Discussiones Mathematicae
General Algebra and Applications 20 (2000) 233–253

LOCALIZATION IN SEMICOMMUTATIVE (m, n)-RINGS

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Abstract

We give a construction for (m, n)-rings of quotients of a semicommutative (m, n)-ring, which generalizes the ones given by Crombez and Timm and by Paunić for the commutative case. We also study various constructions involving reduced rings and rings of quotients and give some functorial interpretations. **Keywords:** (m, n)-rings semicommutative (m, n)-rings, (m, n)-rings of quotients, (m, n)-division rings, (m, n)-semidomains, n-groups, n-semigroups.

1991 Mathematics Subject Classification: 17A42, 20N15.

1 Introduction

Various results involving commutative groups and rings have been generalized by several authors (Timm [15], Crombez and Timm [2], Dudek [4]) for the commutative n-ary case: still, in many cases commutativity may be replaced by a weaker condition, namely semicommutativity (which is a natural generalization of binary commutativity as well). The purpose of this paper is to give a construction for (m, n)-rings of quotients of a semicommutative (m, n)-ring, which generalizes the ones given by Crombez and Timm in [2] and by Paunić in [10]; we also study various constructions involving reduced rings and rings of quotients and give some functorial interpretations. We assume known the standard notions about n-semigroups, n-groups and (m, n)-rings as they appear for example in [3], [13], [1] and [14].

Notations and terminology. Throughout the paper we will make use of similar notations to those in [11] and [8]. In order to simplify notations we will often write $\sum_{i=1}^{m} a_i$ instead of $[a_1, \ldots, a_m]_+$ and $(a_1^n)_\circ$ instead of $(a_1, \ldots, a_n)_\circ$; if k consecutive terms (factors) coincide we use the short notation $\overset{(k)}{a}$. An n-semigroup $(R, \circ)$ is called:
commutative if for any permutation $\sigma \in S_n, a_1, a_2, \ldots, a_n \in R$, we have

$$(a_1, a_2, \ldots, a_n)_\circ = (a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)})_\circ;$$

semicommutative if for any $a_1, a_2, \ldots, a_n \in R$, we have

$$(a_1, a_2^{n-1}, a_n)_\circ = (a_n, a_2^{n-1}, a_1)_\circ;$$

entropic or medial if for $a_{ij} \in R, i, j \in \{1, 2, \ldots, n\}$, we have

$$((a_{11}^{1n})_\circ, (a_{21}^{2n})_\circ, \ldots, (a_{n1}^{nn})_\circ)_\circ = ((a_{11}^{n1})_\circ, (a_{12}^{n2})_\circ, \ldots, (a_{1n}^{nn})_\circ)_\circ;$$

i-cancellative with respect to $S \subseteq R$, where $i \in \{1, 2, \ldots, n\}$, if for $s_j \in S, j \in \{1, 2, \ldots, n\} \setminus \{i\}, a, b \in R$ the following implication holds:

$$(s_1^{i-1}, a, s_{i+1}^n)_\circ = (s_1^{i-1}, b, s_{i+1}^n)_\circ \Rightarrow a = b;$$

cancellative with respect to $S \subseteq R$ if it is i-cancellative for every $i \in \{1, 2, \ldots, n\}$.

11 . (a) A semicommutative n-semigroup is entropic. (b) A right and left cancellative n-semigroup (with respect to some subset) is cancellative (with respect to the same subset).

12 . An algebraic structure $(R, +, \circ)$ is called (m, n)-*ring* if:

(a) $(R, +)$ is a commutative m-group,

(b) $(R,\circ)$ is an n-semigroup,

(c) multiplication $\circ$ is distributive with respect to addition $+$, i.e. for any $i \in \{1,2,\ldots,n\}, a_j, b_k \in R, j \in \{1,2,\ldots,n\} \setminus \{i\}, k \in \{1,2,\ldots,m\}$, we have

$$(a_1^{i-1},[b_1^m]_+,a_{i+1}^n)_\circ = \sum_{j=1}^{m}(a_1^{i-1},b_j,a_{i+1}^n)_\circ.$$

13. An (m,n)-ring $(R,+,\circ)$ is called *commutative, semicommutative, entropic* or *cancellative with respect to* $S \subseteq R$ if the n-semigroup $(R,\circ)$ has that property. A *zero* in R (if it exists) is an element $z \in R$ such that for all $a_1,\ldots,a_{n-1} \in R$, we have

(a) $$(z,a_1^{n-1})_\circ = (a_1,z,a_2^{n-1})_\circ = \ldots = (a_1^{n-1},z)_\circ = z$$

R^* denotes the set $R \setminus \{0\}$ if 0 exists and R otherwise. In an (m,n)-ring the following relations hold:

$$\overline{(a_1,\ldots,a_n)_\circ} = (a_1^{i-1},\overline{a_i},a_{i+1}^n)_\circ, \quad \overline{[a_1,\ldots,a_m]_+} = [\overline{a}_1,\ldots,\overline{a}_m]_+,$$

where $\overline{a}$ denotes the *querelement* (skewelement in other terminology) of a in $(R,+)$. If $x \in R$ satisfies the equation $(\overset{(n-1)}{a},x)_\circ = a$ then x is the multiplicative querelement of a and will be denoted by $\underline{a}$.

2 (m,n)-rings of quotients of a semicommutative (m,n)-ring

We shall first state two lemmas which will prove useful in the sequel. Note that Lemma 22 appeared already, in a slightly different formulation, in [7].

Lemma 21. *If an n-semigroup $(R,\circ)$ is entropic then it is also true that:*

$$\Big((a_{11}^{1k})_\circ,(a_{21}^{2k})_\circ,\ldots,(a_{p1}^{pk})_\circ\Big)_\circ = \Big((a_{11}^{p1})_\circ,(a_{12}^{p2})_\circ,\ldots,(a_{1k}^{pk})_\circ\Big)_\circ,$$

where $p,k \equiv 1 (\mathrm{mod}\ n-1)$.

Proof. The proof of this lemma is based on the following facts: for $p = n, k = 2n-1$, we have

$$\begin{aligned}
&\left((a_{11}^{1,2n-1})_\circ, (a_{21}^{2,2n-1})_\circ, \ldots, (a_{n1}^{n,2n-1})_\circ\right)_\circ \\
&= \left(((a_{11}^{1n})_\circ, a_{1,n+1}^{1,2n-1})_\circ, ((a_{21}^{2n})_\circ, a_{2,n+1}^{2,2n-1})_\circ, \ldots, ((a_{n1}^{nn})_\circ, a_{n,n+1}^{n,2n-1})_\circ\right)_\circ \\
&= \left(((a_{11}^{1n})_\circ, (a_{21}^{2n})_\circ, (a_{n1}^{nn})_\circ)_\circ, (a_{1,n+1}^{n,n+1})_\circ, \ldots, (a_{1,2n-1}^{n,2n-1})_\circ\right)_\circ \\
&= \left((a_{11}^{n1})_\circ, (a_{12}^{n2})_\circ, \ldots, (a_{1n}^{nn})_\circ, (a_{1,n+1}^{n,n+1})_\circ, \ldots, (a_{1,2n-1}^{n,2n-1})_\circ\right)_\circ,
\end{aligned}$$

