ON THE SPECIAL CONTEXT OF INDEPENDENT SETS*

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Abstract

In this paper the context of independent sets $J^p_L$ is assigned to the complete lattice $(\mathcal{P}(M), \subseteq)$ of all subsets of a non-empty set $M$. Some properties of this context, especially the irreducibility and the span, are investigated.

Keywords: context, complete lattice, join-independent and meet-independent sets.

2000 Mathematics Subject Classification: 06B23, 08A02, 08A05.

Let us denote by $(L, \leq)$ a complete lattice in which $\lor, \land$ mean the supremum and the infimum of any subset of $L$, respectively. The least and the greatest elements in $(L, \leq)$ are denoted by 0, 1, respectively. If $a, b \in L$, then $a \parallel b$ means that $a, b$ are incomparable in $(L, \leq)$.

For a subset $A \subseteq L$ we put $U(A) = \{x \in L \mid (\forall a \in A)[a \leq x]\}$ and $L(A) = \{x \in L \mid (\forall a \in A)[x \leq a]\}$. Obviously, $U(A) = U(\lor A)$ and $L(A) = L(\land A)$. Moreover, let us put $|A| := \text{card } A$ and $A_a := A \setminus \{a\}$, $s_a := \lor A_a$, $i_a := \land A_a$ for all $a \in A$.

Definition 1 (F. Machala, [6]). A subset $A \subseteq L$ is said to be join-independent if and only if $a \not\leq s_a$ for all $a \in A$. A subset $B \subseteq L$ is said to be meet-independent if and only if $i_b \not\leq b$ for all $b \in B$.

*Supported by the Council of Czech Government J14/98: 153100011.
Remark 1. The concepts of join-independent and meet-independent sets are the special cases of the definition of independent sets in a context † (an incidence structure) or, more precisely, in two closure spaces associated to each context. Any complete lattice $(L, \leq)$ (and even any partially ordered set) can be understood as the context $(L, L, \leq)$, where (under the denotation established for contexts) $A^\uparrow = U(A), A^{\uparrow \downarrow} = L U(A)$ and $B^\downarrow = L(B), B^{\downarrow \uparrow} = U L(B)$ for $A, B \subseteq L$. The closure operators are given by $A \mapsto A^{\uparrow \downarrow}, B \mapsto B^{\downarrow \uparrow}$ for $A, B \subseteq L$.

The notion of an independent set in a lattice appears in various approaches in literature (see [1], [3], [4], [7], [9] and [10]). In fact, in this paper irredundant sets in complete lattices are discussed, but we prefer to use the terms "join-independent" and "meet-independent" with respect to connections with closure systems and incidence structures.

Remark 2. The notions of join- and meet-independent sets are dual in complete lattices. In the following we will only investigate join-independent sets. Analogous results for meet-independent sets can be obtained dually.

Propositions 1–7 are easy consequences of the definitions of join- and meet-independencies. Thus, the proofs of them are omitted.

Proposition 1. Every singleton $A = \{a\}, a \neq 0, a \in L$, is join-independent.

Proposition 2. A subset $A \subseteq L, |A| \geq 2$, is join-independent if and only if $a \| s_a$ for all $a \in A$.

Proposition 3. If a subset $A \subseteq L$ is join-independent, then $a \| b$ for all $a, b \in A, a \neq b$.

Let us introduce one more denotation: If $A \subseteq L$, then for $a \in A$ we put $X^A(a) := U(s_a) \setminus U(a), Y^A(a) := L(i_a) \setminus L(a)$.

Proposition 4. If $A \subseteq L$ is join-independent, then $X^A(a) \cap X^A(b) = \emptyset$ for any $a, b \in A, a \neq b$.

Proposition 5. The subset $A \subseteq L$ is join-independent if and only if $X^A(a) \neq \emptyset$ for all $a \in A$.

†A context is the triple $(G, H, I)$, where $G$ and $H$ are sets and $I \subseteq G \times H$ (see [2]).
Proposition 6. If the subset \( A \subseteq L \) is join-independent, then every choice \( Q^A = \{ m_a \in X^A(a) \mid a \in A \} \) is a meet-independent set.

