

## EQUIVALENT CONDITIONS FOR $P$ -NILPOTENCE

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### Abstract

In the first part of this paper we prove without using the transfer or characters the equivalence of some conditions, each of which would imply  $p$ -nilpotence of a finite group  $G$ . The implication of  $p$ -nilpotence also can be deduced without the transfer or characters if the group is  $p$ -constrained. For  $p$ -constrained groups we also prove an equivalent condition so that  $O^q(G)P$  should be  $p$ -nilpotent. We show an example that this result is not true for some non- $p$ -constrained groups.

In the second part of the paper we prove a generalization of a theorem of Itô with the help of the knowledge of the irreducible characters of the minimal non-nilpotent groups.

**Keywords:**  $p$ -nilpotent group,  $p$ -constrained group, character of a group, Schmidt group, Thompson-ordering, Sylow  $p$ -group.

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## 1 Definitions and known results

We know the remarkable theorems of Frobenius which tell that in Theorem 2.1 (i) and (iv) both imply that the finite group  $G$  has a normal  $p$ -complement. All existing proofs of them use the transfer homomorphism or characters.

The well-studied minimal non-nilpotent groups, i.e. non-nilpotent groups, each of whose subgroups are nilpotent, sometimes are called Schmidt groups or  $(p, q)$ -groups. They can be described without using the transfer, see 5.1 Satz and 5.2 Satz in pp. 280-281 of [6]. Let  $G$  be a minimal non-nilpotent group. Then it can be proved without using the transfer or characters, that

1.  $G$  is solvable.
2.  $|G|$  is divisible only by 2 primes, say  $|G| = p^a q^b$ .
3.  $G' = P \in \text{Syl}_p(G)$ .
4. If  $p > 2$ , then  $\exp(P) = p$ ; if  $p = 2$ , then  $\exp(P) \leq 4$ ;  $\exp(Z(P)) = p$  in all cases.
5.  $P$  is either abelian or  $P' = \Phi(P) = Z(P) \leq Z(G)$ .
6. If  $Q \in \text{Syl}_q(G)$ , then  $Q$  is cyclic; and if  $Q = \langle x \rangle$ , then  $\langle x^q \rangle \leq Z(G)$ .
7. If  $P$  is abelian, then  $Q$  acts irreducibly on  $P$ ; if  $P$  is nonabelian, then  $Q$  acts irreducibly on  $P/Z(P)$ . If  $P$  is abelian, then  $P$  is of exponent  $p$ ; and if  $P$  is nonabelian, then  $P/Z(P)$  is also of exponent  $p$ . So, they can be considered as vector spaces over  $GF(p)$ . Their dimension is  $o(p)(\text{mod}(q))$ , which is even in the nonabelian case.
8.  $G$  is generated by its Sylow  $q$ -subgroups.

These groups are non- $p$ -nilpotent. It can also be proved using the transfer or characters that a group is  $p$ -nilpotent if and only if, it does not contain such a subgroup.

We shall use the following:

**Notation 11.** We shall write  $(p, q) \not\leq G$  if the group  $G$  does not contain a  $(p, q)$ -group, otherwise we write  $(p, q) \leq G$ .

Let us recall the definition of Thompson-ordering:

**Definition 12.** Let  $G$  be a finite group. Let  $\mathcal{P}$  be a property of subgroups of  $G$ . Let  $\mathcal{A} = \{A \mid A \text{ } p\text{-subgroup of } G, N_G(A) < G, A \text{ } p\text{-group, } N_G(A) \text{ has the property } \mathcal{P}\}$ . We tell for  $A_1, A_2 \in \mathcal{A}$  that  $A_1$  is smaller than  $A_2$  in the Thompson-ordering if either  $|N_G(A_1)|_p < |N_G(A_2)|_p$ , or  $|N_G(A_1)|_p = |N_G(A_2)|_p$  and  $|A_1| < |A_2|$ .

