*Discussiones Mathematicae*
*General Algebra and Applications 22 (2002) 73–86*

# ON THE LATTICE OF ADDITIVE HEREDITARY PROPERTIES OF FINITE GRAPHS*

Ján Jakubík
*Matemathical Institute, Slovak Academy of Sciences*
*Grešákova 6, 040–01 Košice, Slovakia*
**e-mail:** kstefan@saske.sk

**Abstract**

In this paper it is proved that the lattice of additive hereditary properties of finite graphs is completely distributive and that it does not satisfy the Jordan-Dedekind condition for infinite chains.

**Keywords:** Lattice, complete distributivity, finite graph, additive hereditary property, generalized Jordan-Dedekind condition.

**2000 AMS Mathematics Subject Classifications:** 06D10, 05C99.

## 0. Introduction

The lattice $L^a$ of additive hereditary properties of finite graphs has been introduced by Mihók [9] and investigated in connection with generalized colorings of graphs; cf. also Borowiecki and Mihók [3] and Borowiecki, Broere, Frick, Mihók and Semanišin [2].

In the present paper we prove that the lattice $L^a$ is completely distributive. An analogous result is valid in the case when instead of finite graphs we consider finite partially ordered sets.

Further, we deal with chains in intervals of the lattice $L^a$. We show that the Jordan-Dedekind condition for infinite chains fails to be valid in $L^a$. Namely, we prove that there exist elements $A_1, A_2 \in L^a$ with $A_1 < A_2$ and maximal chains $C_1, C_2$ of the interval $[A_1, A_2]$ such that $C_1$ is denumberable and $C_2$ has the power of the continuum.

*Supported by grant VEGA 2/1131/21.

In [5], the following result has been proved:

(∗) Let $\alpha$ be a cardinal, $\alpha \geqq c$. There exists a complete and completely distributive lattice $S_\alpha$ such that for each cardinal $\beta$ with $c \leqq \beta \leqq \alpha$ there exists a maximal chain $C_\beta$ in $S_\alpha$ such that $\operatorname{card}(C_\beta) = \beta$.

No chain in $L^a$ can have a cardinality larger than $c$, since the power of $L^a$ is equal to $c$. Let $\alpha$ be an infinite cardinal. We can consider the powers of maximal chains in sublattices of the lattice $(L^a)^\alpha$.

Let $Q_0$ be the set of all rationals $q$ with $0 \leqq q \leqq 1$ (under the natural linear order). Further, let $\alpha$ be an infinite cardinal. Put $Q_0^\alpha = L_\alpha$. Then $L_\alpha$ is a bounded completely distributive lattice which is isomorphic to a sublattice of the lattice $(L^a)^\alpha$. Moreover, we have

(∗∗) For each cardinal $\beta$ with $\aleph_0 \leqq \beta \leqq \alpha$ there exists a maximal chain $C_\beta$ in $L_\alpha$ such that $\operatorname{card} C_\beta = \beta$.

Fundamental results on completely distributive lattices have been obtained by Raney [10], [11], [12]; cf. also Birkhoff [1] (Chapter V). Higher degrees of distributivity (including complete distributivity) of Boolean algebras have been dealt with by several authors; for the bibliography, cf. Sikorski [13].

The original version of the Jordan-Dedekind condition has been dealing with finite chains (cf. e.g., Birkhoff [1]); the case of infinite chains was considered by Szász [14], Grätzer and Schmidt [4] and by the author [5]–[8].

## 1. Preliminaries

We start by recalling some definitions and conventions.

Let $\mathcal{C}$ be the class of all finite graphs without multiple edges and without loops. From technical reasons, the empty graph is also considered as a member of $\mathcal{C}$; it has no vertices and no edges. For $G \in \mathcal{C}$, we denote by $V(G)$ and $E(G)$ the set of all vertices or the set of all edges of $G$, respectively.