Remark 3. Let \( A \subseteq L \) be join-independent. Then for any choice \( Q^A = \{ m_a \in X^A(a) \mid a \in A \} \) the mapping \( \alpha : a \mapsto m_a \) is a one-to-one mapping of the join-independent set \( A \) onto the meet-independent set \( Q^A \). Analogously for a meet-independent subset. This is called a norming mapping of the set \( A \) (see [5]). If we denote by \( L^p_j (L^p_m) \) the set of all \( p \)-element join-independent (meet-independent) sets of \((L, \leq)\) \((p \) is any cardinal number), then it is possible to define the context of independent sets \( J^p_L = (L^p_j, L^p_m, I^p) \), where the relation \( I^p \) is given by the following: For \( A \in L^p_j, B \in L^p_m \) we put \( AI^pB \) if and only if there exists a norming mapping \( \alpha : A \to B \). (If \( L^p_j = \emptyset \), then \( L^p_m = \emptyset \) and \( J^p_L = (\emptyset, \emptyset, \emptyset) \).) If \( A \in L^p_j \), then obviously \( AI^pS_A \) where \( S_A = \{ s_a \mid a \in A \} \).

Proposition 7. If a set \( A \subseteq L \) is join-independent, then every subset of \( A \) is join-independent.

Now we recall some basic notions from the general theory of contexts (see [8]):

Definition 2. Let \( J = (G, H, I) \) be a context. A sequence \( (g_0, m_0, g_1, m_1, \ldots, g_{r-1}, m_{r-1}, g_r) \), where \( g_i \in G \) for \( i \in \{0, \ldots, r\} \), \( m_j \in H \) for \( j \in \{0, \ldots, r-1\} \) and \( g_i Im_j, g_{j+1} Im_j \) for all \( j \in \{0, \ldots, r-1\} \), is called a path between elements \( g_0 \) and \( g_r \). In a similar way we can define a path between two elements of \( H \).

A positive integer \( r \) is said to be a length of a path between elements \( g_0, g_r \). We suppose that the path \( (g, m, g) \) has a length 0. If a path between two elements of \( G \) exists, then we say that they are joinable. The context \( J \) is said to be irreducible if every two elements of \( G \) are joinable. The minimal length of all paths between elements \( g, h \in G \) we call a distance of these elements and denote by \( v(g, h) \). The maximal distance of any two elements of \( G \) in an irreducible context \( J \) is said to be a span of \( G \) and denoted by \( d(G) \). Similarly for the set \( H \).

We will investigate the contexts of independent sets (their joinability, distances, irreducibility, spans) associated to the lattice \((\mathcal{P}(M), \subseteq)\) where \( \mathcal{P}(M) \) denotes the power set of a non-empty set \( M \). Thus \((\mathcal{P}(M), \subseteq)\) is the complete (boolean) lattice of all subsets of \( M \).
Let us denote by \( \mathcal{M} = \{\{a\} \mid a \in M\} \subseteq \mathcal{P}(M) \) the set of all atoms of \( (\mathcal{P}(M), \subseteq) \). This set (and every its subset) is obviously join-independent.

Further we put \( \mathcal{N} = \{s_a \mid a \in M\} \) where \( s_a = \bigvee \mathcal{M}_{\{a\}} = \bigvee (\mathcal{M} \setminus \{\{a\}\}) \). Then \( \mathcal{N} \) is the set of all coatoms of \( (\mathcal{P}(M), \subseteq) \) and it is meet-independent (also every its subset).

In what follows, \( \mathcal{J}_L^p = (L_j^p, L_m^p, P) \) denotes the context of the \( p \)-element independent sets associated to the lattice \( (\mathcal{P}(M), \subseteq) \), where \( M \) is a non-empty set and \( p \) is any cardinal number.

**Proposition 8.** The following statements are equivalent:

1. \( |M| < p \),
2. \( L_j^p = \emptyset \).

**Proof.** \( 1 \iff 2 : \) Let \( A = \{A_i \mid i \in J\} \in L_j^p \) where \( A_i \subseteq M \) and \( |J| = p \), \( |M| < p \). If we put \( J_i := J \setminus \{i\} \), then \( A_i \subseteq \bigcup_{j \in J_i} A_j \) for all \( i \in J \). This implies \( (A_i \setminus \bigcup_{j \in J_i} A_j) = A_i \neq \emptyset \). For each \( a \in A_i \) we have \( a \notin A_j \) for all \( j \in J_i \). Then we can make a choice \( M' = \{a^i \in A_i \mid i \in J\} \subseteq M \) and \( \alpha : a^i \mapsto i \) is a one-to-one mapping of the subset \( M' \) of \( M \) onto \( J \). However, this is a contradiction to \( |M| < p \).

