CANTOR EXTENSION OF A HALF LINEARLY CYCLICALLY ORDERED GROUP

Štefan Černák

Department of Mathematics, Faculty of Civil Engineering, Technical University,
Vysokoškolská 4, SK–042 02 Košice, Slovakia
e-mail: svfkm@tuke.sk

Abstract

Convergent and fundamental sequences are studied in a half linearly cyclically ordered group $G$ with the abelian increasing part. The main result is the construction of the Cantor extension of $G$.

Keywords: convergent sequence, fundamental sequence, $C$-complete half $lc$-group, Cantor extension of a half $lc$-group.

2000 Mathematics Subject Classification: Primary 06F15; Secondary 20F60.

M. Giraudet and F. Lucas [3] introduced and investigated the notion of a half linearly ordered group (cf. also D.R. Ton [14], J. Jakubík [6], [7]). J. Jakubík [8] defined and studied the notion of a half linearly cyclically ordered group ($lc$-group) generalizing the notion of a half linearly ordered group.

The author [1] investigated the Cantor extension of an abelian $lc$-group. We remark that the Cantor extension of lattice ordered groups was studied by C.J. Everett [2].

Let $G$ be a half $lc$-group such that its increasing part is abelian and its decreasing part is nonempty (thus $G$ fails to be an $lc$-group). The notions of a convergent sequence and a fundamental sequence are defined in a natural way. If every fundamental sequence in $G$ is convergent in $G$, then $G$ is said to be $C$-complete.

In the present paper necessary and sufficient conditions are found under which $G$ is $C$-complete. Further, we define the notion of a Cantor extension and we prove that every half $lc$-group has a Cantor extension which is uniquely determined up to isomorphisms leaving all elements of $G$ fixed.
1. \textit{l-cyclically ordered sets and groups}

We recall the definitions and some results concerning \(l\)-cyclically ordered sets (cf. Novák and Novotný [10], Novák [9], Quilot [11]) and \(l\)-cyclically ordered groups (cf. Rieger [12], Świerczkowski [13], Jakubík and Pringerová [4], [5]).

\textbf{Definition 1.1.} Let \(M\) be a nonempty set and \(T\) a ternary relation on \(M\) such that the following conditions are satisfied:

\begin{enumerate}
  \item[(I)] if \([x, y, z] \in T\) then \([y, x, z] \notin T\).
  \item[(II)] \([x, y, z] \in T\) implies \([y, z, x] \in T\).
  \item[(III)] \([x, y, z] \in T, [y, u, z] \in T\) imply \([x, u, z] \in T\).
\end{enumerate}

Then \(T\) is said to be a \textit{cyclic order} on \(M\) and \((M, T)\) is called a \textit{cyclically ordered set}.

Let \(T\) be a cyclic order on \(M\) satisfying the condition:

\begin{enumerate}
  \item[(IV)] if \(x, y, z \in M, x \neq y \neq z \neq x\), then either \([x, y, z] \in T\) or \([z, y, x] \in T\).
\end{enumerate}

Then \(T\) is said to be an \textit{\(l\)-cyclic order} on \(M\) and \((M, T)\) is called an \textit{\(l\)-cyclically ordered set}.

Several terms are used in papers for the term \(l\)-cyclic order. For instance "\(l\)-cyclic order" is called "linear cyclic order" in [9], "complete cyclic order" in [11] and simply "cyclic order" in [12] and [13].

\textbf{Definition 1.2.} Let \((H; +)\) be a group and \((H; T)\) an \(l\)-cyclically ordered set such that the following condition is fulfilled:

\begin{enumerate}
  \item[(V)] if \([x, y, z] \in T, u, v \in H\), then \([u + x + v, u + y + v, u + z + v] \in T\).
\end{enumerate}

Then \((H; +, T)\) is said to be an \textit{\(l\)-cyclically ordered group} or \textit{lc-group} (linearly cyclically ordered group).

We often write \(H\) or \((H; T)\) instead of \((H; +, T)\).

Every subgroup of an \(lc\)-group is considered as an \(lc\)-group under the induced \(l\)-cyclic order.

\textbf{Example 1.3.} Let \((L; \leq)\) be a linearly ordered group \(x, y, z \in L\). Define the ternary relation \(T_L\) on \(L\) by putting
\[ [x, y, z] \in T_L \text{ if } x < y < z \text{ or } y < z < x \text{ or } z < x < y. \]

Then \((L; T_L)\) is an \(lc\)-group. \(T_L\) is called the \(l\)-cyclic order generated by the linear order \(\leq\) on \(L\). Hence every linearly ordered group is an \(lc\)-group (under the \(l\)-cyclic order generated by its linear order).

**Example 1.4.** Let \(K\) be the group of all reals \(k\) such that \(0 \leq k < 1\) with the group operation defined as the addition mod 1. Consider the natural linear order \(\leq\) on \(K\) and the ternary relation \(T_1\) on \(K\) defined in the same way as \(T_L\) in 1.3. Then \((K; T_1)\) is an \(lc\)-group.