Let $G \in \mathcal{C}$. We denote by $S(G)$ the set of all $G^1 \in \mathcal{C}$ such that (i) $V(G^1) \subseteq V(G)$, and (ii) for $(g_1, g_2) \in E(G)$ we have $(g_1, g_2) \in E(G^1)$ iff $g_1$ and $g_2$ belong to $V(G^1)$. Further, let $c(G)$ be the set of all connected components of $G$.

A nonempty class $A$ of elements of $\mathcal{C}$ is said to be an additive hereditary property of finite graphs if it satisfies the following conditions:

(i) $A$ is closed with respect to isomorphisms;

(ii) if $G \in \mathcal{C}$ and $c(G) \subseteq A$, then $G \in A$;

(iii) if $G \in A$, then $S(G) \subseteq A$.

Let $L^a$ be the system of all $A$ with the mentioned properties. For $A_1, A_2 \in L^a$ we put $A_1 \leqq A_2$ if $A_1 \subseteq A_2$. In [9] it was proved that, under the relation $\leqq$, $L^a$ is a complete lattice. Consider a nonempty system $\{A_i\}_{i \in I}$ of elements of $L^a$. Then

$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i. \tag{1}$$

Let $B$ be the class of all graphs $G \in \mathcal{C}$ such that, whenever $G^1 \in c(G)$, then there is $i \in I$ with $G^1 \in A_i$. We have

$$\bigvee_{i \in I} A_i = B. \tag{2}$$

(Cf. [9].)

Now let $L$ be a complete lattice. We say that $L$ is infinitely distributive if it satisfies the identities

$$x \wedge \left( \bigvee_{i \in I} y_i \right) = \bigvee_{i \in I} (x \wedge y_i), \tag{3a}$$

$$x \vee \left( \bigwedge_{i \in I} y_i \right) = \bigwedge_{i \in I} (x \vee y_i), \tag{3b}$$

where $x \in L$, $(y_i)_{i\in I}$ is any indexed system of elements of $L$, and $I$ is a nonempty set of indices.

Further, let $S$ and $T$ be nonempty sets of indices. The lattice $L$ is called completely distributive if it satisfies the conditions

$$\bigwedge_{t\in T} \bigvee_{s\in S} x_{t,s} = \bigvee_{\varphi\in S^T} \bigwedge_{t\in T} x_{t,\varphi(t)}, \tag{4a}$$

$$\bigvee_{t\in T} \bigwedge_{s\in S} x_{t,s} = \bigwedge_{\varphi\in S^T} \bigvee_{t\in T} x_{t,\varphi(t)} \tag{4b}$$

for each indexed system $(x_{ts})_{t\in T, s\in S}$ of elements of $L$.

## 2. Infinite distributivity

Let $(A_i)_{i\in I}$ be an indexed system of elements of $L^a$ and $C \in L^a$. Denote

$$X = C \wedge \left(\bigvee_{i\in I} A_i\right), \quad Y = \bigvee_{i\in I} (C \wedge A_i).$$

We have clearly $X \geqq Y$. Let $G \in X$ and let $\{G_1, G_2, \dots, G_n\}$ be the system of all connected components of $G$. Then, in view of (1), we have $G \in C$ and $G \in \bigvee_{i\in I} A_i$. Hence, according to (2), there are $i(1), \dots, i(n) \in I$ such that $G_1 \in A_{i(1)}, \dots, G_n \in A_{i(n)}$. The definition of the additive hereditary property yields

$$G_1 \in C \wedge A_{i(1)}, \dots, G_n \in C \wedge A_{i(n)},$$

thus according to (2) we obtain $G \in Y$. Hence $X = Y$. Thus $L^a$ satisfies the condition (3a).

Now we denote

$$X_1 = C \vee \left( \bigwedge_{i \in I} A_1 \right), \quad Y_1 = \bigwedge_{i \in I} (C \vee A_i).$$

Then $X_1 \leqq Y_1$. Let $G \in Y_1$. Hence $G \in C \vee A_i$ for each $i \in I$.