2 \( \iff 1 : \) Let us assume that \( L_j^p = \emptyset \) and \( p \leq |M| \). Then there exists a subset \( \mathcal{M'} \subseteq \mathcal{M} \) such that \( |\mathcal{M'}| = p \). Since every subset of \( \mathcal{M} \) is join-independent, we get \( \mathcal{M'} \in L_j^p \) and \( L_j^p \neq \emptyset \). Thus \( |M| < p \). \( \blacksquare \)

**Proposition 9.** Let \( p \) be a finite cardinal number. Then the following statements are equivalent:

1. \( |M| = p \),
2. \( L_j^p = \{\mathcal{M}\} \).

**Proof.** \( 1 \iff 2 : \) Let \( A = \{A_i \mid i \in J\} \in L_j^p \), \( A_i \subseteq M \) and \( |J| = p = |M| \). Then \( A_i \subseteq \bigcup_{j \in J_i} A_j \) for all \( i \in J \), where \( J_i = J \setminus \{i\} \) again. Hence \( A_i \cap \bigcup_{j \in J_i} A_j = A_i \neq \emptyset \) for all \( i \in J \).

Assume that \( x \in A^r \cap A^s \) for some \( r, s \in J, r \neq s \). Then \( x \in A^r, x \notin A_j \) for all \( j \in J, j \neq r \). Thus \( x \notin A^s \) which is a contradiction to \( x \in A^s \subseteq A_s \).

We have obtained \( A^i \cap A^j = \emptyset \) for all \( i, j \in J, i \neq j \).

If we make a choice \( M' = \{a^i \in A_i \mid i \in J\} \subseteq M \), then \( \alpha : a^i \mapsto i \) is a bijection of \( M' \) onto \( J \) and because of \( |M| = |J| \) we have \( M' = M \).

Therefore \( |A^i| = 1 \) for all \( i \in J \). Let \( A^t = \{a\} \) for a certain \( t \in J \). Then \( a \in A_t \) if \( b \in A_t, b \neq a \), then at the same time \( b \in A^u \) for a certain \( u \neq t \).
This yields \( b \notin A_i \) which is a contradiction. Hence, \( |A_i| = 1 \) for all \( i \in J \). We have proved that \( A_i = \{ a_i \} \) for all \( i \in J \). It means that the only \( p \)-element join-independent set is \( M \).

2 \( \iff \) 1: According to the previous proposition, \( p \leq |M| \). Every \( p \)-element set of atoms \( \{ \{ a_i \} \mid i \in I \} \subseteq \mathcal{P}(M), |I| = p \), is join-independent.

If \( p < |M| \), then there exist at least two distinct \( p \)-element sets of atoms. Thus \( |L^p_j| > 1 \). ■

**Example.** If \( |M| = 3 \), then \( |L^2_j| = 9 \) and \( |L^3_j| = 1 \). If \( |M| = 4 \), then \( |L^2_j| = 55, |L^3_j| = 26 \) and \( |L^4_j| = 1 \).

**Proposition 10.** The set \( \{ A_i \mid i \in J \} \) is join-independent in \( (\mathcal{P}(M), \subseteq) \) if and only if the set \( \{ M\setminus A_i \mid i \in J \} \) is meet-independent in \( (\mathcal{P}(M), \subseteq) \).

**Proof.** For all \( i \in J \) we put \( J_i = J \setminus \{ i \} \). Then it is easy to see that

\[
A_i \not\subseteq \bigcup_{j \in J_i} A_j \iff M \setminus \bigcup_{j \in J_i} A_j \not\subseteq M \setminus A_i \iff \bigcap_{j \in J_i} (M \setminus A_j) \not\subseteq M \setminus A_i.
\]

■

**Remark 4.** It follows from Propositions 8 – 10 that \( p > |M| \) if and only if \( \mathcal{J}_L^p = (\emptyset, \emptyset, \emptyset) \), and \( p = |M| \) if and only if \( |L^p_j| = |L^p_m| = 1 \). Also in the case \( p < |M| \) we get \( |L^p_j| = |L^p_m| \).

