $\phi(y)$ is of the form

$$\begin{aligned} \phi(x) \vee \phi(y) = \bigwedge\{Y \in \mathcal{L}(C_M) :\ & (L_{Sk}(x \vee x) \cup (\{x\})) \cap Sk\mathcal{L} \subseteq Y \cap Sk\mathcal{L}, \\ & (L_{Sk}(y \vee y) \cup (\{y\})) \cap Sk\mathcal{L} \subseteq Y \cap Sk\mathcal{L}\} \cap Sk\mathcal{L} = \\ = \bigcap\{Y \in \mathcal{L}(C_M) :\ & (L_{Sk}(x \vee x) \cup L_{Sk}(y \vee y) \subseteq Y \cap Sk\mathcal{L}\} \cap Sk\mathcal{L}. \end{aligned}$$

Since $L_{Sk}(x \vee x) \cup L_{Sk}(y \vee y) \subseteq L_{Sk}(x \vee y)$, we deduce

$$\phi(x) \vee \phi(y) \subseteq \phi(x \vee y).$$

Similarly as in the Case 1, we have to prove

$$L_{Sk}(x \vee y) \subseteq Y \cap Sk\mathcal{L}$$

for each $Y \in \mathcal{L}(C_M)$ with $L_{Sk}(x \vee x) \cup L_{Sk}(y \vee y) \subseteq Y \cap Sk\mathcal{L}$.

We distinguish again two cases with respect to Y.

If $Y = L_{Sk}(z)$ for some $z \in Sk\mathcal{L}$, then $x \vee x \leq z,\ y \vee y \leq z$ and hence $x \vee y \leq z$, i.e.

$$L_{Sk}(x \vee y) \subseteq L_{Sk}(z) = Y = Y \cap Sk\mathcal{L}.$$

If $Y = L_{Sk}(u \vee u) \cup \{u\}$ for some $u \notin Sk\mathcal{L}$, we obtain $x \vee x \leq u \vee u,\ y \vee y \leq u \vee u$ and $x \vee y \leq u \vee u$, i.e.

$$L_{Sk}(x \vee y) \subseteq L_{Sk}(u \vee u) = Y \cap Sk\mathcal{L},$$

finishing the Case 2.

Case 3. Suppose that $x \in Sk\mathcal{L},\ y \notin Sk\mathcal{L}$. Then

$$\begin{aligned} \phi(x) \wedge \phi(y) &= (L_{Sk}(x) \cap (L_{Sk}(y \vee y) \cup \{y\})) \cap Sk\mathcal{L} = \\ &= L_{Sk}(x) \cap L_{Sk}(y \vee y) = L_{Sk}(x \wedge (y \vee y)) \\ &= L_{Sk}(x \wedge y) = \phi(x \wedge y). \end{aligned}$$

To prove that ϕ is $\vee$-preserving, we start with

$$\begin{aligned} \phi(x) \vee \phi(y) = \bigwedge\{Y \in \mathcal{L}(C_M) :\ & L_{Sk}(x) \subseteq Y \cap Sk\mathcal{L}, \\ & (L_{Sk}(y \vee y) \cup (\{y\})) \cap Sk\mathcal{L} \subseteq Y \cap Sk\mathcal{L}\} \cap Sk\mathcal{L} = \\ = \bigwedge\{Y \in \mathcal{L}(C_M) :\ & L_{Sk}(x) \cup L_{Sk}(y \vee y) \subseteq Y\} \cap Sk\mathcal{L}. \end{aligned}$$

Analogously as in previous cases, we have

$$L_{Sk}(x) \cup L_{Sk}(y \vee y) \subseteq L_{Sk}(x \vee y) = \phi(x \vee y).$$

To finish the proof it is enough to show

$$L_{Sk}(x \vee y) \subseteq Y \cap Sk\mathcal{L}$$

whenever $Y \in \mathcal{L}(C_M)$ with $L_{Sk}(x) \cup L_{Sk}(y \vee y) \subseteq Y \cap Sk\mathcal{L}$.

Considering $Y = L_{Sk}(z)$ for some $z \in Sk\mathcal{L}$, we obtain $x \leq z, y \vee y \leq z$ and $x \vee (y \vee y) = x \vee y \leq z$, i.e.

$$L_{Sk}(x \vee y) \subseteq L_{Sk}(z) = Y \cap Sk\mathcal{L}.$$

Finally, the case $Y = L_{Sk}(u \vee u) \cup \{u\}$ for some $u \notin Sk\mathcal{L}$ yields $x \leq u \vee u$, $y \vee y \leq u \vee u$, i.e. $x \vee y \leq u \vee u$ and

$$L_{Sk}(x \vee y) \subseteq L_{Sk}(u \vee u) = Y \cap Sk\mathcal{L},$$

finishing the Case 3 and the proof of Theorem. ■

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