and for $p = 2n-1, k = n$, we have

$$\begin{aligned}
&\left((a_{11}^{1n})_\circ, (a_{21}^{2n})_\circ, \ldots, (a_{2n-1,1}^{2n-1,n})_\circ\right)_\circ \\
&= \left(((a_{11}^{1n})_\circ, (a_{21}^{2n})_\circ, \ldots, (a_{n1}^{nn})_\circ)_\circ, (a_{n+1,1}^{n+1,n})_\circ, \ldots, (a_{2n-1,1}^{2n-1,n})_\circ\right)_\circ \\
&= \left(((a_{11}^{n1})_\circ, (a_{12}^{n2})_\circ, \ldots, (a_{1n}^{nn})_\circ)_\circ, (a_{n+1,1}^{n+1,n})_\circ, \ldots, (a_{2n-1,1}^{2n-1,n})_\circ\right)_\circ \\
&= \left(((a_{11}^{n1})_\circ, a_{n+1,1}^{2n-1,1})_\circ,)_\circ, \ldots, ((a_{1n}^{nn})_\circ, a_{n+1,n}^{2n-1,n})_\circ\right)_\circ \\
&= \left((a_{11}^{2n-1,1})_\circ, (a_{12}^{2n-1,2})_\circ, \ldots, (a_{1n}^{2n-1,n})_\circ\right)_\circ.
\end{aligned}$$

■

Lemma 22 (see also [7], Corrollary 1). *If $(R, \circ)$ is a semicommutative n-semigroup and $p \geq 2$, $p \in \mathbb{Z}$ then for every permutation $\sigma \in S_p$ and for any a, $b_{ij} \in R$, $i \in \{1, \ldots, p\}$, $j \in \{2, \ldots, n\}$, we have*

$$\begin{aligned}
&\left((\ldots((a, b_{12}^{1n})_\circ, b_{22}^{2n})_\circ \ldots)_\circ, b_{p2}^{pn}\right)_\circ \\
&= \left((\ldots((a, b_{\sigma(1),2}^{\sigma(1),n})_\circ, b_{\sigma(2),2}^{\sigma(2),n})_\circ \ldots)_\circ, b_{\sigma(p),2}^{\sigma(p),n}\right)_\circ.
\end{aligned}$$

Proof. Recall that every permutation can be written as a product of transpositions; now it suffices to note that, the operation $\circ$ being associative and semicommutative, we have: $\left((a, b_{12}^{1n})_\circ, b_{22}^{2n}\right)_\circ = \left((a, b_{22}^{2n})_\circ, b_{12}^{1n}\right)_\circ$. ■

23. As consequences of the above two lemmas note that:

(a) for $k \equiv 0 \pmod{n-1}$ we have

$$\Big((a, s_1^k)_\circ, t_1^k\Big)_\circ = \Big((a, t_1^k)_\circ, s_1^k\Big)_\circ ;$$

(b) for any $\sigma \in S_p$:

$$\frac{a}{s_{12}^{1n}\, s_{22}^{2n} \dots s_{p2}^{pn}} = \frac{a}{s_{\sigma(1),2}^{\sigma(1),n}\, s_{\sigma(2),2}^{\sigma(2),n} \dots s_{\sigma(p),2}^{\sigma(p),n}};$$

(c)
$$\frac{(a, s_2^n)_\circ}{u_2^n s_2^n} = \frac{a}{u_2^n};$$

(d)
$$\left[\frac{a_1}{s_2^n}, \dots, \frac{a_m}{s_2^n}\right]_\oplus = \frac{[a_1, \dots, a_m]_+}{s_2^n}.$$

24 . Let $(R,+,\circ)$ be a semicommutative (m,n)-ring cancellative with respect to a non-empty n-subsemigroup S of $(R^*, \circ)$. On the set $R \times S^{n-1}$ define the equivalence relation (see [2]) "$\sim$" by: $(a, s_2, \dots, s_n) \sim (b, t_2, \dots, t_n)$ if $(a, t_2^n)_\circ = (b, s_2^n)_\circ$. The equivalence class of $(a, s_2, \dots, s_n)$ is denoted by $\frac{a}{s_2^n}$ and we write $R_{S^{n-1}}$ for $R \times S^{n-1}/\sim$. Note that the notation $\frac{a}{s_{12}^{1n} \dots s_{p2}^{pn}}$ is unambiguous (see the definition of "$\sim$"). Define now the n-ary multiplication in $R_{S^{n-1}}$ by:

$$\left(\frac{a_1}{s_{12}^{1n}}, \dots, \frac{a_n}{s_{n2}^{nn}}\right)_\star = \frac{(a_1, \dots, a_n)_\circ}{(s_{12}^{n2})_\circ (s_{13}^{n3})_\circ \dots (s_{1n}^{nn})_\circ}.$$

25 . In [12] M.S. Pop and M. Câmpian show that:

(i) $(R_{S^{n-1}}, \star)$ is an n-semigroup with unit (as an $(n-1)$-ad) and there is an injective homomorphism $f\colon R \to R_{S^{n-1}}$ such that for every $s \in S$ the element $f(s) \in R_{S^{n-1}}$ has a querelement $\underline{f(s)}$.

(ii) if R' is an n-semigroup with unit as an $(n-1)$-ad and $\alpha\colon R \to R'$ is a homomorphism having the property that for every $s \in S$, $\alpha(s)$ has a querelement in R', then there exists a unique homomorphism $\beta\colon R_{S^{n-1}} \to R'$ such that the diagram

$$\begin{tikzcd} R \arrow[r, "\alpha"] \arrow[d, "f"'] & R' \\ R_{S^{n-1}} \arrow[ur, "\beta"'] & \end{tikzcd}$$

commutes.

As a consequence of this result note that every element of $R_{S^{n-1}}$ can be described as:

$$\text{(iii)} \qquad \frac{a}{s_2^n} = \left(f(a), \overset{(n-3)}{f(s_n)}, \underline{f(s_n)}, \dots, \overset{(n-3)}{f(s_2)}, \underline{f(s_2)} \right)_{\star}.$$

We can state now the following

Theorem 26. *Let $(R, +, \circ)$ be a semicommutative (m, n)-ring cancellative with respect to a non-empty n-subsemigroup S of $(R^*, \circ)$, containing no zerodivisors. Then there exists a unique m-ary operation $\oplus$ on $R_{S^{n-1}}$ such that the following conditions are satisfied:*

(a) *$(R_{S^{n-1}}, \oplus, \star)$ is a semicommutative (m, n)-ring with unit as an $(n-1)$-ad,*

(b) *the mapping $f\colon R \to R_{S^{n-1}}, f(a) = \frac{(a, s_2^n)_\circ}{s_2^n}$ is an injective ring-homomorphism.*

