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SOLUTION OF BELOUSOV'S PROBLEM

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Abstract

The authors prove that a local n-quasigroup defined by the equation

$$x_{n+1} = F(x_1, \dots, x_n) = \frac{f_1(x_1) + \dots + f_n(x_n)}{x_1 + \dots + x_n},$$

where $f_i(x_i),\ i, j = 1, \dots, n$, are arbitrary functions, is irreducible if and only if any two functions $f_i(x_i)$ and $f_j(x_j)$, $i \neq j$, are not both linear homogeneous, or these functions are linear homogeneous but $\frac{f_i(x_i)}{x_i} \neq \frac{f_j(x_j)}{x_j}$. This gives a solution of Belousov's problem to construct examples of irreducible n-quasigroups for any $n \geq 3$.

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1. Introduction

An *n-quasigroup* is a set Q with an n-ary operation A such that each equation $A(a_1, a_2, \dots a_{i-1}, x, a_{i+1}, \dots, a_n) = b$ is uniquely solvable with respect to x $(i = 1, \dots, n)$. An n-quasigroup A, $n > 2$, is *reducible* if there exist a

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k-ary operation B and $(n-k+1)$-ary operation C such that $A(x_1, \dots, x_n) = B(C(x_1, \dots, x_k), x_{k+1}, \dots, x_n)$. Otherwise, an n-quasigroup A is said to be *irreducible.*

In his monograph [1], Belousov posed the following problem (p. 217, problem 5): *Construct examples of irreducible n-quasigroups, $n > 3$. Do there exist irreducible n-quasigroups for any $n > 3$?* We describe now the development in the solution of this problem.

- Belousov and Sandik (see [2]) constructed an example of an irreducible 3-quasigroup of order 4 (see also [1], p. 115).

- Frenkin (see [5]) proved that for any $n \geq 3$, there exist irreducible n-quasigroups of order 4.

- Using methods of web geometry, Goldberg proved in [8], [9] (see also the book [10], Ch. 4) an existence of infinite local irreducible n-quasigroups for any $n \geq 3$. It is well-known that the theory of $(n+1)$-webs is equivalent to the theory of local differentiable n-quasigroups. Goldberg proved that in general an arbitrary $(n+1)$-web (or a local differentiable n-quasigroup) is irreducible.

- One year later, independently, using algebraic methods, Glukhov in [6] (see also [7]) proved an existence of infinite irreducible n-quasigroups for any $n \geq 3$.

- For any $n \geq 3$, Borisenko (see [4]) constructed examples of irreducible n-quasigroups of finite composite order $t > 4$.

Note that in all these works, *no examples of infinite irreducible n-quasigroups were given.*

In the current paper we present the *simplest* examples of local irreducible and reducible n-quasigroups. They are coordinate n-quasigroups of a series of irreducible and reducible $(n+1)$-webs. We came to these examples from the web theory. However, to make this paper accessible for mathematicians not working in web geometry, in our presentation we formulate the main results and their proofs without using the web geometry terminology.

Note also that we could give much more examples of local irreducible n-quasigroups but they will be more complicated than examples considered in this paper.

2. Preliminaries

Suppose that a local differentiable n-quasigroup A is given on a differentiable manifold Q by the equation

$$x_{n+1} = F(x_1, x_2, \dots, x_n), \tag{1}$$

where F is a C^2-function. First, we indicate invariant conditions for such a local n-quasigroup A to be reducible.

Note that for a ternary quasigroup (i.e., when $n = 3$), Goldberg proved in [8] (see also [10]) that a 3-quasigroup A defined by the equation

$$x_4 = F(x_1, x_2, x_3) \tag{2}$$

is reducible of type

$$x_4 = g(h(x_1, x_2), x_3) \tag{3}$$

(where g and h are differentiable functions) if and only if the function F satisfies the following second-order nonlinear partial differential equation

$$\frac{F_{31}}{F_{32}} = \frac{F_1}{F_2}. \tag{4}$$

Here and in what follows, we use the notation

$$F_i = \frac{\partial F}{\partial x_i}, \quad F_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j}, \quad i, j = 1, \dots, n.$$

Equation (4) was noticed by Goursat [11], as far back as 1899, who indicated that function (3), where g and h are arbitrary functions of two variables each, is a general solution of equation (4).