Define the ternary relation \(T\) on the direct product \(L \times K\) of groups \(L\) and \(K\) as follows: for elements \(u_1, u_2, u_3 \in L \times K\), \(u_1 = (x, k_1), \ u_2 = (y, k_2), \ u_3 = (z, k_3)\) we put \([u_1, u_2, u_3] \in T\) if some of the following conditions is valid:

(i) \([k_1, k_2, k_3] \in T_1;\)
(ii) \(k_1 = k_2 \neq k_3\) and \(x < y;\)
(iii) \(k_2 = k_3 \neq k_1\) and \(y < z;\)
(iv) \(k_3 = k_1 \neq k_2\) and \(z < x;\)
(v) \(k_1 = k_2 = k_3\) and \([x, y, z] \in T_L.\)

Then \((L \times K; T)\) is an \(lc\)-group which will be denoted by \(L \otimes K\).

The notion of an isomorphism of \(lc\)-groups is defined in a natural way.

**Theorem 1.5** (´Świerczkowski [13]). Let \(H\) be an \(lc\)-group. Then there exists a linearly ordered group \(L\) such that \(H\) is isomorphic to a subgroup of \(L \otimes K\).

Assume that \((H; T)\) is an \(lc\)-group. By 1.5, there exists an isomorphism \(f\) of \(H\) into \(L \otimes K\). Let \(H_o\) be the set of all \(h \in H\) such that there exists \(x \in L\) with the property \(f(h) = (x, 0)\). Then \(H_o\) is a subgroup of \(H, H_o = \{0\}\) or \(H_o \neq \{0\}\). Let \(H_o \neq \{0\}, h \in H_o, h \neq 0\). There exists \(x \in L\) such that \(f(h) = (x, 0)\). \(H_o\) turns out to be a linearly ordered group if we put \(h > 0\) if \(x > 0\). The \(l\)-cyclic order \(T_{H_o}\) on \(H_o\) coincides with the \(l\)-cyclic order induced by \(T\).

2. **Cantor extension of an abelian \(lc\)-group**

Let \((H; T)\) be an abelian \(lc\)-group. A construction of a Cantor extension of \(H\) will be described (cf. [1]) and some results from [1] will be presented.
Definition 2.1. Let \((x_n)\) be a sequence in \(H\) and \(x \in H\).

\(a)\) We say that \((x_n)\) \textit{converges} to \(x\) (or \(x\) is a limit of \((x_n)\)) in \(H\) and we write \(x_n \to x\) (or \(\lim_{n \to \infty} x_n = x\))

(i) if \(\text{card } H = 2\) and there exists \(n_0 \in \mathbb{N}\) such that \(x_n = x\) for each \(n \in \mathbb{N}, n \geq n_0\),

or

(ii) if \(\text{card } H > 2\) and for each \(\varepsilon \in H, \varepsilon \neq 0\) with the property \([-\varepsilon, 0, \varepsilon] \in T\) there exists \(n_0 \in \mathbb{N}\) such that \([-\varepsilon, x_n - x, \varepsilon] \in T\) for each \(n \in \mathbb{N}, n \geq n_0\).

\(b)\) The sequence \((x_n)\) is called \textit{fundamental} in \(H\) if for each \(\varepsilon \in H, \varepsilon \neq 0\) with the property \([-\varepsilon, 0, \varepsilon] \in T\) there exists \(n_0 \in \mathbb{N}\) such that \([-\varepsilon, x_n - x, \varepsilon] \in T\) for each \(m, n \in \mathbb{N}, m, n \geq n_0\).

\(c)\) By a \textit{zero sequence} we understand a sequence \((x_n)\) such that \(x_n \to 0\).

\(d)\) \(H\) is called \textit{C-complete} if each fundamental sequence in \(H\) is convergent in \(H\).

The set of all fundamental (zero) sequences in \(H\) will be denoted by \(F_H(E_H)\).

Definition 2.2. Let \(H_1\) be an abelian \(lc\)-group satisfying the following conditions:

\(a)\) \(H_1\) is \(C\)-complete.

\(b)\) \(H\) is a subgroup of \(H_1\).

\(c)\) Every element of \(H_1\) is a limit of some fundamental sequence in \(H\).

\(d)\) Let \((x_n)\) be a sequence in \(H\) such that \(x_n \to 0\) in \(H\). Then \(x_n \to 0\) in \(H_1\).

Then \(H_1\) is said to be a \textit{Cantor extension} of \(H\).

Now we consider two cases: \(H_0 \neq \{0\}\) and \(H_0 = \{0\}\).

1) Assume that \(H_0 \neq \{0\}\). Let \((x_n), (y_n) \in F_H\). Under the natural definition of the operation \(+\) on \(F_H, (x_n) + (y_n) = (x_n + y_n), F_H\) is a group and \(E_H\) is a subgroup of \(F_H\). We form the factor group \(H^* = F_H/E_H\). Symbol \((x_n)^*\) will denote the coset of \(H^*\) containing the sequence \((x_n)\) in \(F_H\).