Proof. Suppose first that there exists an operation $\oplus$ on $R_{S^{n-1}}$ satisfying conditions (a) and (b); we prove that it is then unique. Indeed, the above conditions (a) and (b), 25 (iii) and 22 imply that for any $\frac{a_i}{s_{i2}^{in}}$, $i = 1, \dots, m$, we have

$$\left[\frac{a_1}{s_{12}^{1n}}, \dots, \frac{a_m}{s_{m2}^{mn}} \right]_{\oplus} = \sum_{i=1}^{m} \left(f(a_i), \overset{(n-3)}{f(s_{in})}, \underline{f(s_{in})}, \dots, \overset{(n-3)}{f(s_{i2})}, \underline{f(s_{i2})} \right)_{\star}$$

and that for $i = 1, \ldots, m$ the following holds:

$$\left(f(a_i), \overset{(n-3)}{f(s_{in})}, \underline{f(s_{in})}, \ldots, \overset{(n-3)}{f(s_{i2})}, \underline{f(s_{i2})}\right)_\star$$
$$\overset{\text{def}}{=} (f(a_i), A)_\star = \left(f(a_i), f(s_{12}), \overset{(n-3)}{f(s_{12})}, \underline{f(s_{12})}, A\right)_\star$$
$$= \left(f(a_i), f(s_{12}), f(s_{13}), \overset{(n-3)}{f(s_{13})}, \underline{f(s_{13})}, \overset{(n-3)}{f(s_{12})}, \underline{f(s_{12})}, A\right)_\star = \ldots$$
$$= \left(f((a_i, s_{12}^{1n})_\circ), \overset{(n-3)}{f(s_{1n})}, \underline{f(s_{1n})}, \ldots, \overset{(n-3)}{f(s_{12})}, \underline{f(s_{12})}, A\right)_\star = \ldots$$
$$= \Bigg(f(\,(a_i, s_{12}^{1n}, \ldots, s_{i-1,2}^{i-1,n}, s_{i+1,2}^{i+1,n}, \ldots, s_{m2}^{mn})_\circ\,), \overset{(n-3)}{f(s_{mn})}, \underline{f(s_{mn})}, \ldots,$$
$$\overset{(n-3)}{f(s_{i+1,2})}, \underline{f(s_{i+1,2})}, \overset{(n-3)}{f(s_{i-1,n})}, \underline{f(s_{i-1,n})}, \ldots, \overset{(n-3)}{f(s_{12})}, \underline{f(s_{12})}, A\Bigg)_\star$$
$$\overset{22}{=}\left(f((a_i, s_{12}^{1n}, \ldots, s_{i-1,2}^{i-1,n}, s_{i+1,2}^{i+1,n}, \ldots, s_{m2}^{mn})_\circ), \overset{(n-3)}{f(s_{mn})}, \underline{f(s_{mn})}, \ldots, \overset{(n-3)}{f(s_{12})}, \underline{f(s_{12})}\right)_\star$$

and so, by using distributivity, the sum will be equal to:

$$\Bigg(\sum_{i=1}^{m} f((a_i, s_{12}^{1n}, \ldots, s_{i-1,2}^{i-1,n}, s_{i+1,2}^{i+1,n}, \ldots, s_{m2}^{mn})_\circ), \overset{(n-3)}{f(s_{mn})}, \underline{f(s_{mn})}, \ldots,$$
$$\overset{(n-3)}{f(s_{12})}, \underline{f(s_{12})}\Bigg)_\star = \left(f\left(\sum_{i=1}^{m}(a_i, s_{12}^{1n}, \ldots, s_{i-1,2}^{i-1,n}, s_{i+1,2}^{i+1,n}, \ldots, s_{m2}^{mn})_\circ\right), \overset{(n-3)}{f(s_{mn})},\right.$$
$$\left.\underline{f(s_{mn})}, \ldots, \overset{(n-3)}{f(s_{12})}, \underline{f(s_{12})}\right)_\star = \frac{\sum_{i=1}^{m}(a_i, s_{12}^{1n}, \ldots, s_{i-1,2}^{i-1,n}, s_{i+1,2}^{i+1,n}, \ldots, s_{m2}^{mn})_\circ}{s_{12}^{1n} \ldots s_{m2}^{mn}}.$$

and this proves the uniqueness of the operation $\oplus$. In order to prove the existence of an operation $\oplus$ on $R_{S^{n-1}}$ which satisfies (a) and (b), we define $\oplus$ by:

$$\left[\frac{a_1}{s_{12}^{1n}}, \cdots, \frac{a_m}{s_{m2}^{mn}}\right]_\oplus = \frac{\sum_{i=1}^{m}(a_i, s_{12}^{1n}, \ldots, s_{i-1,2}^{i-1,n}, s_{i+1,2}^{i+1,n}, \ldots, s_{m2}^{mn})_\circ}{s_{12}^{1n} \ldots s_{m2}^{mn}}.$$

By using Lemmas 21, 22 and their consequences one proves that addition in $R_{S^{n-1}}$ is well defined, it is an associative and commutative operation and the

querelement of $\frac{a}{s_2^n}$ in $(R_{S^{n-1}}, \oplus)$ is $\frac{\bar{a}}{s_2^n}$; therefore $(R_{S^{n-1}}, \oplus)$ is a commutative m-group. It is shown in [12] that $(R_{S^{n-1}}, \star)$ is a semicommutative n-semigroup
with unit as an $(n-1)$-ad, for example

$$\left(\frac{(\overset{(n)}{s})_\circ}{\overset{(n-1)}{s}}, \ldots, \frac{(\overset{(n)}{s})_\circ}{\overset{(n-1)}{s}}, \frac{s}{\overset{(n-1)}{s}} \right)$$

is a lateral unit in $R_{S^{n-1}}$ for every $s \in S$; $R_{S^{n-1}}$ is cancellative with respect to $S_{S^{n-1}} = \{\frac{s_1}{s_2^n} | s_i \in S, i = 1, 2, \ldots, n\}$; f is an injective homomorphism of n-semigroups and for every $s \in S$ the element $f(s) \in R_{S^{n-1}}$ has a multiplicative querelement $\underline{f(s)} = \frac{s}{\overset{(n-1)}{s}}$. The fact that $f([a_1^m]_+) = [f(a_1), \ldots, f(a_m)]_\oplus$ follows from the properties of the operations in R and by the quoted two lemmas, so f is a ring-homomorphism. Distributivity laws in $R_{S^{n-1}}$ also follow by the properties of the operations in R and by the lemmas. ■

The (m, n)-ring $(R_{S^{n-1}}, \oplus, \star)$ is called *the ring of quotients of R with respect to S (*or *with denominators in S^{n-1}).* It has the following universal property which determines it up to isomorphism:

Theorem 27. *Let $(R, +, \circ)$ be a semicommutative (m, n)-ring cancellative with respect to a non-empty n-subsemigroup S of $(R^*, \circ)$. If $R_{S^{n-1}}$ is the ring of quotients of R with denominators in S^{n-1} and $f: R \to R_S$ is the canonical homomorphism defined in* (26)*, then for any ring-homomorphism $\alpha: R \to R'$, where $(R', [\,], \odot)$ is a semicommutative (m, n)-ring with unit as an $(n-1)$-ad such that for every $s \in S$ $\alpha(s)$ has a querelement in $(R', \odot)$, there exists a unique ring-homomorphism $\beta: R_S \to R'$ such that $\beta \circ f = \alpha$.*

Definition 28. An (m, n)-ring R is called (m, n)-*semidomain* if it is semicommutative, it has unit as a system of $n-1$ elements and it is cancellative with respect to R^*.