We define a local reducible n-quasigroup. Without loss of generality, we can define a local *reducible* n-quasigroup A as an n-quasigroup for which the function $F(x_1, \dots, x_n)$ has the following form:

$$x_{n+1} = g(h(x_1, \dots, x_k), x_{k+1}, \dots, x_n), \tag{5}$$

(i.e., its operation is reduced to a k-ary and $(n+1-k)$-ary operations, $2 \leq k \leq n-1$). In terminology of [2] (see also [1]), such an n-quasigroup is $(1, k)$-reducible. Goldberg (see [9] or [10]) found necessary and sufficient conditions for an n-quasigroup (1) to be reducible. For reducibility of type (5), these conditions are: the function F must satisfy the following system of second-order nonlinear partial differential equations

$$\frac{F_{pa}}{F_{pb}} = \frac{F_a}{F_b}, \quad a, b = 1, \dots k, \ a \neq b; \ p, q = k+1, \dots, n. \tag{6}$$

The proof of this statement is straightforward: conditions (6) can be obtained from (5) by differentiation, and it can be shown that the function defined by equation (5) is a general solution of the system (6).

Let us consider a few examples.

Example 1. If an n-quasigroup is (1, 2)-reducible, i.e., if

$$x_{n+1} = g(h(x_1, x_2), x_3 \dots, x_n), \tag{7}$$

then conditions (6) take the form

$$\frac{F_{p1}}{F_{p2}} = \frac{F_1}{F_2}, \quad p = 3, \dots, n. \tag{8}$$

Example 2. If an n-quasigroup is $(1, 2)$- and $(3, 5)$-reducible, i.e., if

$$x_{n+1} = g(h(x_1, x_2), k(x_3, x_4, x_5), x_6, \dots, x_n), \tag{9}$$

then conditions (6) take the form

$$\begin{cases} \dfrac{F_{p1}}{F_{p2}} = \dfrac{F_1}{F_2}, \quad p = 3, \dots, n; \\[2ex] \dfrac{F_{\sigma a}}{F_{\sigma b}} = \dfrac{F_a}{F_b}, \quad a, b = 3, 4, 5, \ a \neq b; \ \sigma = 1, 2, 6, 7, \dots, n. \end{cases} \tag{10}$$

Example 3. If an n-quasigroup is $(1,2,3)$- and $(1,2)$-reducible, i.e., if

$$x_{n+1} = g(h(k(x_1, x_2), x_3)), x_4, \dots, x_n), \tag{11}$$

then conditions (6) take the form

$$\begin{cases} \dfrac{F_{pa}}{F_{pb}} = \dfrac{F_a}{F_b}, \ \ a, b = 1, 2, 3, \ a \neq b; \ p = 4, 5, \dots, n; \\[2ex] \dfrac{F_{31}}{F_{32}} = \dfrac{F_1}{F_2}. \end{cases} \tag{12}$$

These three examples show how to get conditions (6) for different kinds of reducibilities.

Now we will formulate our main theorem.

Theorem 1. *A local n-quasigroup defined by the equation*

$$x_{n+1} = F(x_1, \dots, x_n) = \frac{f_1(x_1) + \dots + f_n(x_n)}{x_1 + \dots + x_n} \tag{13}$$

is irreducible if and only if any two functions $f_i(x_i)$ and $f_j(x_j)$, $i \neq j$, are not both linear homogeneous, or these functions are both linear homogeneous but $\frac{f_i(x_i)}{x_i} \neq \frac{f_j(x_j)}{x_j}$.