Let $i_1$ be a fixed element of $I$. In view of the above relation and according to (2) we can express $G$ as a disjoint sum

$$G = G_{i_1} + H_{i_1}$$

such that $G_{i_1} \in C$, $H_{i_1} \in A_{i_1}$ and no nonzero connected component of $A_{i_1}$ belongs to $C$. In other words, $G_{i_1}$ is the disjoint sum of all connected components of $G$ which belong to $C$. Hence if $i_2$ is another element of $I$ and if we have the analogous expression

$$G = G_{i_2} + H_{i_2}, \tag{5}$$

then we must have $G_{i_2} = G_{i_1}$, and thus $H_{i_2} = H_{i_1}$. We get

$$G_{i_1} \in \bigwedge_{i \in I} A_i$$

and hence, according to (5) and (2), we have $G \in X_1$. Thus $X_1 = Y_1$. Therefore $L^a$ satisfies the identity (3b). We obtain

**Lemma 2.1.** *The lattice $L^a$ is infinitely distributive.* ■

We remark that the distributivity of $L^a$ has been proved in [9].

**Lemma 2.2.** *Let $L$ be a lattice which is complete and infinitely distributive. Assume that $L$ does not satisfy the identity* (4a). *Then there are elements $u, v \in L$ and an indexed system $(x'_{t,s})_{t \in T, s \in S}$ of elements of $L$ such that $u < v$ and*

$$\bigvee_{s \in S} x'_{ts} = v \quad \textit{for each } t \in T, \tag{6a}$$

$$\bigwedge_{t\in T} x'_{t,\varphi(t)} = u \quad \textit{for each } \varphi \in S^T. \tag{6b}$$

***Proof.*** In view of the assumption, there exists an indexed system $(x_{ts})_{t\in T, s\in S}$ of elements of $L$ such that the relation (4a) fails to be satisfied. Denote

$$v = \bigwedge_{t\in T}\bigvee_{s\in S} x_{ts}, \quad u = \bigvee_{\varphi\in S^T}\bigwedge_{t\in T} x_{t,\varphi(t)}.$$

Then $v > u$. Put

$$x'_{ts} = (x_{ts} \vee u) \wedge v.$$

In view of the infinite distributivity of $L$ we obtain

$$v = (v \vee u) \wedge v = \bigwedge_{t\in T}\bigvee_{s\in S} (x_{ts} \vee u) \wedge v = \bigwedge_{t\in t}\bigvee_{s\in S} x'_{ts}. \tag{7}$$

For each $t \in T$ and each $s \in S$, we have $x'_{ts} \leqq v$, whence

$$\bigvee_{s\in S} x'_{t,s} \leqq v \quad \text{for each } t \in T.$$

If there is $t(1) \in T$ with

$$\bigvee_{s\in S} x'_{t(1),s} < v,$$

then the relation (7) cannot hold, which is a contradiction. Therefore the relation (6a) is valid.

Further, $x'_{ts} = (x_{ts} \wedge v) \vee u \geqq u$, whence

$$\bigwedge_{t\in T} x'_{t,\varphi(t)} \geqq u \quad \text{for each } \varphi \in S^T.$$

We have

$$u = (u \wedge v) \vee u = \bigvee_{\varphi\in S^T} \bigwedge_{t\in T} x'_{t,\varphi(t)}. \tag{8}$$

If there is $\varphi_1 \in S^T$ with

$$\bigwedge_{t\in T} x'_{t,\varphi_1(t)} > u,$$

then (8) does not hold and so we arrived at a contradiction. Thus (6b) is valid. ∎

Analogously we can prove

**Lemma 2.3.** *Let $L$ be a complete and infinitely distributive lattice. Assume that $L$ does not satisfy the identity* (4b). *Then there are elements $u, v \in L$ and an indexed system $(x'_{ts})_{t\in T, s\in S}$ of elements of $L$ such that $u < v$ and*

$$\bigwedge_{s\in S} x'_{t,s} = u \quad \textit{for each} \ \ t \in T; \tag{9a}$$

$$\bigvee_{t\in T} x'_{t,\varphi(t)} = v \quad \textit{for each} \ \ \varphi \in S^T. \tag{9b}$$