Suppose that \((x_n)^*, (y_n)^*, (z_n)^*\) are mutually distinct elements of \(H^*\). Let \(T^*\) be the set of all triples \([ (x_n)^*, (y_n)^*, (z_n)^* ]\) of elements of \(H^*\) such that there exists \(n_0 \in \mathbb{N}\) with \([x_n, y_n, z_n] \in T\) for each \(n \in \mathbb{N}, n \geq n_0\). Then \((H^*, T^*)\) is an \(lc\)-group.
Let $\varphi$ be the mapping from $H$ into $H^*$ defined by $\varphi(x) = (x, x, \ldots)^*$ for each $x \in H$. Then $\varphi$ is an isomorphism of the $lc$-group $H$ into $H^*$. We identify $x$ and $\varphi(x)$ for each $x \in H$. Then $H$ is a subgroup of $H^*$ and $H^*$ is a Cantor extension of $H$.

If we denote $(x_n)^* = X$ and $(x_n, x_n, \ldots)^* = X_n$, then we have (cf. [1], the proof of Lemma 3.12)

(A) $X_n \to X$ in $H^*$.

**Lemma 2.3** ([1], Lemma 3.9). $H$ is $C$-complete if and only if $H_o$ is $C$-complete.

2) Now assume that $H_o = \{0\}$. Then $H$ can be considered as a subgroup of $K$.

**Lemma 2.4** ([1], Lemma 4.2). If $H$ is a finite subgroup of $K$, then $H$ is $C$-complete.

**Lemma 2.5** ([1], Lemma 4.5). If $H$ is an infinite subgroup of $K$, then $K$ is a Cantor extension of $H$.

The following result is valid in both cases 1) and 2).

**Theorem 2.6** ([1], Theorem 4.9). Let $H$ be an abelian $lc$-group. Then

(i) there exists a Cantor extension of $H$,

(ii) if $H_1$ and $H_2$ are Cantor extensions of $H$, then there exists an isomorphism $\Phi$ from the $lc$-group $H_1$ onto $H_2$ such that $\Phi(x) = x$ for each $x \in H$.

3. Half $lc$-groups

The notion of a half $lc$-group was introduced by Jakubík [8]. Now we recall the definitions and results that will be applied in the next sections.

Let $(G; +, T)$ be a system such that $(G; +)$ is a group and $(G; T)$ is a cyclically ordered set. Assume that $x, y, z \in G$. Denote

$$G \upharpoonright = \{u \in G : [x, y, z] \in T \Rightarrow [u + x, u + y, u + z] \in T\},$$

$$G \downharpoonright = \{u \in G : [x, y, z] \in T \Rightarrow [u + z, u + y, u + x] \in T\}.$$
Definition 3.1. Let \((G;+,T)\) be as above. Assume that the following conditions are fulfilled:

1. The system \(T\) is nonempty.
2. If \([x,y,z]\in T\), then \([x+u,y+u,z+u]\in T\) for each \(u\in G\).
3. \(G=G^\uparrow \cup G^\downarrow\).
4. If \([x,y,z]\in T\), then either \(\{x,y,z\}\subseteq G^\uparrow\) or \(\{x,y,z\}\subseteq G^\downarrow\).

Then \((G;+,T)\) is said to be a **half cyclically ordered group**.

Let \((G;+,T)\) be a half cyclically ordered group. The definition implies that \(G^\uparrow\) is a cyclically ordered group. If \(G^\uparrow\) is an lc-group then \((G;+,T)\) is called a **half lc-group** (half linearly cyclically ordered group).

There are elements \(x, y, z \in G\) with \([x, y, z] \in T\). This is an immediate consequence of (1).

Again, we often write \(G\) or \((G;T)\) instead of \((G;+,T)\).

In the next, let \(G\) be a half lc-group. \(G^\uparrow\) \((G^\downarrow)\) is called the increasing \((decreasing, \text{ resp.})\) part of \(G\).

A subgroup \(G'\) of \(G\) is said to be a **half lc-subgroup** of \(G\) if the induced l-cyclic order on \(G'\) is nonempty.

Each lc-group \(G\) with \(\text{card } G \geq 3\) is a half lc-group (with \(G^\uparrow=G\) and \(G^\downarrow=\emptyset\)). Every linearly ordered group is an lc-group. Hence every half linearly ordered group (for the definition cf. [3]) is a half lc-group.

The notion of an isomorphism of half lc-groups is defined in a natural way.

From the definition 3.1 it follows (cf. [8]):

(i) If \(x, y \in G^\downarrow\), then \(x+y \in G^\uparrow\);
(ii) If \(x \in G^\uparrow, y \in G^\downarrow\), then \(x+y \in G^\downarrow\) and \(y+x \in G^\downarrow\).

4. Cantor extension of a half lc-group

In what follows, we assume that \((G,T)\) is a half lc-group such that \(G^\uparrow\) is abelian and \(G^\downarrow\neq\emptyset\). Hence \(G\) is neither abelian group nor lc-group.