Note that if a (m, n)-*semidomain* R has a zero, then cancellativity is equivalent to the non-existence of zerodivisors (see [2]). For (m, n)-semidomains, we can also state the following:

Theorem 29. *Any (m, n)-semidomain can be embedded in a semicommutative (m, n)-division ring; the ring of quotients $R_{R^{*\,n-1}}$ is (up to isomorphism) the unique minimal semicommutative (m, n)-division ring with this property.*

3 (m,n)-rings of quotients and their reducts

31 . In the study of (m,n)-rings, as well as in the case of n-semigroups or n-groups, it is always of interest to establish connections between the given structure and its (binary) reduct. In the sequel we shall refer to the Hosszù-type reduced operations and we shall make the following notations:

(a) $\mathrm{red}_{\mathrm{u}_1^{\mathrm{n}-2}}(\mathrm{A},\circ)$ (where $(A,\circ)$ is an n-semigroup and $u_1,\dots,u_{n-2}\in A$ are fixed arbitrary elements) denotes the (binary) semigroup $(A,\cdot)$, with $x\cdot y=(x,u_1^{n-2},y)_\circ$.

(b) $\mathrm{red}_{\mathrm{a}}(\mathrm{G},\circ)$ (where $(G,\circ)$ is an n-group, $a\in G$) denotes the (binary) group $(G,\cdot)$, with $x\cdot y=(x,\overset{(n-3)}{a},\overline{a},y)_\circ$.

(c) $\mathrm{red}^{(\mathrm{m},2)}_{\mathrm{u}_1^{\mathrm{n}-2}}(\mathrm{R},+,\circ)$ (where $(R,+,\circ)$ is an (m,n)-ring, $u_1,\dots,u_{n-2}\in R$) denotes the $(m,2)$-ring $(R,+,\cdot)$ in which addition remains unchanged while binary multiplication is defined as above, i.e., $x\cdot y=(x,u_1^{n-2},y)_\circ$.

(d) $\mathrm{red}^{(\mathrm{m},\mathrm{k})}_{\mathrm{u}_1^{\mathrm{t}-1}}(\mathrm{R},+,\circ)$ (where $(R,+,\circ)$ is an (m,n)-ring, $n-1=t(k-1)$ and $u_1,\dots,u_{t-1}\in R$) denotes the (m,k)-ring $(R,+,\star)$ in which addition remains unchanged and $(x_1^k)_\star=(x_1,u_1^{t-1},x_2,u_1^{t-1},\dots,x_k)_\circ$. Similar k-reducts for n-groups were introduced and studied by Dudek and Michalski in [5] where a generalization of the Hosszú theorem was also proved.

(e) $\mathrm{red}^{(2,\mathrm{n})}_0(\mathrm{R},+,\circ)$ (where $(R,+,\circ)$ is an (m,n)-ring with zero 0) denotes the $(2,n)$-ring $(R,\oplus,\circ)$ in which multiplication remains unchanged, while binary addition is given by: $x\oplus y=[x,\overset{(m-2)}{0},y]_+$.

First reducing and then constructing a ring of quotients or first constructing a ring of quotients and then reducing it – we prove that these two procedures lead to isomorphic results.

Theorem 32. *Let $(R,+,\circ)$ be a semicommutative (m,n)-ring, cancellative with respect to an n-subsemigroup S of $(R,\circ)$. Let $u_1,\dots,u_{n-2}\in S$ and $(R,+,\cdot)=\mathrm{red}^{(\mathrm{m},2)}_{\mathrm{u}_1^{\mathrm{n}-2}}(\mathrm{R},+,\circ)$. Then the $(m,2)$-ring of quotients of $(R,+,\cdot)$*

with denominators in S is isomorphic to $\mathrm{red}^{(m,2)}_{v_1^{n-2}}(R_{S^{n-1}}, \oplus, \star)$*, where*

$$v_i = \frac{(u_i, \overset{(n-1)}{s})_\circ}{\overset{(n-1)}{s}}, \qquad i = 1, \ldots, n-2$$

(and $(R_{S^{n-1}}, \oplus, \star)$ denotes, as in Section 2, the (m,n)-ring of quotients of R with denominators in S^{n-1}).

Proof. The mapping $\varphi\colon (R_S, \cdot) \to \mathrm{red}_{\mathrm{v}_1^{\mathrm{n}-2}}(\mathrm{R}_{\mathrm{S}^{\mathrm{n}-1}}, \star)$, $\quad \varphi(\frac{\mathrm{a}}{\mathrm{s}}) = \frac{\mathrm{a}}{\mathrm{u}_1^{\mathrm{n}-2}\,\mathrm{s}}$ is an isomorphism of semigroups (see [8]); one also proves then that

$$\varphi\Big(\left[\frac{a_1}{s_1}, \ldots, \frac{a_m}{s_m}\right]_+\Big) = \left[\varphi\left(\frac{a_1}{s_1}\right), \ldots, \varphi\left(\frac{a_m}{s_m}\right)\right]_\oplus \text{ and } \varphi\left(\left(\overline{\frac{a}{s}}\right)\right) = \overline{\varphi\left(\frac{a}{s}\right)}.$$

■

A similar result can be stated for the more general case of (m,k)-reducts:

Theorem 33. *Let $(R, +, \circ)$ be a semicommutative (m,n)-ring, cancellative with respect to an n-subsemigroup S of $(R, \circ)$ and $n-1 = t(k-1)$. Let $u_1, \ldots, u_{t-1} \in S$ and $(R, +, \cdot) = \mathrm{red}^{(\mathrm{m,k})}_{\mathrm{u}_1^{\mathrm{t}-1}}(\mathrm{R}, +, \circ)$. Then the (m,k)-ring of quotients of $(R, +, \cdot)$ with denominators in S^{k-1} is isomorphic to* $\mathrm{red}^{(\mathrm{m,k})}_{\mathrm{v}_1^{\mathrm{t}-1}}(\mathrm{R}_{\mathrm{S}^{\mathrm{n}-1}}, \oplus, \star)$*, where*

$$v_i = \frac{(u_i, \overset{(n-1)}{s})_\circ}{\overset{(n-1)}{s}}, \qquad i = 1, \ldots, t-1\,.$$