**Proposition 11.** Let \( A, B \in L^p_m, A = \{ A_i \mid i \in J \}, B = \{ B_i \mid i \in J \}, |J| = p \). If we denote \( C = \{ M\setminus A_i \mid i \in J \}, D = \{ M\setminus B_i \mid i \in J \} \), then \( v(A, B) = 1 \) if and only if \( v(C, D) = 1 \).

**Proof.** Assume that \( v(A, B) = 1 \). Then there exists \( \tilde{A} \in L^p_m \) such that \( AIP\tilde{A}, BIP\tilde{A} \). Let us put \( \tilde{A} = \{ \tilde{A}_i \mid i \in J \} \) and \( J_i = J \setminus \{ i \} \). Under a suitable enumeration we get \( \tilde{A}_i \in X^A(A_i) \cap X^B(B_i) \) for all \( i \in J \). Thus \( \bigcup_{j \in J_i} A_j \subseteq \tilde{A}_i, A_i \not\subseteq \tilde{A}_i \) and \( \bigcup_{j \in J_i} B_j \subseteq A_i, B_i \not\subseteq A_i \) for all \( i \in J \). Let us put \( \tilde{C}_i = M\setminus \tilde{A}_i \). Then we have \( \tilde{C}_i \subseteq M \setminus A_i, \tilde{C}_i \not\subseteq M \setminus A_i \). This yields \( \tilde{C}_i \in Y^C(M \setminus A_i) \) and \( \tilde{C}_i \in Y^D(M \setminus B_i) \). Thus \( \tilde{C}_i \in Y^C(M \setminus A_i) \cap Y^D(M \setminus B_i) \) for all \( i \in J \). If we denote \( \tilde{C} = \{ \tilde{C}_i \mid i \in J \} \), then \( \tilde{C}IP\tilde{C}, \tilde{C}IPD \) and \( v(C, D) = 1 \). Similarly for the converse assertion. ■

**Proposition 12.** The sets \( A = \{ A_i \mid i \in J \}, B = \{ B_i \mid i \in J \} \in L^p_j \) are joinable in \( \mathcal{J}_L^p \) if and only if the sets \( C = \{ M\setminus A_i \mid i \in J \}, D = \{ M\setminus B_i \mid i \in J \} \) are joinable in \( \mathcal{J}_L^p \).
Proof. The sets \( A, B \in L_p^p \) are joinable if and only if there exist sets \( A_1', \ldots, A_r' \in L_p^p, A_1'', \ldots, A_{r+1}'' \in L_{p_0}^p \) such that \( A^{IP} A_1'', A_1' IP A''_1, A_1' IP A''_2, A_2' IP A''_2, \ldots, A_r' IP A''_r, A_r' IP A''_{r+1}, B^{IP} A''_{r+1} \). Thus, \( v(A, A_1') = v(A_1', A_2') = \cdots = v(A_r', B) = 1 \). It follows from propositions 10 and 11 that there exist meet-independent sets \( A_1', A_2', \ldots, A_r' \) such that \( v(C, A_1') = v(A_1', A_2') = \cdots = v(A_r', D) = 1 \). Hence, the sets \( C, D \) are joinable. Similarly for the converse assertion.

Remark 5. If \( A \subseteq M \) (the subset of atoms), then for \( \{a\} \in A \) we will write just \( X^A(a), A_a, U(a) \) etc. instead of (more correct) \( X^A(\{a\}), A_{\{a\}}, U(\{a\}) \) etc. Then \( X^A(a) = U(\forall A_a) \setminus U(a) \) and hence, \( Y_a \in X^A(a) \) if and only if \( A_a \subseteq Y_a, a \notin Y_a \).

Proposition 13. If \( A, B \subseteq M, A \neq B, |A| = |B| = p, \) then \( v(A, B) = 1 \).