We will use the notation \(G^\uparrow=H\) and \(G^\downarrow=H'\).
Definition 4.1. Let \((x_n)\) be a sequence in \(G\) and \(x \in G\).

a) We say that \((x_n)\) converges to \(x\) (or \(x\) is a limit of \((x_n)\)) in \(G\) and we write \(x_n \to x\) (or \(\lim x_n = x\)) if for each \(\varepsilon \in G, \varepsilon \neq 0\) with the property \([-\varepsilon, 0, \varepsilon] \in T\) there exists \(n_0 \in N\) such that \([-\varepsilon, x_n - x, \varepsilon] \in T\) and \([-\varepsilon, -x + x_n, \varepsilon] \in T\) for each \(n \in N, n \geq n_0\).

b) The sequence \((x_n)\) is said to be fundamental if for each \(\varepsilon \in G, \varepsilon \neq 0\) with \([-\varepsilon, 0, \varepsilon] \in T\) there exists \(n_0 \in N\) such that \([-\varepsilon, x_n - x_m, \varepsilon] \in T\) and \([-\varepsilon, -x_m + x_n, \varepsilon] \in T\) for each \(m, n \in N, m, n \geq n_0\).

c) If \(x_n \to 0\) in \(G\), then \((x_n)\) is called a zero sequence in \(G\).

d) \(G\) is said to be \(C\)-complete if every fundamental sequence in \(G\) is convergent in \(G\).

Definition 4.2. Let \(G_1\) be a half \(lc\)-group with the following properties:

(\(\alpha\)) \(G_1\) is \(C\)-complete;

(\(\beta\)) \(G\) is a half \(lc\)-subgroup of \(G_1\);

(\(\gamma\)) Every element of \(G_1\) is a limit of some fundamental sequence in \(G\);

(\(\delta\)) Let \((x_n)\) be a sequence in \(G\) such that \(x_n \to 0\) in \(G\). Then \(x_n \to 0\) in \(G_1\).

Then \(G_1\) is said to be a Cantor extension of \(G\).

We prove that \(G\) has a Cantor extension and this is uniquely determined up to isomorphisms leaving all elements of \(G\) fixed.

Denote by \(F(E)\) the set of all fundamental (zero) sequences in \(G\). Symbols \(F_H\) and \(E_H\) have the same meaning as in the section 2.

The following two lemmas are easy to prove.

Lemma 4.3. Let \((x_n)\) be a sequence in \(G\). Then \(x_n \to x\) in \(G\) if and only if \(x_n - x \to 0\) and \(-x + x_n \to 0\) in \(G\).

For a fixed element \(n_0 \in N\) and a sequence \((x_n)\) in \(G\) we apply the notation \(x'_n = x_{n_0 + n - 1}\) for each \(n \in N\).
Lemma 4.4. Let \((x_n)\) be a sequence in \(G\).

(i) \((x_n)\) \(\in E\) if and only if there exists \(n_o \in N\) such that \((x_n^o)\) is a sequence in \(H\) and \((x_n^o)\) \(\in E_H\).

(ii) Let \(x \in G\) such that \(x_n \to x\) in \(G\). Then there exists \(n_o \in N\) such that either \((x_n^o)\) is a sequence in \(H\) (and then \(x \in H\)) or \((x_n^o)\) is a sequence in \(H'\) (and then \(x \in H'\)).

(iii) Let \((x_n)\) \(\in F\). Then there exists \(n_o \in N\) such that either \((x_n^o)\) is a sequence in \(H\) (and then \((x_n^o) \in F_H\)) or \((x_n^o)\) is a sequence in \(H'\).

Let \((x_n)\) be a sequence in \(H, x \in H\). Then

(iv) \(x_n \to x\) in \(H\) if and only if \(x_n \to x\) in \(G\).

Let \(\varepsilon \in G, \varepsilon \neq 0\). If \([-\varepsilon, 0, \varepsilon] \in T\), then \(\varepsilon \in H\). Thus we have:

Lemma 4.5. \(E_H \subseteq E\) and \(F_H \subseteq F\).

Let \(a\) be a fixed element of \(H'\). Every element of \(H'\) can be expressed in the form \(a + x\) for some \(x \in H\).

Lemma 4.6. Let \((x_n)\) be a sequence in \(H, x \in H\). Then

(i) \(x_n \to x\) in \(H\) if and only if \(a + x_n \to a + x\) in \(G\).

(ii) \(x_n \to x\) in \(H\) if and only if \(a + x_n + a \to a + x + a\) in \(H\).

(iii) \((x_n)\) \(\in F_H\) if and only if some of the following conditions is satisfied \((a + x_n) \in F, (a + x_n + a) \in F_H, (-a + x_n + a) \in F_H\).

Proof. (i) and (ii) are easy to verification.