**Definition 13.** Let  $P \in \text{Syl}_p(G)$ . A subgroup  $U \leq P$  is called strongly closed if for every  $u \in U$  if  $u^x \in P$  then  $u^x \in U$ .

## 2 Main results

The aim of this paper is to prove that the equivalence of the following four conditions can be proved without the use of transfer or characters:

**Theorem 21.** *Let  $G$  be a finite group, let  $P \in \text{Syl}_p(G)$ . Then the following are equivalent:*

- (i) *If  $x, y \in P$  are  $G$ -conjugate, then they are conjugate already in  $P$ ;*
- (ii) *If  $x, y \in P$  of order  $p$  or  $4$  and are  $G$ -conjugate, then they are conjugate already in  $P$ ;*
- (iii)  *$(p, q) \not\leq G$  for every prime  $q \neq p$ ;*
- (iv) *For every  $p$ -subgroup  $U \leq G$ ,  $N_G(U)/C_G(U)$  is a  $p$ -group.*

As a corollary we get:

**Theorem 22.** *Let  $G$  be a  $p$ -constrained group,  $P \in \text{Syl}_p(G)$ . If any of the conditions of the above theorem holds for  $G$ , then one can deduce without using the transfer that  $G$  has a normal  $p$ -complement.*

As an application of Theorem 2.1 we prove also the following:

**Theorem 23.** *Let  $G$  be a finite group, let  $p \neq q$  be primes with  $p, q \in \pi(G)$ . Let  $P \in \text{Syl}_p(G)$  and let  $O^{q'}(G)$  denote the subgroup of  $G$  generated by the  $q$ -elements of  $G$ . If  $G$  is  $p$ -constrained then the following are equivalent:*

- (i)  *$(p, q) \not\leq G$ .*
- (ii)  *$O^{q'}(G)P$  has a normal  $p$ -complement.*

Another application of Theorem 2.2 is to prove without using the transfer the following generalization of a theorem of Itô:

**Theorem 24.** *Let  $G$  be a finite  $p$ -constrained group, let  $P \in \text{Syl}_p(G)$ . Let us suppose that  $\chi$  is a character of  $G$  satisfying the following conditions:*

- $\alpha)$   $\chi(1) \leq 2p - 2$ ,
- $\beta)$  *for every subgroup  $H \leq G$ ,  $\chi_H$  does not have a constituent of degree  $p$ ,*
- $\gamma)$   $\text{Ker}(\chi) = 1$ .

*Then one of the following two possibilities holds:*

- (i)  *$P$  is abelian and  $P$  is normal in  $G$ ;*
- (ii)  *$p$  is a Fermat-prime and one of the constituents of  $\chi$  has degree at least  $p - 1$ .*

*The inequality in  $\alpha)$  is sharp. There is a solvable group  $G_0$  having a character  $\chi_0 \in \text{Char}(G_0)$  satisfying  $\beta)$  and  $\gamma)$  with degree  $\chi_0(1) = 2p - 1$ , such that for this pair the assertion of the Theorem does not hold.*

■

### 3 Preliminary lemmas

In the proof of Theorem 2.1 we will need the following lemma, which is Lemma 2 in [4].

**Lemma 31.** *Let  $G$  be a group with  $H \in \text{Hall}_\pi(G)$  and with the property that every  $\pi$ -subgroup  $Y$  of  $G$  can be conjugated into  $H$ . Let  $\mathcal{K}$  be a class of elements of  $H$ , which is closed under conjugation inside  $H$  with elements of  $G$  such that if two elements of  $\mathcal{K}$  are conjugate in  $G$  then they are already conjugate in  $H$ . Then if  $G_1 \triangleleft G$  and  $|G : G_1| = q$ , where  $q \in \pi$ , then for  $H_1 = H \cap G_1$  it holds that each pair of elements of  $H_1 \cap \mathcal{K}$  that are conjugate in  $G_1$  are already conjugate in  $H_1$ . ■*

For the proof of Theorem 2.1 we will also need the following lemma, which generalizes both Lemma 5 in [4] and Lemma 3.2 in [3].