## 3. Complete distributivity

**Theorem 3.1.** *The lattice $L^a$ is completely distributive.*

***Proof.*** By way of contradiction, assume that $L^a$ fails to be completely distributive. Then there exists an indexed system $(X_{t,s})_{t\in T, s\in S}$ of elements of $L^a$ such that either (4a) or (4b) fails to be valid.

a) At first assume that (4a) does not hold. Then in view of Lemma 2.2 there are elements $U, V \in L^a$, $U < V$, and an indexed system $(X'_{ts})_{t\in T, s\in S}$ of elements of $L^a$ such that the condition (6a) and (6b) are satisfied (with $x$ replaced by $X$).

There exists $G \in V \setminus U$. If all connected components of $G$ belong to $U$, then $G$ would belong to $U$; thus there is a connected component $G_1$ which does not belong to $U$. On the other hand, $G_1$ is an element of $V$.

Let $s \in S$. In view of (6a), we have $G_1 \notin X'_{ts}$ for each $t \in T$. Put $\varphi_0(t) = s$ for each $t \in T$. According to (6b), we obtain $G_1 \in X'_{t,\varphi_0(t)} = X'_{t,s}$ for each $t \in T$; we arrived at a contradiction.

b) Now suppose that (4b) does not hold. We apply Lemma 2.3 with the notation analogous to that in a).

Again, there is $G \in V \setminus U$ and a connected component $G_1$ of $G$ with $G_1 \in V \setminus U$.

Let $t \in T$. In view of (9a) there exists $x \in S$ such that $G_1 \notin X'_{t,s}$; we put $\varphi_0(t) = s$. Hence $G_1 \notin X'_{t,\varphi_0(t)}$ for each $t \in T$. Then according to (2) we infer that $G_1$ does not belong to

$$\bigvee_{t\in T} X'_{t,\varphi_0(t)} = V,$$

which is a contradiction. ∎

We conclude this section by the following remarks.

1) Let $\alpha$ be an infinite cardinal. The above considerations remain valid if instead of finite graphs we take into account the graphs $G$ such that the set $V(G)$ of vertices of $G$ has the cardinality less or equal to $\alpha$. In this way we obtain the lattice $L^a(\alpha)$ constructed analogously as $L^a$. Since $L^a(\alpha)$ is completely distributive, the well-known result of Raney [11] concerning the subdirect product representation can be applied for $L^a(\alpha)$.

2) The same method as above can be used for proving analogous results dealing with finite partially ordered sets (or with partially ordered sets having the cardinality less or equal to $\alpha$, where $\alpha$ is a given infinite cardinal).

## 4. On chains in the lattice $L^a$

Let $L$ be a lattice and $a, b \in L$, $a < b$. The interval of $L$ with the endpoints $a$ and $b$ is denoted by $[a, b] = L_1$. Let $\mathcal{C}(L_1)$ be the sytstem of all chains of the lattice $L_1$; the system $\mathcal{C}(L_1)$ is partially ordered by the set-theoretical inclusion.

We say that the lattice $L$ satisfies the Jordan-Dedekind condition (shortly: condition (JD)) if, whenever $a, b, L_1$ are as above and $C_1, C_2$ are maximal elements of $\mathcal{C}(L_1)$, then

$$\operatorname{card}(C_1) = \operatorname{card}(C_2).$$

In this section we prove that the lattice $L^a$ does not satisfy the condition (JD).

We need some lemmas.