(iii): Let \((x_n) \in F_H\). We intend to show that \((a + x_n) \in F\). Assume that \(\varepsilon \in G, \varepsilon \neq 0, [-\varepsilon, 0, \varepsilon] \in T\). Then \(\varepsilon \in H\) and so \(-a - \varepsilon + a \in H\). Since \((x_n) \in F_H, [-a + \varepsilon + a, 0, -a - \varepsilon + a] \in T\) implies that there exists \(n_o \in N\) such that \([-a + \varepsilon + a, x_n - x_m, -a - \varepsilon + a] \in T\) for each \(m, n \in N, m, n \geq n_o\).

Therefore \([-\varepsilon, a + x_n - (a + x_m), \varepsilon] \in T\). From \([-\varepsilon, -x_m + x_n, \varepsilon] \in T\) it follows that \([-\varepsilon, -(a + x_m) + a + x_n, \varepsilon] \in T\). We conclude that \((a + x_n) \in F\). The converse and remaining cases are similar.
Lemma 4.7. \( G \) is \( C \)-complete if and only if \( H \) is \( C \)-complete.

**Proof.** Let \( G \) be \( C \)-complete and let \( (x_n) \in F_H \). In view of Lemma 4.5, we get \( (x_n) \in F \). Hence there exists \( x \in G \) with \( x_n \to x \) in \( G \). Applying Lemma 4.4 (ii) and Lemma 4.4 (iv), we obtain \( x \in H \) and \( x_n \to x \) in \( H \). Hence \( H \) is \( C \)-complete.

Let \( H \) be \( C \)-complete and let \( (x_n) \in H \). In view of Lemma 4.5, we get \( (x_n) \in F \). Hence there exists \( x \in G \) with \( x_n \to x \) in \( G \). Applying Lemma 4.4 (ii) and Lemma 4.4 (iv), we obtain \( x \in H \) and \( x_n \to x \) in \( H \). Hence \( H \) is \( C \)-complete.

The following result is an immediate consequence of Lemmas 4.6 and 4.7.

**Lemma 4.8.** Let \( G \) be a subgroup of a half \( lc \)-group \( G_1 \). Then \( G_1 \) is a Cantor extension of \( G \) if and only if \( G_1 \uparrow H \) is a Cantor extension of \( H \).

5. The case \( H_0 \neq \{0\} \)

In the whole section we suppose that \( H_0 \neq \{0\} \). Since \( H_0 \) is infinite, \( G \) is infinite as well.

We form the sets

\[
(B) \quad a + H^* = \{a + (x_n)^*: (x_n)^* \in H^*\}, \\
C_h(G) = H^* \cup (a + H^*).
\]

Assume that \( (x_n) \in F_H \). With respect to Lemma 4.6 (iii), we get \( (a + x_n + a) \in F_H \) and \( (-a + x_n + a) \in F_H \).

We intend to define a group operation \(+\) and a ternary relation \( T_h \) on \( C_h(G) \). Let \( (x_n)^*, (y_n)^*, (z_n)^* \in H^* \).

The operation \(+\) on \( C_h(G) \) is defined to coincide with the operation \(+\) on \( H^* \) defined in the section 2, i.e., we put

\[(x_n)^* + (y_n)^* = (x_n + y_n)^*.
\]

Further, we put
(a + (x_n)^*) + (a + (y_n)^*) = (a + x_n + a + y_n)^*,
(x_n)^* + (a + (y_n)^*) = a + (-a + x_n + a + y_n)^*,
(a + (x_n)^*) + (y_n)^* = a + (x_n + y_n)^*.

We define the ternary relation $T^h$ on $C_h(G)$ in such a way that $T^h$ coincides with $T^*$ on $H^*$.

Further, we put

$$[a + (x_n)^*, a + (y_n)^*, a + (z_n)^*] \in T^h \text{ if } [(z_n)^*, (y_n)^*, (x_n)^*] \in T^*.$$ 

If $p, q$ and $r$ are distinct elements of $C_h(G)$ such that $[p, q, r] \in T^h$, then either $\{p, q, r\} \subseteq H^*$ or $\{p, q, r\} \subseteq a + H^*$.

**Lemma 5.1.** $(C_h(G); +)$ is a group.

**Proof.** First, we verify that the operation $+$ is associative. Only three cases are considered. The remaining cases are similar.

$$(a + (x_n)^*) + (a + (y_n)^*) + (a + (z_n)^*) = (a + x_n + a + y_n)^* + (a + (z_n)^*) = a + (-a + a + x_n + a + y_n + a + z_n)^* = a + (x_n + a + y_n + a + z_n)^*,$$

$$(a + (x_n)^*) + ((a + (y_n)^*) + (a + (z_n)^*)) = a + (x_n + a + y_n + a + z_n)^* = (a + (x_n)^*) + (a + y_n + a + z_n)^* = a + (x_n + a + y_n + a + z_n)^*.$$

Hence,

$$(a + (x_n)^*) + (a + (y_n)^*) + (a + z_n)^*) = (a + (x_n)^*) + ((a + (y_n)^*) + (a + (z_n)^*).$$