Proof. The mapping

$$\varphi\colon R_{S^{k-1}} \to \mathrm{red}^{(\mathrm{m,k})}_{\mathrm{v}_1^{\mathrm{t}-1}}\mathrm{R}_{\mathrm{S}^{\mathrm{n}-1}}, \quad \varphi\left(\frac{\mathrm{a}}{\mathrm{s}_2^{\mathrm{k}}}\right) = \frac{\mathrm{a}}{\mathrm{u}_1^{\mathrm{t}-1}\mathrm{s}_2\mathrm{u}_1^{\mathrm{t}-1}\mathrm{s}_3 \ldots \mathrm{u}_1^{\mathrm{t}-1}\mathrm{s}_\mathrm{k}}$$

is an isomorphism of (m,k)-rings; note that an arbitrary fraction $\frac{a}{s_2^n}$ in $R_{S^{n-1}}$ is the image under φ of the (uniquely determined) fraction

$$\frac{(a, u_1^{t-1}, s, \ldots, u_1^{t-1}, s)_\circ}{\overset{(k-2)}{s} \quad (s, s_2^n)_\circ} \in R_{S^{k-1}}\,.$$

■

4 The case of (m, n)-semidomains with zero

41 . In the special case of (m, n)-semidomains with zero some more results can be given and even in a functorial way. We denote by:

(a) (m, n)SDom-the category which consists of (m, n)-semidomains with zero as objects and injective ring-homomorphisms as morphisms;

(b) (m, n)SDR-the category which consists of semicommutative (m, n)-division rings with zero as objects and injective ring-homomorphisms as morphisms;

(c) (m, n)Fd-the category which consists of (m, n)-fields as objects and ring-homomorphisms as morphisms.

In all cases, multiplication of morphisms is ordinary composition of maps. Note that if R_1, R_2 are (m, n)-rings with zero and $\varphi\colon R_1 \to R_2$ is a ring-homomorphism, then φ is injective if and only if $\varphi(x) = 0$ implies $x = 0$.

42 . Define now the following covariant functors:

(a) Quot: (m, n)SDom $\to$ (m, n)SDR sends an (m, n)-semidomain with zero into its (m, n)-division ring of fractions, $\mathrm{Quot}\, R = R_{R^{*\,(n-1)}}$ and, for $\varphi \in \mathrm{Hom}\,(R_1, R_2)$, $\mathrm{Quot}\,\varphi \in \mathrm{Hom}\,(\mathrm{Quot}\, R_1, \mathrm{Quot}\, R_2)$ is defined by
$$\mathrm{Quot}\,\varphi\left(\frac{a}{s_2^n}\right) = \frac{\varphi(a)}{\varphi(s_2)\ldots\varphi(s_n)}\,.$$

(b) Red: (m, n)SDom $\to$ $(2, n)$SDom sends an (m, n)-semidomain with zero into its $(2, n)$-reduced ring with respect to 0, $\mathrm{Red}R = \mathrm{red}_0^{(2,n)}\mathrm{R}$, and it leaves the morphisms unchanged, i.e. for $\varphi \in \mathrm{Hom}\,(R_1, R_2)$, we have
$$\mathrm{Red}\,\varphi \in \mathrm{Hom}(\mathrm{Red}R_1, \mathrm{Red}R_2), \quad \mathrm{Red}\,\varphi\ (x) = \varphi(x)\,.$$

Note that the functor Red was already introduced and studied for the case of n-groups in the papers [9] and [6].

(c) F_m: (m, n)SDom $\to$ $(m, 2)$SDR; for any (m, n)-semidomain R with zero, $\mathrm{F}_m R$ is obtained as the division ring of quotients of its $(m, 2)$-reduct with respect to $n-2$ nonzero elements and, for $\varphi \in \mathrm{Hom}(R_1, R_2)$,

$$\mathrm{F}_m\varphi \in \mathrm{Hom}\,(\mathrm{F}_m R_1, \mathrm{F}_m R_2) \text{ is defined by } \quad \mathrm{F}_m\varphi\left(\frac{a}{b}\right) = \frac{\varphi(a)}{\varphi(b)}\,.$$

The functor F_m is well-defined; indeed, let $(R, +, \circ)$ be an (m, n)-semidomain with zero,

$$u_1, \dots, u_{n-2}, v_1, \dots, v_{n-2} \in R^*,\ A = \mathrm{red}_{\mathrm{u}_1^{\mathrm{n-2}}}\mathrm{R},$$

$$(Q_1, +, \cdot) = A_{A^{*\,n-1}},\ B = \mathrm{red}_{\mathrm{v}_1^{\mathrm{n-2}}}\mathrm{R},\ (\mathrm{Q}_2, \oplus, \odot) = \mathrm{B}_{\mathrm{B}^{*\,\mathrm{n-1}}};$$

we prove that, in fact, $Q_1 \equiv Q_2$. The fraction $\frac{a}{b}$ means in Q_1 :

$$\begin{aligned} \tfrac{a}{b} &= \{\tfrac{x}{y} \mid x \in R, y \in R^*, a \cdot y = x \cdot b\} \\ &= \{\tfrac{x}{y} \mid x \in R, y \in R^*, (a, u_1^{n-2}, y)_\circ = (x, u_1^{n-2}, b)_\circ\}\,, \end{aligned}$$

while in Q_2 :

$$\frac{a}{b} = \left\{\frac{x}{y} \mid x \in R,\ y \in R^*, (a, v_1^{n-2}, y)_\circ = (x, v_1^{n-2}, b)_\circ\right\};$$

note now that – since R is an (m, n)-semidomain – we have by cancellation laws and semicommutativity:

$$\begin{aligned} &(a, u_1^{n-2}, y)_\circ = (x, u_1^{n-2}, b)_\circ \\ &\Leftrightarrow (a, u_1^{n-2}, y, v_1^{n-2}, y)_\circ = (x, u_1^{n-2}, b, v_1^{n-2}, y)_\circ \\ &\Leftrightarrow (a, v_1^{n-2}, y, u_1^{n-2}, y)_\circ = (x, v_1^{n-2}, b, u_1^{n-2}, y)_\circ \\ &\Leftrightarrow (a, v_1^{n-2}, y)_\circ = (x, v_1^{n-2}, b)_\circ, \end{aligned}$$

i.e. the fraction $\frac{a}{b}$ means the same thing in Q_1 as in Q_2 (and so $Q_1 = Q_2 = Q$). Furthermore, we have

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = \frac{a_1}{b_1} \odot \frac{a_2}{b_2} \quad \text{and} \quad \left[\frac{a_1}{b_1}, \dots, \frac{a_m}{b_m}\right]_+ = \left[\frac{a_1}{b_1}, \dots, \frac{a_m}{b_m}\right]_\oplus$$

for any
$$\frac{a_i}{b_i} \in Q, \quad i = 1, \ldots, m.$$

(d) $\mathrm{P}_m\colon (m,n)\mathrm{SDR} \to (m,2)\mathrm{Fd}$. Indeed, for any semicommutative (m,n)-division ring D with zero, $\mathrm{P}_m D$ is the $(m,2)$-field $(\mathcal{M}/\sim, +, \cdot)$, where $\mathcal{M}$ is the set of all sequences of $(n-1)$ elements of D,

$$\mathcal{M} = \{(a_1, \ldots, a_{n-1}) \mid a_1, \ldots, a_{n-1} \in D\};$$

"$\sim$" is the equivalence on $\mathcal{M}$ (see [13]) defined as:

$$[a_1, \ldots, a_{n-1}] \sim [b_1, \ldots, b_{n-1}] \Leftrightarrow \forall c \in D : (c, a_1^{n-1})_\circ = (c, b_1^{n-1})_\circ$$