**Lemma 4.1** (Cf. Borowiecki and Mihók [3].) *There exist $A, B_n \in L^a$ $(n = 1, 2, \dots)$ such that for each $n \in N$ we have $A < B_n$ and $[A, B_n]$ is a prime interval (i.e., $[A, B_n]$ is a two-element set).*

**Lemma 4.2.** *Let $L$ be a lattice which is complete and completely distributive. Assume that $a, b_n \in L$ such that for each $n \in N$ we have $a < \bar{b}_n$ and $[a, \bar{b}_n]$ is a prime interval. Put $b = \bigvee_{n \in N} \bar{b}_n$. Then the interval $[a, b]$ is a Boolean algebra isomorphic to $2^N$.*

***Proof.*** The interval $[a, b]$ is a complete and completely distributive lattice. Put $B_n = [a, \bar{b}_n]$. For each $x \in B$ we set

$$\varphi(x) = (x \wedge \bar{b}_n)_{n \in N}.$$

It is a routine to verify that $\varphi$ is an isomorphism of $B$ onto the direct product $\prod_{n \in N} B_n$ which is isomorphic to $2^N$. Hence $B$ is a Boolean algebra. ■

Under the notation as above, we put $x_n = x \wedge \bar{b}_n$ for each $x \in B$ and $n \in N$.

For each $m \in N$, we denote by $y^m$ the element of $B$ such that $y^m_n = \bar{b}_n$ if $n \leqq m$ and $y^m_n = a$ otherwise. We put

$$C_1 = \{y^m\}_{m \in N} \cup \{a, b\}.$$

Then $C_1$ is a chain in $[a, b]$.

**Lemma 4.3.** *$C_1$ is a maximal chain in $[a, b]$.*

***Proof.*** By way of contradiction, assume that $x$ is an element of $[a, b] \setminus C_1$ such that $x$ is comparable with each element of $C_1$. Hence $a \neq x \neq b$. Put

$$C_{11} = \{c \in C_1 : a < c < x\},\ C_{12} = \{c \in C_1 : b > c > x\}.$$

If $C_{12} = \emptyset$, then $x_n = a$ for each $n \in N$, whence $x = a$, which is impossible. Similarly, if $C_{11} = \emptyset$, then we get $x = b$, a contradiction. Hence $C_{11} \neq \emptyset \neq C_{12}$. Then $C_{11}$ is finite, whence it has the greatest element; we denote it by $y^{m(1)}$. Thus $y^{m(1)+1}$ is the least element of $C_{12}$ and $y^{m(1)} < x < y^{m(1)+1}$. But $[y^{m(1)}, y^{m(1)+1}]$ is a prime interval and so we arrived at a contradiction. ∎

Let $I$ be the set of all rationals $i$ with $0 < i < 1$. There exists a bijection $\psi : N \to I$. Then we have an isomorphism of $[a, b]$ onto $\prod_{i \in I} B_i$. For $x \in [a, b]$ and $j \in I$, let $x_j$ be the component of $x$ in $B_j$ under the isomorphism under consideration.

For each $i \in I$, we denote by $z^i$ the element of $B$ such that $z^i_j = 1$ for $j \in I, j \leqq i$, and $z^i_j = 0$ otherwise.

Consider the interval $[0, 1]$ of reals and let $0 \neq k \neq 1$, $k \in [0, 1] \setminus I$. The set of all such $k$ will be denoted by $K$. For $k \in K$ let $t^k$ be the element of $B$ such that

$$t^k_i = b_i \quad \text{for } i \in I, i < k,$$

$$t^k_i = a \quad \text{for } i \in I, i > k.$$

We put

$$C_2 = \{z^i\}_{i \in I} \cup \{t^k\}_{k \in K} \cup \{a, b\}.$$

It is obvious that $C_2$ is a chain in $[a, b]$.

**Lemma 4.4.** *$C_2$ is a maximal chain in $[a, b]$.*

***Proof.*** By way of contradiction, assume that an element $x \in [a, b]$ is comparable with all elements of $C_2$ and does not belong to $C_2$. Denote

$$C_{21} = \{c \in C_2 : a < c < x\}, \quad C_{22} = \{c \in C_2 : b > c > x\}.$$

Hence $C_2 = C_{21} \cup C_{22}$. Similarly as in the proof of 4.3 we can verify that $C_{21} \neq \emptyset \neq C_{22}$.