The proof of this theorem follows from the next theorem on necessary and sufficient conditions for an n-quasigroup defined by equation (13) to be reducible.

Note that the function (13) defines an n-cone system (see [12]).

Theorem 2. *A local n-quasigroup defined by equation* (13) *is* $(1, k)$*-reducible if and only if*

$$f_a(x_a) = c x_a, \ \ a = 1, \dots, k, \tag{14}$$

where c is the same constant for all $a = 1, \dots, k$.

Proof. A necessary and sufficient condition for an n-quasigroup to be $(1,k)$-reducible is that the function F from (13) satisfies the system of partial differential equations (6). We write equations (6) for any two fixed different values a and b, $a, b = 1, \dots, k$:

$$\frac{F_{pa}}{F_{pb}} = \frac{F_a}{F_b}. \tag{15}$$

For the n-quasigroup defined by equation (13), equation (15) takes the form

$$(f'_a(x_a) - f'_b(x_b))[f'_p(x_p)(x_1 + \ldots + x_n) - (f_1(x_1) + \ldots + f_n(x_n))] = 0. \tag{16}$$

First, we assume that

$$f'_p(x_p)(x_1 + \ldots + x_n) - (f_1(x_1) + \ldots + f_n(x_n)) = 0.$$

Then

$$f'_p(x_p) = \frac{f_1(x_1) + \ldots + f_n(x_n)}{x_1 + \ldots + x_n} (= F).$$

Since the left-hand side does not depend on x_a, then $F_a = 0$. But $F_p = 0$ implies that $f'_p(x_p) \sum_i x_i - \sum_i f_i(x_i) = 0$, i.e., $f'_a(x_a) = F$. This along with $f'_p(x_p) = F$ leads to $f'_i(x_i) = F, \; i = 1, \dots, n$. These equalities are possible if and only if

$$F = A = (\text{const.})$$

However, in this case equation (13) does not define a local n-quasigroup since it is not solvable with respect to the variables $x_i, \; i = 1, \dots, n$.

Leaving this case aside, we find from condition (16) that

$$f'_a(x_a) = f'_b(x_b).$$

We can see again that both sides of this equation are constant:

$$f'_a(x_a) = f'_b(x_b) = c \; (= \text{const.}).$$

Integration gives

$$f_a(x_a) = cx_a, \quad f_b(x_b) = cx_b.$$

Since a and b are arbitrary numbers from $1, \dots, k$, this proves that

$$f_a(x_a) = cx_a, \quad a = 1, \dots, k.$$

Thus,

$$F = \frac{c(x_1 + \ldots + x_k) + f_{k+1}(x_{k+1}) + \ldots + f_n(x_n)}{x_1 + \ldots + x_n}. \tag{17}$$

■

The following corollary gives a geometric meaning of local reducible n-quasigroups defined by equation (13). First, note that in $\mathbb{R}^n$ equation (13) determines an $(n+1)$-web W formed by hyperplanes $x_i = c_i, \ i = 1, \dots, n$, parallel to the coordinate hyperplanes of a Cartesian coordinate system of $\mathbb{R}^n$ and by a family λ_{n+1} of hypersurfaces V defined by the equations $F = \alpha (= \text{const.})$

Corollary 3. *A local n-quasigroup* (13) *is reducible if and only if in a Cartesian coordinate system of $\mathbb{R}^n$, the normal vector to any hypersurface of the $(n+1)$-web W has at least two equal coordinates* (*i.e., if at least two projections of this vector onto the coordinate axes are equal*).

Proof. In fact, the equation of a hypersurface V is

$$f_1(x_1) + \ldots + f_n(x_n) - \alpha(x_1 + \ldots + x_n) = 0,$$

where α is a constant.