(in fact, it is easy to prove that it is sufficient that the relation above is verified for one $c \in D$); it is also easy to see that the relation above implies $\forall c \in D : (a_1^{n-1}, c)_\circ = (b_1^{n-1}, c)_\circ$; the equivalence class of the sequence $(a_1, \ldots, a_{n-1})$ is denoted by $\langle a_1, \ldots, a_{n-1}\rangle$; the operations in $\mathcal{M}/\sim$ are defined by:

$$\langle a_1, \ldots, a_{n-1}\rangle \cdot \langle b_1, \ldots, b_{n-1}\rangle = \langle (a_1^{n-1}, b_1)_\circ, b_2, \ldots, b_{n-1}\rangle$$

$$[\langle a_{11}, \ldots, a_{1,n-1}\rangle, \ldots, \langle a_{m1}, \ldots, a_{m,n-1}\rangle]_+$$
$$= \left\langle \left[(a_{11}^{1,n-1}, u)_\circ, \ldots, (a_{m1}^{m,n-1}, u)_\circ\right]_+, u, \ldots, u, \underline{u} \right\rangle \quad \text{with } u \in D^*.$$

For $\varphi \in \mathrm{Hom}(D_1, D_2)$, we have $\mathrm{P}_m\varphi \in \mathrm{Hom}(\mathrm{P}_m D_1, \mathrm{P}_m D_2)$,

$$\mathrm{P}_m\varphi(\langle a_1, \ldots, a_{n-1}\rangle) = \langle \varphi(a_1), \ldots, \varphi(a_{n-1})\rangle.$$

Theorem 43. *The diagram*

$$\begin{tikzcd}
(m,n)\mathrm{SDom} \arrow[r, "\mathrm{Quot}"] \arrow[d, "\mathrm{Red}"'] & (m,n)\mathrm{SDR} \arrow[d, "\mathrm{Red}"] \\
(2,n)\mathrm{SDom} \arrow[r, "\mathrm{Quot}"] & (2,n)\mathrm{SDR}
\end{tikzcd}$$

with categories and functors is commutative, i.e. $\mathrm{Red} \circ \mathrm{Quot} = \mathrm{Quot} \circ \mathrm{Red}$.

Proof. Let $(R,[\,]_+,\circ)$ be an (m,n)-semidomain with zero. Then

$$\mathrm{Red}R=\mathrm{red}_0^{(2,\mathrm{n})}\left(\mathrm{R},[\,]_+,\circ\right)=\left(\mathrm{R},+,\circ\right),$$

and for

$$\varphi\colon R_1\to R_2,\ \ \text{wehave}\ \ \mathrm{Red}\varphi\colon\ \mathrm{Red}R_1\to\ \mathrm{Red}R_2,\quad \mathrm{Red}\varphi(x)=\varphi(x)\,.$$

Now Quot $\circ$ Red acts on objects as follows: $\mathrm{Quot}(\mathrm{Red}R)=(Q,+,\odot)$, where

$$\begin{aligned}
Q&=\left\{\frac{a}{s_2^n}\mid a\in R,s_2,\dots,s_n\in R^*\right\},\\
\frac{a}{s_2^n}&=\left\{\frac{x}{y_2^n}\mid x\in R,y_2,\dots,y_n\in R^*,\ (a,y_2^n)_\circ=(x,s_2^n)_\circ\right\},\\
\frac{a}{s_2^n}+\frac{b}{t_2^n}&=\frac{(a,t_2^n)_\circ+(b,s_2^n)_\circ}{s_2^n\,t_2^n}=\frac{[(a,t_2^n)_\circ,\overset{(m-2)}{0},(b,s_2^n)_\circ]_+}{s_2^n\,t_2^n},\\
\left(\frac{a_1}{s_{12}^{1n}},\dots,\frac{a_n}{s_{n2}^{nn}}\right)_\odot&=\frac{(a_1,\dots,a_n)_\circ}{(s_{12}^{n2})_\circ\dots,(s_{1n}^{nn})_\circ}\,.
\end{aligned}$$

On morphisms Quot $\circ$ Red acts as follows:

$$\begin{aligned}
&\mathrm{Quot}(\mathrm{Red}\varphi)\colon Q_1\to Q_2,\\
&\mathrm{Quot}(\mathrm{Red}\varphi)\left(\frac{a}{s_2^n}\right)=\frac{\mathrm{Red}\varphi(a)}{\mathrm{Red}\varphi(s_2)\dots\ \mathrm{Red}\varphi(s_n)}=\frac{\varphi(a)}{\varphi(s_2)\dots\varphi(s_n)}\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathrm{Quot}R&=R_{R^{*\,n-1}}\ \stackrel{\mathrm{def}}{=}(\overline{Q},[\,]_\oplus,\star),\\
\overline{Q}&=\left\{\frac{a}{s_2^n}\mid a\in R,s_2,\dots,s_n\in R^*\right\},\\
\frac{a}{s_2^n}&=\left\{\frac{x}{y_2^n}\mid x\in R,\ y_2,\dots,y_n\in R^*,\ (a,y_2^n)_\circ=(x,s_2^n)_\circ\right\},
\end{aligned}$$

so the sets $\overline{Q}$ and Q coincide and

$$\left[\frac{a_1}{s_{12}^{1n}},\dots,\frac{a_m}{s_{m2}^{mn}}\right]_{\oplus} = \frac{\sum_{i=1}^{m}(a_i, s_{12}^{1n},\dots,s_{i-1,2}^{i-1,n}, s_{i+1,2}^{i+1,n},\dots,s_{m2}^{mn})_\circ}{s_{12}^{1n}\dots s_{m2}^{mn}},$$

$$= \left(\frac{a_1}{s_{12}^{1n}},\dots,\frac{a_n}{s_{n2}^{nn}}\right)_{\star} = \frac{(a_1,\dots,a_n)_\circ}{(s_{12}^{n2})_\circ\dots(s_{n2}^{nn})_\circ},$$

and the zero in $\overline{Q}$ is $0 = \{\frac{0}{s_2^n} \mid s_2,\dots,s_n \in R^*\}$.