Let $I_1$ be the set of all $i \in I$ such that there exists $c \in C_{21}$ with $c_i = b_i$. Let $i \in I_1$. Then $c_{i(1)} = b_{i(1)}$ for each $i(1) \in I$ with $i(1) < i$. Hence $I_1$ is an ideal of $I$. Put $I_2 = I \setminus I_1$; the set $I_2$ is a dual ideal of $I$. Thus for each $i(1) \in I_1$ we must have $x_{i(1)} = b_{i(1)}$.

If $i(2) \in I_2$, then there is $i(3) \in I_2$ with $i(3) < i(2)$. If $x_{i(2)} = b_{i(2)}$, then $x_{i(2)} > 2^{i(3)}_{i(2)} = a$, thus $x \not< z^{i(3)} \in C_{22}$, which is a contradiction. Thus $x_{i(2)} = a$ for each $i \in I_2$.

Denote $r = \sup I_1$. Hence $r = \inf I_2$. In view of the properties of $x$ we get $x = t^r$, hence $x \in C_2$, which is a contradiction. ∎

It is obvious that $C_1$ is denumerable and that the power of $C_2$ is equal to the power of the continuum. Hence we have

**Proposition 4.5.** *The lattice $L^a$ does not satisfy the condition* (JD). ∎

## 5. THE LATTICE $L_\alpha$

The aim of the present section is to prove the assertion $(**)$ formulated in the introduction above.

We recall that $Q_0$ is the set of all rationals $q$ with $0 \leqq q \leqq 1$. We consider the natural linear order on $Q_0$. Then $Q_0$ is a completely distributive lattice. Let $I$ be an infinite set of indices; for each $i \in I$ we put $T_i = Q_0$. Further, we set

$$L = \prod_{i \in I} T_i.$$

If $\mathrm{card}(I) = \gamma$, then, under another notation, we can write $L = (Q_0)^\gamma$. The elements of $L$ are written as $x = (x_i)_{i\in I}$.

In particular, if the cardinality of the set $I$ is equal to $\alpha$, then we have

$$\prod_{i\in I} T_i = (Q_0)^\alpha = L_\alpha.$$

Since each linearly ordered set is a completely distributive lattice and since a direct product of such lattices is again completely distributive, we conclude that $L_\alpha$ is completely distributive. It is obvious that $L_\alpha$ is bounded. Further, in view of 4.4, $Q_0$ is isomorphic to a sublattice of the lattice $L^a$. Hence $L_\alpha = Q_0^\alpha$ is isomorphic to a sublattice of $(L^a)^\alpha$.

For $q \in Q_0$, let $x^q$ be the element of $L$ such that $(x^q)_i = q$ for each $i \in I$. Put $C_d = \{x^q\}_{q\in Q_0}$. It is obvious that $C_d$ is a chain in $L$.

**Lemma 5.1.** *$C_d$ is a maximal chain in $L$ and* $\mathrm{card}(C_d) = \aleph_0$.

***Proof.*** The method of verifying the first assertion is the same as in the proof of Lemma 1 in [5]; the second assertion is obvious. ■

We say that $C_d$ is the diagonal chain in $L$.

By applying the Axiom of Choice, we can assume that the set $I$ is well-ordered and possesses the greatest element.

For each $i \in I$, let $R^i$ be the set of all $f \in L$ such that

$$j \in I, j < i \Rightarrow f_j = b_j,$$

$$j \in I, j > i \Rightarrow f_j = a.$$

Put $C_s = \bigcup_{i\in I} R^i$. It is clear that $C_s$ is a chain in $L$ and that $\mathrm{card}(C_s) = \gamma$.

**Lemma 5.2.** *$C_s$ is a maximal chain in $L$.*

***Proof.*** It suffices to apply the same method as in the proof of Lemma 2 in [5] (instead of $M$ we have now the set $I$). ■

We say that $C_s$ is a superficial chain in $L$.