In a Cartesian coordinate system of $\mathbb{R}^n$, the normal vector $\mathbf{N}$ at an arbitrary point of the hypersurface V has the coordinates

$$f_1'(x_1) - \alpha, \dots, f_n'(x_n) - \alpha.$$

Thus, if two of these coordinates are equal, this implies $f_i'(x_i) = f_j'(x_j), i \neq j$. As we saw in the proof of Theorem 2, this implies that $f_i(x_i) + f_j(x_j) = c(x_i + x_j)$, and the local n-quasigroup is reducible.

The converse statement is trivial: it follows from equation (17) if one calculate the coordinates of a normal vector. ■

By Theorem 2, for reducibilities of types (7), (9), and (11), the function F defined by (13) has the forms

$$F = \frac{c(x_1 + x_2) + f_3(x_3) + \ldots + f_n(x_n)}{x_1 + \ldots + x_n},$$

$$F = \frac{c(x_1 + x_2) + e(x_3 + x_4 + x_5) + f_6(x_6) + \ldots + f_n(x_n)}{x_1 + \ldots + x_n},$$

and

$$F = \frac{c(x_1 + x_2 + x_3) + f_4(x_4) + \ldots + f_n(x_n)}{x_1 + \ldots + x_n}$$

(where c and e are constants), respectively.

Example 4. It follows from Theorem 1 that we obtain the simplest example of an irreducible n-quasigroup of type (13) by taking the functions $f_i(x_i) = c_i x_i, \ i, j = 1, \ldots, n$, where $c_i \neq c_j$ if $i \neq j$.

One can produce numerous examples of irreducible n-quasigroups of this kind. For example, the local n-quasigroup defined by the equation

$$x_{n+1} = \frac{x_1 + 2x_2 + 3x_3 + \ldots + nx_n}{x_1 + \ldots + x_n}, \quad n \geq 3, \tag{18}$$

is irreducible. An $(n+1)$-web in $\mathbb{R}^n$ corresponding to this n-quasigroup is formed by n pencils of hyperplanes parallel to the coordinate hyperplanes of a Cartesian coordinate system of $\mathbb{R}^n$ and a pencil of hyperplanes whose axis is an $(n-2)$-plane defined by the equations

$$x_1 + 2x_2 + 3x_3 + \ldots + nx_n = 0, \quad x_1 + \ldots + x_n = 0.$$

Note that for $n = 2$, equation (18) defines a parallelizable web in a 2-plane.

It is easy to see that a local n-quasigroup defined by equation (13) is isotopic to the n-quasigroup defined by the equation

$$x_{n+1} = F(x_1, \ldots, x_n) = \frac{f_1(x_1) + \ldots + f_n(x_n) + A}{x_1 + \ldots + x_n + a}, \tag{19}$$

where A and a are constants. In fact, we can make two successive isotopic transformations for n-quasigroup (19):

$$x_n + a \to x_n.$$

and

$$f_n(x_n - a) + A \to f_n(x_n).$$

As a result, we will get the n-quasigroup defined by equation (13). Thus, the local n-quasigroups defined by equations (13) and (19) are isotopic.

Example 5. If we take $f(x_i) = x_i^2$ in (21), then we obtain the local n-quasigroup defined by

$$x_{n+1} = F(x_1, \dots, x_n) = \frac{(x_1)^2 + \dots + (x_n)^2 + A}{x_1 + \dots + x_n + a}. \tag{20}$$

By Theorem 1, this n-quasigroup is irreducible.

It is easy to see that $F = \alpha$ (= const) defines a family λ_{n+1} of hyperspheres in $\mathbb{R}^n$. If all these hyperspheres pass through the points $(1, 0, \dots, 0)$, $(0, 1, \dots, 0), (0, 0, \dots, 1)$, then (20) implies that $1 + A = \alpha(1 + a)$. Since the last equation must be valid for any α, it follows that $A = a = -1$. The family λ_{n+1} along with the families $\lambda_i,\ i = 1, \dots, n$, of hyperplanes $x_i = c_i$ parallel to the coordinate hyperplanes of a Cartesian coordinate system of $\mathbb{R}^n$ form an irreducible $(n+1)$-web.