**Lemma 32.** *Let  $q \in \pi(G) \setminus \{p\}$  be a fixed prime,  $P \in \text{Syl}_p(G)$ ,  $U < P$  abelian and strongly closed in  $P$ . Then if  $(p, q) \not\leq N_G(U)$  then  $(p, q) \not\leq G$ , as well.*

**Proof.** Let  $G$  be a counterexample of minimal order.

First we prove that we may assume that  $O_p(G) = 1$ .

Let  $O_p(G) > 1$  and let  $B \triangleleft G$  be a  $p$ -subgroup. Let  $\overline{G} = G/B$ , and the images of  $U$  and  $P$  in this factor group let  $\overline{U}$  and  $\overline{P}$ , respectively. Then  $\overline{P} \in \text{Syl}_p(\overline{G})$  and the triple  $(\overline{G}, \overline{P}, \overline{U})$  satisfies the conditions set for  $(G, P, U)$ . To see this we have to show only that  $(p, q) \not\leq N_G(U)$  implies  $(p, q) \not\leq N_{\overline{G}}(\overline{U})$ . Let  $M$  be the inverse image of  $N_{\overline{G}}(\overline{U})$  in  $G$ . Here  $M < G$ , since if  $\overline{U} \triangleleft \overline{G}$  then  $U^G \leq P$ , and as  $U \leq P$  is strongly closed,  $U^G = U$  would follow. This would imply  $(p, q) \not\leq N_G(U) = G$ , which is a contradiction. So  $M < G$ . But  $P \leq M$  and  $N_G(U) = N_M(U)$ . The triple  $(M, P, U)$  satisfies the conditions of the Lemma. By induction  $(p, q) \not\leq M$ . By [2],  $(p, q) \not\leq \overline{M} = N_{\overline{G}}(\overline{U})$ , as well. Hence the conditions of the Lemma are satisfied by the triple  $(\overline{G}, \overline{P}, \overline{U})$  and by induction  $(p, q) \not\leq \overline{G}$ .

Let  $V$  be a  $(p, q)$ -group in  $G$ . Then  $V' = V_p \in \text{Syl}_p(V)$  and by the above result, its image  $\overline{V}$  in  $\overline{G}$  is nilpotent. Hence  $V_p \leq B$ . There are two cases:

- (i)  $U \cap O_p(G) = 1$ ,
- (ii)  $U \cap O_p(G) \neq 1$ .

Ad (i): We know that  $U \triangleleft P$  and we may assume that  $V_p \leq P$ , by replacing  $V$  with a suitable conjugate of it. Hence  $[U, V_p] \leq U$ . On the other hand as  $V_p \leq O_p(G)$ ,  $[U, V_p] \leq O_p(G) \cap U = 1$ .

Ad (ii): If  $U \cap O_p(G) \neq 1$ , then we may choose  $B$  equal to it, because  $U \cap O_p(G)$  is normal in  $G$  as  $U$  is strongly closed. Hence  $V_p \leq B \leq U$  by the above results. But  $U$  is abelian, so  $[V_p, U] = 1$  in this case, too.