***Proof of*** $(**)$ Let $\alpha$ be an ifninite cardinal. If $\alpha = \aleph_0$, then the assertion of $(**)$ is a consequence of Lemma 4.3 and of the fact that $\operatorname{card}(C_1) = \aleph_0$. Hence it suffices to suppose that $\alpha > \aleph_0$. Also, it suffices to consider the case when $\aleph_0 < \beta \leqq \alpha$. For $\beta = \alpha$ we can apply Lemma 5.2. Now suppose that $\aleph_0 < \beta < \alpha$. Let $I$ be a set of indices with $\operatorname{card}(I) = \alpha$ and let $\beta$ be a cardinal with $\aleph_0 < \beta < \alpha$.

There exist nonempty subsets $I_1, I_2$ of $I$ such that

$$I_1 \cap I_2 = \emptyset, \quad I_1 \cup I_2 = I \quad \text{and } \operatorname{card}(I_1) = \beta.$$

Put

$$L_1 = \prod_{i \in I_1} T_i, \quad L_2 = \prod_{i \in I_2} T_i.$$

Thus we have $L_\alpha = L_1 \times L_2$. For $x \in L_\alpha$ we denote by $x(1)$ and $x(2)$ the component of $x$ in $L_1$ or in $L_2$, respectively. Further, let $0^i, 1^i$ be the least or the greatest element in $L_i$, respectively $(i = 1, 2)$.

There exists a superficial chain $C_s$ in $L_1$; also, there exists the diagonal chains $C_d$ in $L_2$. Put

$$C_1 = \{x \in L : x(1) \in C_s, x(2) = 0^2\},$$

$$C_2 = \{x \in L : x(1) = 1^1, x(2) \in C_d\},$$

$$C = C_1 \cup C_2.$$

Then we have $\operatorname{card}(C_1) = \beta$, $\operatorname{card}(C_2) = \aleph_0$, whence $\operatorname{card}(C) = \beta$. According to Lemma 5 in [5], $C$ is a maximnal chain in $L_\alpha$.

This completes the proof. ∎

## References


[1] G. Birkhoff, *Lattice Theory*, (the 3-rd ed.), Amer. Math. Soc., Providence, RI, 1967.

[2] M. Borowiecki, I. Broere, M. Frick, P. Mihók, and G. Semanišin, *A survey of hereditary properties of graphs*, Discussiones Math.- Graph Theory **17** (1997), 5–50.

[3] M. Borowiecki and P. Mihók, *Hereditary properties of graphs*, p. 41–68 in: "*Advances in Graph Theory*", Vishwa International Publications, Gulbarga 1991.

[4] G. Grätzer and E. T. Schmidt, *On the Jordan-Dedekind chain condition*, Acta Sci. Math. **18** (1957), 52–56.

[5] J. Jakubík, *On the Jordan-Dedekind chain condition*, Acta Sci. Math. **16** (1955), 266–269.

[6] J. Jakubík, *A remark on the Jordan-Dedekind chain condition in Boolean algebras* (Slovak), Časopis Pěst. Mat. **82** (1957), 44–46.

[7] J. Jakubík, *On chains in Boolean algebras* (Slovak), Mat. Fyz. Časopis SAV **8** (1958), 193–202.

[8] J. Jakubík, *Die Jordan-Dedekindsche Bedingung im direkten Produkt von geordneten Mengen*, Acta Sci. Math. **24** (1963), 20–23.

[9] P. Mihók, *On graphs critical with respect to generalized independence numbers*, Colloq. Math. Soc. J. Bolyai **52** (1987), 417–421.

[10] G.N. Raney, *Completely distributive complete lattices*, Proc. Amer. Math. Soc. **3** (1952), 677–680.

[11] G.N. Raney, *A subdirect-union representation for completely distributive lattices*, Proc. Amer. Math. Soc. **4** (1952), 518–522.

[12] G.N. Raney, *Tight Galois connection and complete distributivity*, Trans. Amer. Math. Soc. **97** (1960), 418–426.

[13] R. Sikorski, *Boolean Algebras* (the second edition), Springer-Verlag, Berlin 1964.

[14] G. Szász, *Generalization of a theorem of Birkhoff concerning maximal chains of a certain type of lattices*, Acta Sci. Math. **16** (1955), 89–91.


Received 29 February 2002