Proof. Let us denote \( C = A \cap B \). There exists a bijective mapping \( \varphi : A \rightarrow B \) such that \( \varphi(c) = c \) for all \( c \in C \). Further we put \( Y_a = A_a \cup B_\varphi(a) \) for all \( a \in A \). If \( a \in C \), then \( a = \varphi(a) \) and \( a \notin A_a, B_a \). Thus \( a \notin Y_a \). If \( a \notin C \), then \( a \notin B \) and \( a \notin Y_a \). Similarly, \( \varphi(a) \notin A \) and \( \varphi(a) \notin Y_a \). It follows that \( Y_a \in X^A(a) \cap X^B(\varphi(a)) \). If we put \( Y = \{Y_a | a \in A\} \), then \( A \rightarrow Y : a \mapsto Y_a \) and \( B \rightarrow Y : \varphi(a) \mapsto Y_a \) are norming mappings. Thus, \( A^{IP} Y, B^{IP} Y \) and \( v(A, B) = 1 \).

Proposition 14. If \( A, B \subseteq N, A \neq B, |A| = |B| = p, \) then \( v(A, B) = 1 \).

Proof. Dual to the previous one.

Theorem 1. Let \( J_L^p \) be a context of independent sets associated to the complete lattice \( (P(M), \subseteq) \), where \( M \) is a non-empty set and \( p \) is a cardinal number with the property \( 3 \leq p < |M| \). Then \( J_L^p \) is irreducible and (the span) \( d(L_p^p) = 2 \).

Proof. Consider join-independent sets \( A = \{A_i | i \in J\}, B = \{B_i | i \in J\} \), where \( A_i, B_i \subseteq M \) for all \( i \in J, |J| = p \). For each \( i \in J \), we put \( J_i = J \setminus \{i\} \) and \( A^i = \bigcup_{j \in J_i} A_j \). Then \( Y \in X^A(A_i) \) if and only if \( A^i \subseteq Y, A_i \not\subseteq Y \). Since \( A \) is join-independent, we have \( A_i \not\subseteq A^i \) for all \( i \in J \). It follows that there always exists an element \( a_i \in A_i \) such that \( a_i \notin A^i \). Then \( A^i \subseteq M_{a_i} = s_{a_i} \). From \( a_i \notin s_{a_i} \), we get \( A_i \not\subseteq s_{a_i} \) and hence \( s_{a_i} \in X^A(A_i) \). We can make a choice \( Y_1 = \{s_{a_i} | i \in J\} \). The set \( Y_1 \subseteq N \) is meet-independent and \( A^{IP} Y_1 \).

In a similar way, we can proceed in the case of the set \( B \) and we obtain...
BI\(p\)Y\(_2\) for a certain set \(Y_2 \subseteq N\). According to Proposition 14, there exists a set \(C \subseteq \mathcal{M}, |C| = p\) such that \(CIPY_1, CIPY_2\). Thus \(v(A, B) \leq 2\).

It remains to find join-independent sets \(A = \{A_i \mid i \in J\}, B = \{B_i \mid i \in J\}, |J| = p\), such that \(v(A, B) = 2\). We determine them in the following way:

Consider three distinct elements \(a, b, c \in M\). Let us put \(A_1 = \{a, b\} = B_1, A_2 = \{a, c\}, B_2 = \{b, c\}\) and \(A_i = B_i = \{x_i\}\) for the other sets where \(x_i \in M\) are pairwise distinct elements not equal to \(a, b, c\). Moreover, we denote \(C = \{a, b, c\}\) and \(X = \{x_i \mid i \in J'\}\).

It is easy to verify that the sets \(A, B\) defined above are join-independent. Obviously, \(X^A(x_i) = X^B(x_i)\) for all \(i \in J'\) and \(X^A(A_2) = X^B(B_2)\). It is also clear that \(Y \subseteq X^A(A_1)\) if and only if \(\{a, c\} \cup X \subseteq Y, A_1 \not\subseteq Y\), and \(Y \subseteq X^B(B_1)\) if and only if \(\{b, c\} \cup X \subseteq Y, B_1 \not\subseteq Y\). Let \(Y \in X^A(A_1) \cap X^B(B_1)\). Then \(C \cup X \subseteq Y\) which is a contradiction to \(A_1, B_1 \subseteq Y\). Therefore, there is no meet-independent set \(Z\) such that \(AIPZ, BIPZ\). Thus \(v(A, B) = 2\).

Remark 6. Dually we can prove that also every two meet-independent sets are joinable and \(d(L^p_n) = 2\).

References


Received 2 August 2000
Revised 3 April 2001