Hence in both cases (i) and (ii)  $[V_p, U] = 1$ . Let  $N = N_G(V_p)$ . We claim that  $N = G$ . If  $N < G$  then if we choose  $S \in \text{Syl}_p(N)$  with the property  $U \leq S$ , then the triple  $(N, S, U)$  satisfies the conditions of the Lemma. So by induction  $(p, q) \not\leq N$ . This contradicts the fact that  $V \leq N$ . Thus  $V_p$  is normal in  $G$ . Let  $C = C_G(V_p)$ . Then  $C \triangleleft G$ . Let us choose  $Q \in \text{Syl}_q(G)$  so that it should contain a Sylow  $q$ -subgroup of  $V$ . Set  $L = CQ$ . Then  $P \cap L = P \cap C \in \text{Syl}_p(L)$  and  $P \cap C \triangleleft P$ . Then the triple  $(L, P \cap L, U)$  satisfies the conditions of the Lemma. Hence if  $L < G$  then by induction  $(p, q) \not\leq L$ , which is impossible as  $V \leq L$ . Thus  $G = L = CQ$ . Since  $P = P \cap L = P \cap C$ ,  $P \leq C$  and as  $C \triangleleft G$ , so by the Frattini argument we have that  $G = CN_G(P)$ . As  $U$  is a strongly closed subgroup of  $P$ ,  $N_G(P) \leq N_G(U)$  and thus  $(p, q) \not\leq N_G(P)$ . Since  $V_p \leq C_G(P) \leq N_G(P)$  and  $V_p \triangleleft G$ , hence  $V_p \triangleleft N_G(P)$ . Thus  $|N_G(P) : C \cap N_G(P)| \not\equiv 0 \pmod{q}$ . But then, since  $|N_G(P) : C \cap N_G(P)| = |CN_G(P) : C| = |G : C| = |CQ : C| = |Q : C \cap Q|$ ,  $Q \leq C = G$  follows. This is a contradiction, since  $V \leq G = C_G(V_p)$ .

End of the proof: Let  $\mathcal{A} = \{A \mid N_G(A) < G, A \text{ } p\text{-group}, (p, q) \leq N_G(A)\}$ . Let  $K$  be a maximal element of  $\mathcal{A}$  for the Thompson-ordering. Then  $|N_G(K)|_p$  is maximal and among those with this property  $K$  is also of maximal order. As  $\mathcal{A} \neq \emptyset$ , so such  $K$  exists. Then  $K \leq P_1$  for a suitable Sylow  $p$ -subgroup  $P_1$  of  $G$ . Let  $x \in G$  such that  $P_1^x = P$ . Then  $K^x \leq P$ . Let  $Z(P_1) \leq R \in \text{Syl}_p(N_G(K))$ . Then  $Z(P) \leq R^x \in \text{Syl}_p(N_G(K^x))$ . Let  $R^x \leq P_2 \in \text{Syl}_p(G)$ . Choose  $t \in G$  so that  $P_2^t = P$ . Thus  $R^{xt} \leq P$ . Since  $U \triangleleft P$ ,  $Z(P) \cap U \neq 1$ , thus  $R^x \cap U \neq 1$ , as well. As  $U$  is strongly closed in  $P$ ,  $(R^x \cap U)^t \leq R^{xt} \cap U$  and  $R^{xt} \cap U$  is strongly closed in  $R^{xt}$ . It is enough to prove that the triple  $(N_G(K^{xt}), R^{xt}, R^{xt} \cap U)$  satisfies the conditions of the Lemma. When we prove this, then  $(p, q) \not\leq N_G(K^{xt})$  follows, contradicting our assumption. It is enough to prove that  $(p, q) \not\leq N_G(R^{xt} \cap U)$ . If  $|R| = |P|$  then we have that the triple  $(N_G(K^{xt}), P, U)$  satisfies the conditions of the Lemma, and since  $N_G(K^{xt}) < G$ , by induction we get that  $(p, q) \not\leq N_G(K^{xt})$ , contradicting the choice of  $K$ . Thus  $|R| < |P|$ . Then  $N_P(R^{xt}) > R^{xt}$ , and since  $R^{xt} \cap U$  is strongly closed in  $R^{xt}$ ,  $N_G(R^{xt}) \leq N_G(R^{xt} \cap U)$ .