Now we will see how $\mathrm{Red}\circ\mathrm{Quot}$ acts on objects and morphisms. We have:

$\mathrm{Red}(\ \mathrm{Quot}R) = (\overline{Q},\oplus,\star)$, where

$$\frac{a}{s_2^n}\oplus\frac{b}{t_2^n} = \left[\frac{a}{s_2^n}, \overset{(m-2)}{\frac{0}{u_2^n}}, \frac{b}{t_2^n}\right]_{\oplus} = \frac{[(a, \overset{(m-2)}{u_2^n}, t_2^n)_\circ, \overset{(m-2)}{0}, (b, s_2^n, \overset{(m-2)}{u_2^n})_\circ]_+}{s_2^n\, u_2^n \dots u_2^n\, t_2^n}$$

$$= \frac{\left([(a,t_2^n)_\circ, \overset{(m-2)}{0}, (b,s_2^n)_\circ]_+, \overset{(m-2)}{u_2^n}\right)_\circ}{s_2^n \quad \overset{(m-2)}{u_2^n} \quad t_2^n} = \frac{[(a,t_2^n)_\circ, \overset{(m-2)}{0}, (b,s_2^n)_\circ]_+}{s_2^n\, t_2^n},$$

multiplication $\star$ remaining unchanged; we see now that Q and $\overline{Q}$ coincide, i.e. $\mathrm{Quot}(\mathrm{Red}R) = \mathrm{Red}(\mathrm{Quot}R)$. On morphisms, we have

$$\mathrm{Red}(\mathrm{Quot}\varphi)\left(\frac{a}{s_2^n}\right) = \mathrm{Quot}\varphi\left(\frac{a}{s_2^n}\right) = \frac{\varphi(a)}{\varphi(s_2)\dots\varphi(s_n)} = \mathrm{Quot}(\mathrm{Red}\varphi)\left(\frac{a}{s_2^n}\right)$$

i.e. $\mathrm{Red}(\mathrm{Quot}\varphi) = \mathrm{Quot}(\mathrm{Red}\varphi)$. ∎

Theorem 44. *In the diagram*

$$\begin{tikzcd}
(m,n)\mathrm{SDom} \arrow[r, "\mathrm{Quot}"] \arrow[d, "\mathrm{F_m}"'] & (m,n)\mathrm{SDR} \arrow[dl, "\mathrm{P_m}"] \\
(m,2)\mathrm{Fd} &
\end{tikzcd}$$

the two paths are naturally equivalent, i.e. $\mathrm{P_m} \circ \mathrm{Quot}$ *and* $\mathrm{F_m}$ *are naturally equivalent functors.*

Proof. Define θ: $\mathrm{F}_m \to \mathrm{P}_m \circ \mathrm{Quot}$, as follows: for each $R \in (m,n)\mathrm{SDom}$, θ_R: $\mathrm{F}_m R \to (\mathrm{P}_m \circ \mathrm{Quot})R$ is given by

$$\theta_R\left(\frac{a}{b}\right) = \left\langle \frac{(a, \overset{(n^2-3n+2)}{b})_\circ}{\overset{(n-1)}{b}}, \frac{b}{\overset{(n-1)}{b}}, \dots, \frac{b}{\overset{(n-1)}{b}} \right\rangle .$$

We will prove that θ is a natural equivalence of functors. We will show that for each (m,n)-semidomain with zero R, θ_R is an isomorphism and that for any $\alpha \in \mathrm{Hom}(R_1, R_2)$ (in $(m,n)\mathrm{SDom}$) the diagram:

$$\begin{tikzcd}
\mathrm{F}_m R_1 \arrow[r, "\theta_{R_1}"] \arrow[d, "\mathrm{F}_m\alpha"'] & (\mathrm{P}_m \circ \mathrm{Quot})R_1 \arrow[d, "(\mathrm{P}_m \circ \mathrm{Quot})\alpha"] \\
\mathrm{F}_m R_2 \arrow[r, "\theta_{R_2}"] & (\mathrm{P}_m \circ \mathrm{Quot})R_2
\end{tikzcd}$$

(D)

is commutative.

Let $(R, [\ ]_+, \circ)$ be an (m,n)-semidomain with zero and $\mathrm{F}_m R = (Q, +, \cdot)$, where

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = \frac{(a_1, u_1^{n-2}, a_2)_\circ}{(b_1, u_1^{n-2}, b_2)_\circ};$$

$$\left[\frac{a_1}{b_1}, \dots, \frac{a_m}{b_m}\right]_+ = \frac{\sum_{i=1}^m a_i \cdot b_1 \cdot \ldots \cdot b_{i-1} \cdot b_{i+1} \cdot \ldots \cdot b_m}{b_1 \cdot \ldots \cdot b_m}$$

$$= \frac{\sum_{i=1}^m (a_i, u_1^{n-2}, b_1, u_1^{n-2}, \dots, b_{i-1}, u_1^{n-2}, b_{i+1}, \dots, u_1^{n-2}, b_m)_\circ}{(b_1, u_1^{n-2}, b_2, \dots, u_1^{n-2}, b_m)_\circ} .$$

For $\frac{a}{b}$, $\frac{c}{d} \in Q$ we have:

$$\theta_R\left(\frac{a}{b}\cdot\frac{c}{d}\right)=\theta_R\left(\frac{(a,u_1^{n-2},c)_\circ}{(b,u_1^{n-2},d)_\circ}\right)$$

$$=\left\langle \frac{(a,u_1^{n-2},c,\overset{(n^2-3n+2)}{(b,u_1^{n-2},d)_\circ})_\circ}{\overset{(n-1)}{(b,u_1^{n-2},d)_\circ}},\frac{(b,u_1^{n-2},d)_\circ}{\overset{(n-1)}{(b,u_1^{n-2},d)_\circ}},\dots,\frac{(b,u_1^{n-2},d)_\circ}{\overset{(n-1)}{(b,u_1^{n-2},d)_\circ}}\right\rangle \overset{\text{def}}{=} A$$

and

$$\theta_R\left(\frac{a}{b}\right)\cdot\theta_R\left(\frac{c}{d}\right)$$

$$=\left\langle \frac{(a,\overset{(n^2-3n+2)}{b})_\circ}{\overset{(n-1)}{b}},\frac{b}{\overset{(n-1)}{b}},\dots,\frac{b}{\overset{(n-1)}{b}}\right\rangle\cdot\left\langle \frac{(c,\overset{(n^2-3n+2)}{d})_\circ}{\overset{(n-1)}{d}},\frac{d}{\overset{(n-1)}{d}},\dots,\frac{d}{\overset{(n-1)}{d}}\right\rangle$$

$$=\left\langle \Big(\frac{(a,\overset{(n^2-3n+2)}{b})_\circ}{\overset{(n-1)}{b}},\frac{b}{\overset{(n-1)}{b}},\dots,\frac{b}{\overset{(n-1)}{b}},\frac{(c,\overset{(n^2-3n+2)}{d})_\circ}{\overset{(n-1)}{d}}\Big)_\star,\frac{d}{\overset{(n-1)}{d}},\dots,\frac{d}{\overset{(n-1)}{d}}\right\rangle$$

$$=\left\langle \frac{(a,\overset{(n^2-2n)}{b},c,\overset{(n^2-3n+2)}{d})_\circ}{(\overset{(n-1)}{b},d)_\circ\dots(\overset{(n-1)}{b},d)_\circ},\frac{d}{\overset{(n-1)}{d}},\dots,\frac{d}{\overset{(n-1)}{d}}\right\rangle \overset{\text{def}}{=} B.$$