Thus,

$$((a + (x_n)^*) + (a + (y_n)^*)) + (z_n)^* = (a + (x_n)^*) + ((a + (y_n)^*) + (z_n)^*.$$

Therefore, $(x_n)^* + (y_n)^* + (a + (z_n)^*) = (x_n + y_n + a + z_n)^* = a + (-a + x_n + y_n + a + z_n)^* = (x_n)^* + ((y_n)^* + (a + (z_n)^*)) = (x_n)^* + (y_n)^* + (a + (z_n)^*) = (x_n + a + (-a + y_n + a + z_n)^*) = a + (-a + x_n + y_n + a + z_n)^* = a + (-a + x_n + y_n + a + z_n)^* = a + (-a + x_n + y_n + a + z_n)^*.$$

Now, we show that every element of $C_h(G)$ has an inverse in $C_h(G)$. It suffices to consider elements of $a + H^*$. Assume that $a + (x_n)^* \in a + H^*$. Then $a + (-a - x_n - a)^* \in a + H^*$ and it is the inverse to $a + (x_n)^*$ in $C_h(G)$.

**Lemma 5.2.** Let $(x_n)^*, (y_n)^*, (z_n)^* \in H^*$. Then $[(x_n)^*, (y_n)^*, (z_n)^*] \in T^*$ if and only if some of the following conditions is satisfied:
Lemma 5.3. Let \((x_n)^*, (y_n)^*, (z_n)^*, (u_n)^* \in H^*\).

(i) If \([(x_n)^*, (y_n)^*, (z_n)^*] \in T^*\), then \([(x_n)^* + (u_n)^*, (y_n)^* + (u_n)^*, (z_n)^* + (u_n)^*] \in T^*\) and \([(x_n)^* + (a + (u_n)^*), (y_n)^* + (a + (u_n)^*), (z_n)^* + (a + (u_n)^*)] \in T^h\).

(ii) If \([a + (x_n)^*], a + (y_n)^*, a + (z_n)^*] \in T^h\), then \([a + (x_n)^*] + (u_n)^*, (a + (y_n)^*) + (u_n)^*, (a + (z_n)^*) + (u_n)^* \in T^h\) and \([(a + (x_n)^*) + (a + (u_n)^*), (a + (y_n)^*) + (a + (u_n)^*), (a + (z_n)^*) + (a + (u_n)^*)] \in T^*\).

Proof. (i): Assume that \([(x_n)^*, (y_n)^*, (z_n)^*] \in T^*\). Hence there exists \(n_o \in N\) such that \([x_n, y_n, z_n] \in T\) for each \(n \in N, n \geq n_o\). This yields that \([-a + z_n + a, -a + y_n + a, -a + x_n + a] \in T\) for each \(n \in N, n \geq n_o\). According to Lemma 4.6 (iii), we have \((-a + z_n + a), (-a + y_n + a), (-a + x_n + a) \in F_H\).

We conclude that \([-a + z_n + a)^*, (-a + y_n + a)^*, (-a + x_n + a)^*] \in T^*\).

The converse and (ii) are similar. 

Lemma 5.4. Let \((x_n)^*, (y_n)^*, (z_n)^*, (u_n)^* \in H^*\).

(i) If \([(x_n)^*, (y_n)^*, (z_n)^*] \in T^*\), then \([(u_n)^* + (x_n)^*, (u_n)^* + (y_n)^*, (u_n)^* + (z_n)^*] \in T^*\) and \([(a + (u_n)^*), (a + (u_n)^*), (a + (u_n)^*)] \in T^h\).

(ii) If \([a + (x_n)^*], a + (y_n)^*, a + (z_n)^*] \in T^h\), then \([(u_n)^* + (a + (x_n)^*), (u_n)^* + (a + (y_n)^*), (u_n)^* + (a + (z_n)^*)] \in T^h\) and \([(a + (u_n)^*) + (a + (z_n)^*), (a + (u_n)^*) + (a + (z_n)^*)] \in T^*\).

Proof. (i): Assume that \([(x_n)^*, (y_n)^*, (z_n)^*] \in T^*\). The first part of the assertion follows from the fact that \(H^*\) is an lc-group. Now, we prove the second part. From Lemma 5.2 (i), we infer that \([-a + z_n + a)^*, (-a + y_n + a)^*, (-a + x_n + a)] \in T^*\). Then \([-a + z_n + a)^* + (u_n)^*, (-a + y_n + a)^* + (u_n)^*, (-a + x_n + a)^* + (u_n)^* \in T^*\), \([-a + z_n + a + (u_n)]^* + (a + (u_n)^*), (a + (u_n)^*), (a + (u_n)^*) \in T^*\). Hence \([a + (a + x_n + a + u_n)^*], a + (a + y_n + a + u_n)^*, a + (a + z_n + a + u_n)^*] \in T^h\), i.e., \([(x_n)^* + (a + (u_n)^*), (y_n)^* + (a + (u_n)^*), (z_n)^* + (a + (u_n)^*)] \in T^h\).