So  $|\mathbf{N}_G(R^{xt} \cap U)|_p > |\mathbf{N}_G(K^{xt})|_p = |R|$ . As  $\mathbf{N}_G(R^{xt} \cap U) \neq G$ , thus  $(p, q) \not\leq \mathbf{N}_G(R^{xt} \cap U)$ , by the maximality of  $K$  in the Thompson ordering. The proof is complete.  $\blacksquare$

For the proof of Theorem 2.4 we will need the description of irreducible characters of minimal non-nilpotent groups. As we did not find any reference to it in the literature, for the sake of selfcontainedness we include it here.

**Lemma 33.** *Let  $G$  be a  $(p, q)$ -group,  $P \in \text{Syl}_p(G)$ ,  $Q \in \text{Syl}_q(G)$ ,  $|Q| = q^n$ . Then  $G$  has exactly  $q^n$  linear characters.*

- (i) *If  $P$  is abelian, then all other characters in  $\text{Irr}(G)$  are of degree  $q$ . They are induced from nontrivial characters of the unique index  $q$  subgroup of  $G$ . There are  $(|P| - 1)q^{n-2}$  such characters.*
- (ii) *If  $P$  is extraspecial, then  $|P| = p^{2m+1}$ , where  $2m \equiv o(p) \pmod{q}$ .  $P/Z(P)Q$  is a  $(p, q)$ -group of type (i). So it has  $(p^{2m} - 1)q^{n-2}$  irreducible characters of degree  $q$ . The  $p - 1$  irreducible characters of degree  $p^m$  of  $P$  can be extended to  $G$  giving  $(p - 1)q^n$  irreducible characters of degree  $p^m$ .*
- (iii) *If  $P$  is special and nonabelian, then if  $|Z(P)| = p^k$  then  $Z(P)$  has  $\frac{p^k - 1}{p - 1}$  maximal subgroups. By factoring with one of them we get a  $(p, q)$ -group of type (ii). The union of inverse images of these characters give  $\text{Irr}(G)$ .*

**Proof.** As  $G' = P$ ,  $|G : G'| = q^n$ , so  $G$  has exactly  $q^n$  linear characters. Let  $H = P\langle x^q \rangle$ . Then  $|G : H| = q$  and  $H$  is normal in  $G$ .

Ad (i): If  $P$  is abelian, then so is  $H$ , so if  $\chi \in \text{Irr}(G)$  nonlinear, then  $\chi_H = \sigma_1 + \dots + \sigma_q$  and  $\chi = \sigma_i^G$  for  $i = 1, \dots, q$ . So  $\chi(1) = q$  and  $\chi$  is induced from exactly  $q$  linear characters of  $H$ . As  $|G| = |P|q^n = q^n + q^2(|P| - 1)q^{n-2}$ , we get that each nontrivial character of  $H$  that does not contain  $P$  in its kernel is induced to  $\text{Irr}(G)$ .

Ad (ii): If  $P$  is extraspecial, then  $|P| = p^{2m+1}$ . As  $Q$  acts irreducibly on  $P/Z(P)$ , by Lemma 3.10 in Chapter II. of [6] we get that  $2m \equiv o(p) \pmod{q}$ . The  $p - 1$  faithful irreducible characters of  $P$  are of degree  $p^m$ , they are 0 outside  $Z(P)$ , so they are  $G$ -invariant, and as  $(|P|, |G : P|) = 1$ , they can be extended to  $G$ . By Gallagher's theorem, se e.g. [7], they can be extended in  $q^n$  ways. This way we get  $(p - 1)q^n$  irreducible characters of  $G$ . By taking into consideration those of degree 1 and  $q$  the sum of squares of the degrees gives:

$$q^n + q^2(p^{2m} - 1)q^{n-2} + p^{2m}(p - 1)q^n = q^n p^{2m+1} = |G|,$$

so we determined all  $\text{Irr}(G)$ .