In particular, if $n = 3$, we have

$$x_4 = \frac{(x_1)^2 + (x_2)^2 + (x_3)^2 - 1}{x_1 + x_2 + x_3 - 1}.$$

In this case, the family λ_4 of the web W is the 1-parameter family of hyperspheres in $\mathbb{R}^4$ passing through the points (1, 0, 0), (0, 1, 0), and (0, 0, 1).

Moreover, if $n = 2$, then we have

$$x_3 = \frac{(x_1)^2 + (x_2)^2 - 1}{x_1 + x_2 - 1}.$$

In this case, the equation

$$(x_1)^2 + (x_2)^2 - 1 - \alpha(x_1 + x_2 - 1) = 0, \ \alpha = \text{const.},$$

determines an 1-parameter family of circles in $\mathbb{R}^3$ passing through the points (1, 0) and (0, 1). This family and the two 1-parameter families of straight lines parallel to the coordinate lines of a Cartesian coordinate system of $\mathbb{R}^2$ form a nonhexagonal 3-web (cf. [3], §3, where the same 3-web is considered — the only difference is that in [3], circles pass through the points (0, 0) and (1, 1)).

References

[1] V.D. Belousov, *n-ary quasigroups* (Russian), Izdat. "Shtiintsa", Kishinev 1972, 227 pp.

[2] V.D. Belousov, and M. D. Sandik, *n-ary quasigroups and loops* (Russian), Sibirsk. Mat. Zh. **7** (1966), no. 1, 31–54. (English transl. in: Siberian Math. J. **7** (1966), no. 1, 24–42).

[3] W. Blaschke, *Einführung in die Geometrie der Waben*, Birkhäuser-Verlag, Basel-Stuttgart 1955, 108 pp. (Russian transl. GITTL, Moscow 1959, 144 pp.

[4] V.V. Borisenko, *Irreducible n-quasigroups on finite sets of composite order* (Russian), Mat. Issled., Vyp. **51** (1979), 38–42.

[5] B.R. Frenkin, *Reducibility and uniform reducibility in certain classes of n-groupoids II* (Russian), Mat. Issled., Vyp. **7** (1972), no. 1 (23), 150–162.

[6] M.M. Glukhov, *Varieties of (i, j)-reducible n-quasigroups* (Russian), Mat. Issled., Vyp. **39** (1976), 67–72.

[7] M.M. Glukhov, *On the question of reducibility of principal parastrophies of n-quasigroups* (Russian), Mat. Issled., Vyp. **113** (1990), 37–41.

[8] V.V. Goldberg, *The invariant characterization of certain closure conditions in ternary quasigroups* (Russian), Sibirsk. Mat. Zh. **16** (1975), no. 1, 29–43. (English transl. in: Siberian Math. J. **16** (1975), no. 1, 23–34).

[9] V.V. Goldberg, *Reducible $(n + 1)$-webs, group $(n + 1)$-webs, and $(2n + 2)$-hedral $(n+1)$-webs of multidimensional surfaces* (Russian), Sibirsk. Mat. Zh. **17** (1976), no. 1, 44–57. (English transl. in: Siberian Math. J. **17** (1976), no. 1, 34–44).

[10] V.V. Goldberg, *Theory of Multicodimensional $(n+1)$-Webs*, Kluwer Academic Publishers, Dordrecht, 1988, xxii+466 pp.

[11] E. Goursat, *Sur les équations du second ordre à n variables, analogues à l'équation de Monge–Ampère*, Bull. Soc. Math. France **27** (1899), 1–34.

[12] V.V. Ryzhkov, *Conjugate nets on multidimensional surfaces* (Russian), Trudy Moscow. Mat. Obshch. **7** (1958), 179–226.

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