The proof of (ii) is analogous.
Lemma 5.6. and 5.5 yield:

\[ C \]

Theorem 5.7.

\[ \text{lc} \quad \text{half} \quad \text{f} \quad \text{an isomorphism} \]

\[ \psi \]

Then

\[ \text{Lemma 5.5.} \]

From Lemma 5.4 and (B), we infer the validity of the following result:

\[ 42 \]

ˇS. ˇCern´ak

Theorem 5.8.

\[ \phi \]

By Theorem 2.6, there exists an isomorphism

\[ \text{for each} \]

\[ \text{we identify} \]

\[ x \]

Proof. (i) Assume that \([ (x_n)^*, (y_n)^*, (z_n)^* ] \in T^* \). The first assertion holds because of the fact that \( H^* \) is an \( lc \)-group. Now, we prove the second assertion. The assumption implies that \([ [(u_n)^* + (x_n)^*], [(u_n)^* + (y_n)^*], [(u_n)^* + (z_n)^*] \in T^* \) and so \([ [(u_n + x_n)^*], [(u_n + y_n)^*], [(u_n + z_n)^*] \in T^* \). Whence \([ a + (u_n + z_n)^*, a + (u_n + y_n)^*, a + (u_n + x_n)^* ] \in T^h h \). Thus \([ (a + (u_n)^*) + (z_n)^*, (a + (u_n)^*) + (y_n)^*, (a + (u_n)^*) + (x_n)^* ] \in T^h h \).

To prove (ii), we proceed in a similar way.

From Lemma 5.4 and (B), we infer the validity of the following result:

Lemma 5.5. \( C_h(G) \uparrow = H^*, C_h(G) \downarrow = a + H^* \) and \( C_h(G) = C_h(G) \uparrow \cup C_h(G) \downarrow \).

Since \( T \) is nonempty, \( T^h h \) is nonempty as well. Then Lemmas 5.1, 5.3 and 5.5 yield:

Lemma 5.6. \( (C_h(G), +, T^h h) \) is a half \( lc \)-group.

Let \( x \in H \). Define the mapping \( \psi \) from \( G \) into \( C_h(G) \) by

\[ \psi(x) = (x, x, \ldots)^*, \quad \psi(a + x) = a + \psi(x). \]

Then \( \psi \) is an isomorphism of the half \( lc \)-group \( G \) into \( C_h(G) \). In the next, we identify \( x \) and \( \psi(x) \) for each \( x \in H \). Then \( G \) is a half \( lc \)-subgroup of \( C_h(G) \). Since \( H^* \) is a Cantor extension of \( H \), from Lemma 4.8, we conclude.

Theorem 5.7. \( C_h(G) \) is a Cantor extension of \( G \).

Remark that it is easy to verify that (A) implies \( X_n \rightarrow X \) and \( a + X_n \rightarrow a + X \) in \( G \).

Theorem 5.8. Let \( G_1 \) and \( G_2 \) be Cantor extensions of \( G \). Then there exists an isomorphism \( f \) from the half \( lc \)-group \( G_1 \), onto \( G_2 \) such that \( f(x) = x \) for each \( x \in G \).

Proof. With respect to 4.8, \( G_1 \uparrow \) and \( G_2 \uparrow \) are Cantor extension of \( H \). By Theorem 2.6, there exists an isomorphism \( \phi \) from \( G_1 \uparrow \) onto \( G_2 \uparrow \) with \( \phi(x) = x \) for any \( x \in H \).

Choose an arbitrary element \( z \in G_1 \uparrow \). The mapping \( f : G_1 \rightarrow G_2 \) defined by \( f(z) = \phi(z) \) and \( f(a + z) = a + \phi(z) \) is an isomorphism of the half \( lc \)-group \( G_1 \) onto \( G_2 \) and \( f(a + x) = a + \phi(x) = a + x \) for each \( x \in H \).
A half lc-group \( C_h(G) \) corresponds to an element \( a \in H' \). Let \( a' \in H' \), \( a' \neq a \). Then the half lc-group \( (C'_h(G); +', T') \) corresponding to \( a' \) can be constructed formally in the same way \( (+, T_h \text{ and } a \text{ are replaced by } +', T' \text{ and } a' \) respectively). Therefore, the operations + and +' (relations \( T^h \text{ and } T' \)) coincide on \( G \) and \( H' \). From Theorems 5.7 and 5.8, it follows that \( C_h(G) \) and \( C'_h(G) \) are isomorphic half lc-groups. Moreover, we have:

**Lemma 5.9.** A half lc-group \( C_h(G) = C'_h(G) \).

**Proof.** For each \( (x_n)^* \in H^* \) we get \( a + (x_n)^* = a' +' (a' + a + x_n)^* \). Hence, \( a + H^* \subseteq a' +' H^* \). Analogously, we get \( a' +' H^* \subseteq a + H' \). Therefore, the set \( C_h(G) = C'_h(G) \).