Ad (iii): We calculate the sum of squares of the irreducible characters we produced so far:  $q^n + q^2(p^{2m} - 1)q^{n-2} + p^{2m} \frac{p^k - 1}{p - 1} (p - 1)q^n = q^n p^{2m+k} = |G|$ , so we produced the whole  $\text{Irr}(G)$ . ■

## 4 Proofs of the main results

Now we prove Theorem 2.1:

**Proof of Theorem 2.1.** (i)  $\rightarrow$  (ii) is trivial.

(ii)  $\rightarrow$  (iii): We use induction on  $|G|$ . The following argument is similar to one in the last part of the proof of Theorem 1 in [4]. For the sake of selfcontainedness, we repeat it here.

Let  $A \triangleleft P$  be an abelian normal subgroup in  $P$  such that  $\exp(A) \leq p$  if  $p > 2$ , and  $\exp(A) \leq 4$  if  $p = 2$ , and  $A$  is maximal with these properties.

(a) If  $A \leq Z(P)$ :

then according to Alperin's theorem [1],  $\Omega_j(P) \leq Z(P)$ , where  $j = 1$  if  $p > 2$  and  $j = 2$  if  $p = 2$ . Then  $A$  is strongly closed in  $P$ , as if we take two elements  $a$  and  $a^x$  of order  $p$  or of order 4 in  $A$  and  $P$ , then they are conjugate in  $P$ , and as  $A$  is normal in  $P$ , we get that  $a^x \in A$ , too. Let  $N = N_G(A)$ . If  $N < G$ , then as  $P \in \text{Syl}_p(N)$ , by induction we get that  $(p, q) \not\leq N$ , and by Lemma 3.2,  $(p, q) \not\leq G$ . So we may assume that  $N_G(A) = G$ . Then  $P \leq C_G(A) \triangleleft G$ . As for each  $a \in A^\#$  for the conjugacy class  $K_G(a)$  of  $a$  in  $G$  and for the conjugacy class  $K_P(a)$  in  $P$  it holds that  $K_G(a) = K_G(a) \cap P = K_P(a) = a$ , as  $A \triangleleft G$  and  $A \leq Z(P)$ , so  $|G : C_G(a)| = |P : C_P(a)| = 1$ , and we get that  $A \leq Z(G)$ . Thus, if  $p > 2$ , then  $\Omega_1(P) \leq A \leq Z(G)$ ; if  $p = 2$ , then  $\Omega_2(P) \leq Z(G)$ ; and this means that  $G \not\leq (p, q)$  in cases  $p > 2$  and  $p = 2$ , either, for every prime divisor  $q \neq p$  of  $G$ .

(b) If  $A \not\leq Z(P)$ :

then  $A \cap Z(P) < A$ . Thus  $A/A \cap Z(P)$  contains a central subgroup of  $P/A \cap Z(P)$  of order  $p$ . Let  $A_1$  be its inverse image in  $A$ . According to our assumption,  $A_1 = \langle A \cap Z(P), x \rangle$ , where  $o(x) = p$  or  $o(x) = 4$  and  $x \notin Z(P)$ .  $A_1$  is strongly closed in  $P$  as if  $a_1 \in A_1$  and  $a_1^u \in P$ , then by assumption  $a_1^u$  is conjugate to  $a_1$  in  $P$ . But as  $A_1 \triangleleft P$ ,  $a_1^u \in A_1$ . If