Now the identity $\theta_R(\frac{a}{b}\cdot\frac{c}{d})=\theta_R(\frac{a}{b})\cdot\theta_R(\frac{c}{d})$ is equivalent to the identity:

$$\left(\frac{x}{\overset{(n-1)}{x}},A\right)_\star=\left(\frac{x}{\overset{(n-1)}{x}},B\right)_\star,\quad\text{for some }\ x\in R^*,$$

and further to the identity

$$\Big(x,a,u_1^{n-2},c,\overset{(n^2-2n)}{(b,u_1^{n-2},d)_\circ},(x,\overset{(n-1)}{\overset{(n-1)}{b},\overset{(n-1)}{d}})_\circ\Big)_\circ$$

$$=\Big(x,a,\overset{(n^2-2n)}{b},c,\overset{(n^2-2n)}{d},(x,\overset{(n-1)}{\overset{(n-1)}{(b,u_1^{n-2},d)_\circ}})_\circ\Big)_\circ,$$

which (by cancellative laws and lemmas) holds. After some similar computations one shows that

$$\theta_R\left(\left[\frac{a_1}{b_1},\cdots,\frac{a_m}{b_m}\right]_+\right)=\left[\theta_R\left(\frac{a_1}{b_1}\right),\cdots,\theta_R\left(\frac{a_m}{b_m}\right)\right]_+,$$

and so θ_R is a ring-homomorphism. Now notice that we have

$$\theta_R\left(\frac{a}{b}\right)=0 \;\Leftrightarrow\; \left\langle\frac{(a,\overset{(n^2-3n+2)}{b})_\circ}{\overset{(n-1)}{b}},\frac{b}{\overset{(n-1)}{b}},\cdots,\frac{b}{\overset{(n-1)}{b}}\right\rangle=0$$

in $(\mathrm{P}_m\circ\mathrm{Quot})R$, i.e. one of the components is 0 in $\mathrm{Quot}R$. The only possibility is that

$$\frac{(a,\overset{(n^2-3n+2)}{b})_\circ}{\overset{(n-1)}{b}}=0 \quad \text{in } \mathrm{Quot}R,$$

which means that: $(a,\overset{(n^2-3n+2)}{b})_\circ=0$. Since R is an (m,n)-semidomain, this implies $a=0$, so $\frac{a}{b}=0$. Thus, θ_R is injective.

Let now y be an element of $(\mathrm{P}_m\circ\mathrm{Quot})R$, $y=\langle\frac{a_1}{s_{12}^{1n}},\cdots,\frac{a_{n-1}}{s_{n-1,2}^{n-1,n}}\rangle$; by (23(b)) and (23(c)) we have that

$$y=\left\langle\frac{(a_1,s_{22}^{2n},\ldots,s_{n-1,2}^{n-1,n})_\circ}{s_{12}^{1n}\,s_{22}^{2n}\ldots s_{n-1,2}^{n-1,n}},\cdots,\frac{(a_{n-1},s_{12}^{1n},\ldots s_{n-2,2}^{n-2,n})_\circ}{s_{12}^{1n}\,s_{22}^{2n}\ldots s_{n-1,2}^{n-1,n}}\right\rangle .$$

To abbreviate notation, we set

$$b_i=(a_i,s_{12}^{1n},\ldots,s_{i-1,2}^{i-1,n},s_{i+1,2}^{i+1,n},\ldots,s_{n-1,2}^{n-1,n})_\circ,\quad i=1,\ldots,n-1$$

$$t_2=(s_{12}^{1n},s_{22}^{2n},\ldots,s_{n-2,2}^{n-2,n},s_{n-1,2})_\circ,\ t_i=s_{n-1,i},\quad i=3,\ldots,n\ ,$$

$$\text{and so}\quad y=\left\langle\frac{b_1}{t_2^n},\cdots,\frac{b_{n-1}}{t_2^n}\right\rangle .$$

Then

$$y=\theta_R\left(\frac{(b_1^{n-1},x)_\circ}{(\overset{(n-1)}{t_2},\ldots,\overset{(n-1)}{t_n},x)_\circ}\right),$$

where x is an arbitrary (fixed) element of R^*; thus θ_R is surjective too. Finally, for any $\frac{a}{b} \in \mathrm{F}_m R_1$ we have:

$$(((\mathrm{P}_m \circ \mathrm{Quot})\alpha) \circ \theta_{R_1})(\tfrac{a}{b})$$

$$= (\mathrm{P}_m \circ \mathrm{Quot})\alpha \left(\left\langle \frac{(a, \overset{(n^2-3n+2)}{b})_\circ}{\overset{(n-1)}{b}}, \frac{b}{\overset{(n-1)}{b}}, \cdots, \frac{b}{\overset{(n-1)}{b}} \right\rangle\right)$$

$$= \left\langle \ \mathrm{Quot}\ \alpha \left(\frac{(a, \overset{(n^2-3n+2)}{b})_\circ}{\overset{(n-1)}{b}}\right),\ \mathrm{Quot}\ \alpha\left(\frac{b}{\overset{(n-1)}{b}}\right), \cdots,\ \mathrm{Quot}\ \alpha\left(\frac{b}{\overset{(n-1)}{b}}\right) \right\rangle$$

$$= \left\langle \frac{\alpha((a, \overset{(n^2-3n+2)}{b})_\circ)}{\alpha(b)...\alpha(b)}, \frac{\alpha(b)}{\alpha(b)...\alpha(b)}, \cdots, \frac{\alpha(b)}{\alpha(b)...\alpha(b)} \right\rangle$$

$$\text{and } (\theta_{R_2} \circ \mathrm{F}_m\alpha)\left(\frac{a}{b}\right) = \theta_{R_2}\left(\frac{\alpha(a)}{\alpha(b)}\right)$$

$$= \left\langle \frac{(\alpha(a), \overset{(n^2-3n+2)}{\alpha(b)})_\circ}{\alpha(b)...\alpha(b)}, \frac{\alpha(b)}{\alpha(b)...\alpha(b)}, \cdots, \frac{\alpha(b)}{\alpha(b)...\alpha(b)} \right\rangle,$$

which proves that the diagram (D) is commutative and so θ is a natural equivalence. ∎

45 . As a consequence of the two previous theorems we obtain that the diagram

$$\begin{tikzcd}
(m,n)\mathrm{SDom} \arrow[r, "\mathrm{Quot}"] \arrow[d, "\mathrm{Red}"'] & (m,n)\mathrm{SDR} \arrow[d, "\mathrm{Red}"] \\
(2,n)\mathrm{SDom} \arrow[r, "\mathrm{Quot}"] \arrow[dr, "\mathrm{F}_2"'] & (2,n)\mathrm{SDR} \arrow[d, "\mathrm{P}_2"] \\
 & (2,2)\mathrm{Fd}
\end{tikzcd}$$

is "naturally" commutative (paths are naturally equivalent functors).

Acknowledgements

Part of this paper was written during a visit of the first named author to the Université Paris VII, from February to December 1998. Thanks go to the members of the "Equipe des groupes finis" for their hospitality and their support. The stay was supported by a scholarship from the Romanian Ministry of Education, which is gratefully acknowledged.

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Received 20 January 1999
Revised 20 September 2000