Evidently, that relations \( T^h \text{ and } T' \) coincide. Now we show that group operations + on \( C_h(G) \) and +' on \( C'_h(G) \) coincide.

Let \( (x_n)^*, (y_n)^* \in H^* \). Then
\[
(a + (x_n)^*) + (a + y_n)^* = (a + x_n + a + y_n)^* = (a' - a' + a + x_n + a' - a' + a + y_n)^* = (a' +' (a' + a + x_n)^*) +' (a' +' (a' + a + y_n)^*);
\]
\[
(x_n)^* + (a + y_n)^* = a + (a + x_n + a + y_n)^* = a' +' (a' - a + a + x_n + a + y_n)^* = (a' +' (a' - a + x_n)^*) +' (a' +' (a' + a + y_n)^*);
\]
\[
(a + (x_n)^*) + (y_n)^* = a + (x_n + y_n)^* = a' +' (a' - a + x_n + y_n)^* = a' +' (a' + a + x_n)^* + (y_n)^*.
\]

6. **The case** \( H_o = \{0\} \)

In this section, we assume that \( H_o = \{0\} \). Then \( H \) can be considered as a subgroup of \( K \).

Assume that \( G \) is a finite half lc-group. Then \( H \) is a finite lc-group. With respect to Lemmas 2.4 and 4.7, we obtain:

**Lemma 6.1.** Let \( G \) be a finite half lc-group. Then \( G \) is \( C \)-complete.

Now, assume that \( G \) is an infinite half lc-group. Then \( H \) is an infinite lc-group.

Let \( a \) be a fixed element of \( H' \). We denote
\[
a + K = \{a + x : x \in K\};
\]
\[
C_h(G) = K \cup (a + K).
\]

We will define a group operation + and a ternary relation \( T^h \) on \( C_h(G) \).
Let \( x, y, z \in K \). From Lemma 2.5, we infer that there are fundamental sequences \( (x_n) \) and \( (y_n) \) in \( G \) such that \( \lim x_n = x, \lim y_n = y \) in \( K \).

The operation \( x + y \) on \( C_h(G) \) coincides with \( x + y \) on \( K \).

Further, we put

\[
(a + x) + (a + y) = \lim (a + x_n + a + y_n),
\]

\[
x + (a + y) = a + \lim (-a + x_n + a + y_n),
\]

\[
(a + x) + y = a + \lim (x_n + y_n).
\]

Limits are taken into account in \( K \). The operation \( + \) is correctly defined.

The ternary relation \( T^h \) on \( C_h(G) \) is defined in the following way:

\( T^h \) coincides with \( T_1 \) on \( K \).

Further, we put

\[
[a + x, a + y, a + z] \in T^h \text{ if } [x, y, z] \in T_1,
\]

if \( p, q, r \in C_h(G), [p, q, r] \in T^h \), then either \( \{p, q, r\} \subseteq K \) or \( \{p, q, r\} \subseteq a + K \).

It is a routine to verify that the following assertion is true:

**Lemma 6.2.** \((C_h(G), +, T^h)\) is a half lc-group, \( C_h(G) \uparrow = K \), \( C_h(G) \downarrow = a + K \).

From Lemmas 2.5 and 4.7, it follows:

**Lemma 6.3.** Let \( G \) be an infinite half lc-group. Then \( C_h(G) \) is a Cantor extension of \( G \).

Let \( a' \) and \( C'_h(G) \) be as in the Section 5.

**Remark 6.4.** It is easy to verify that Theorem 5.8 and Lemma 5.9 are valid also in the case \( H_0 = \{0\} \).

From Lemmas 4.7, 2.5 and Theorem 2.6, we obtain:

**Lemma 6.5.** Let \( G \) be an infinite half lc-group. Then \( G \) is \( C \)-complete if and only if \( H \) is isomorphic to \( K \).

Let \( G \) be an arbitrary lc-group as in the section 4 (neither \( H_0 \neq \{0\} \) nor \( H_0 = \{0\} \) is supposed). From Lemmas 6.1, 6.5 and 2.3, we conclude:
Theorem 6.6. Let $G$ be a half lc-group such that $H$ is abelian and $H' \neq \emptyset$. Then $G$ is $C$-complete if and only if some of the following conditions is fulfilled:

(i) $G$ is finite;

(ii) $H$ is isomorphic to $K$;

(iii) $H_o \neq \{0\}$ and $H_o$ is $C$-complete.  

By summarizing Theorem 5.7, Lemma 6.3, Theorem 5.8, and Remark 6.4 we get:

Theorem 6.7. Let $G$ be a half lc-group such that $H$ is abelian and $H' \neq \emptyset$. Then

(i) There exists a Cantor extension of $G$.

(ii) If $G_1$ and $G_2$ are Cantor extensions of $G$, then there exists an isomorphism from the half lc-group $G_1$ onto $G_2$ leaving all elements of $G$ fixed.

References


Received 7 June 1999
Revised 30 April 1996