$N_1 = N_G(A_1) < G$ , then as  $P \leq N_1$ , by induction we have that  $(p, q) \not\leq N_1$ . Then, by Lemma 3.2,  $(p, q) \not\leq G$ . So, we may assume that  $N_1 = G$ . Then  $C_G(A_1) \triangleleft G$  and  $A \cap Z(P) \leq Z(G)$ , as if  $a \in A \cap Z(P)$  and  $g \in G$  then  $a^g \in A_1 \leq P$ , so by assumption there is a  $u \in P$  such that  $a^g = a^u$  and as  $a \in A \cap Z(P)$   $a^u = a$ . So,  $|G : C_G(a)| = 1$  and thus  $A \cap Z(P) \leq Z(G)$  and  $C_G(A_1) = C_G(x)$ . Let  $g \in G$ , then  $x^g \in A_1 \leq P$ , so there exists an  $u \in P$  such that  $x^g = x^u$ , therefore the conjugacy classes  $K_G(x)$  and  $K_P(x)$  coincide and thus  $|G : C_G(A_1)| = |G : C_G(x)| = |P : C_P(x)| > 1$ . So,  $C_G(A_1)$  is a proper normal subgroup of  $p$ -power index in  $G$  and it is contained in some normal subgroup  $G_1$  of index  $p$ . By Lemma 3.1, applied for  $H = P \in \text{Syl}_p(G)$  if we take  $\mathcal{K}$  to be the set of elements of  $P$  of order  $p$  or 4, then induction gives that  $(p, q) \not\leq G_1$  for every prime  $q \neq p$ . As a  $(p, q)$ -group is generated by its Sylow  $q$ -subgroups,  $(p, q) \not\leq G$ , either.

(iii)  $\rightarrow$  (iv): Let  $T$  be a  $p$ -subgroup of  $G$ . Let  $N = N_G(T)$ . Let  $q \in \pi(N) \setminus \{p\}$ ,  $Q \in \text{Syl}_q(N)$ . If  $[Q, T] \neq 1$ , then  $(p, q) \leq QT$ , which cannot happen by assumption. Hence (iv) follows.

(iv)  $\rightarrow$  (i): The proof is similar to the second part of Lemma 5 in [4]. For the sake of selfcontainedness we repeat it here.

Let  $a, b \in P$  such that  $a = b^x$  for some  $x \in G$ . By the theorem of Alperin, see e.g. Chapter 7, Theorem 2.6 in [5], there exist Sylow  $p$ -subgroups  $Q_1, \dots, Q_n$  of  $G$ , elements  $x_1, \dots, x_n$  with  $x_j \in N_G(P \cap Q_j)$ , and  $y \in N_G(P)$  such that  $b \in P \cap Q_1$ ,  $b^{x_1 \dots x_{j-1}} \in P \cap Q_j$ ,  $x = x_1 \dots x_n y$  and  $N_P(P \cap Q_j) \in \text{Syl}_p(N_G(P \cap Q_j))$  for  $j = 1, \dots, n$ . Let  $N_j = N_G(P \cap Q_j)$ ,  $C_j = C_G(P \cap Q_j)$ ,  $P_j = N_P(P \cap Q_j)$ ,  $j = 1, \dots, n$ .

So  $N_j = C_j(P \cap N_j)$  and hence  $x_j = y_j z_j$ , where  $y_j \in C_j$  and  $z_j \in P \cap N_j$ . It is easy to see that  $a = b^x = b^{z_1 \dots z_n y}$ .  $N_G(P) = C_G(P)P$ , so  $y = cz_{n+1}$ , where  $c \in C_G(P)$ ,  $z_{n+1} \in P$ . Thus  $a = b^x = b^{z_1 \dots z_{n+1}}$ , which means that  $a$  and  $b$  are conjugate in  $P$ . The proof is complete. ■

Now we prove Theorem 2.2:

**Proof of Theorem 2.2.** Let  $H = O_{p',p}(G)$ ,  $R = H \cap P$ . As  $G$  is  $p$ -constrained,  $C_G(R) \leq H$ . By the Frattini argument,  $G = O_{p'}(G)N_G(R)$ . Let  $q \neq p$  prime,  $Q \in \text{Syl}_q(N_G(R))$ . Then  $QR = Q \times R$ , as  $(p, q) \not\leq QR$ . Hence  $Q \leq C_G(R) \leq H$ , and so  $Q \leq O_{p'}(G)$ . Thus  $G = O_{p'}(G)P$ . ■

Now we prove Theorem 2.3:

**Proof of Theorem 2.3.** (i)  $\rightarrow$  (ii): Repeating the argument of the previous proof one gets that  $O^{q'}(G) \leq O_{p'}(G)$ . As  $O^{q'}(G) \triangleleft G$  and it is a  $p'$ -group, so  $O^{q'}(G)P$  is a subgroup of  $G$  having normal  $p$ -complement.



(ii)→(i): If  $O^{q'}(G)P$  has a normal  $p$ -complement, then  $O^{q'}(G)$  is a  $p'$ -subgroup. If  $U$  is a  $(p, q)$ -group in  $G$ , then  $U \leq O^{q'}(U) \leq O^{q'}(G)$ . As  $O^{q'}(G)$  is a  $p'$ -subgroup, this is a contradiction. So, the proof is complete. ■

**Remark 41.** In Theorem 2.3 the condition that  $G$  is  $p$ -constrained cannot be omitted. Take  $G = A_5$ ,  $P \in \text{Syl}_5(G)$ . Then  $(5, 3) \not\leq G$ , since 25 does not divide  $|G|$ . As  $G$  is simple,  $G = O^{3'}(G) = O^{3'}(G)P$ .

Now we prove Theorem 2.4:

**Proof of Theorem 2.4.** We use induction on  $|G|$ . We assume that  $G$  is a finite  $p$ -constrained group of minimal order that has character  $\chi \in \text{Char}(G)$  satisfying the conditions in our Theorem such that the assertion is not yet known. We may assume that (ii) is false for  $G$ , and we have to prove that for  $G$  (i) holds. As (ii) does not hold for  $G$ , so it cannot hold for any proper subgroup  $H$  of  $G$ . So by induction (i) holds for all such  $H$ . Using  $\alpha)$  and  $\beta)$  we can deduce that every constituent of  $\chi_P$  is linear, and thus  $P' \leq \text{Ker}(\chi)$ . By  $\gamma)$ ,  $P$  is abelian. As  $G$  is  $p$ -constrained, so  $P' = 1$  implies that  $G$  is  $p$ -solvable. By the choice of  $G$  we get immediately that  $\pi(G) = \{p, q\}$  for a suitable prime  $q \neq p$ . This gives that  $G$  is solvable, hence  $G$  is also  $q$ -constrained.

We have to prove that  $P \triangleleft G$ . As  $\pi(G) = \{p, q\}$ , this means that  $G$  is  $q$ -nilpotent. As  $G$  is  $q$ -constrained, our Theorem 2.1 (iii) and Theorem 2.2 implies, (even without the use of the transfer), that either  $P \triangleleft G$  or  $G$  is a  $(q, p)$ -group. To finish the proof, it is enough to show that the second possibility cannot occur. Assume that  $G$  is a  $(q, p)$ -group. Let  $m$  be  $o(q) \pmod{p}$ . Using  $\beta)$  an appeal to Lemma 33. shows that  $\chi$  can have a nonlinear constituent only in the case when  $m$  is even, say  $m = 2a$ . The degree of a nonlinear irreducible constituent of  $\chi$  is then  $q^a$ . Since  $q^a + 1 \equiv 0 \pmod{p}$ ,  $q^a = pl - 1$ , for a suitable natural number  $l$ . From this one deduces that either  $q^a \geq 2p - 1$  or  $q^a = p - 1$  and  $q = 2$ . If  $q^a = p - 1$ ,  $p = 2$  cannot occur. As (ii) is not true in  $G$ , case  $q^a = p - 1$  cannot hold, either. So  $\chi$  has only linear constituents. But then  $G' \leq \text{Ker}(\chi)$ , so by  $\gamma)$   $G$  has to be abelian in this case, contradicting the assumption that  $G$  is a  $(q, p)$ -group. This completes the proof. ■

Now we give an example showing that in  $\alpha)$   $2p - 2$  cannot be replaced by  $2p - 1$ .

Let  $p$  and  $q = 2p - 1$  be primes, where  $p \geq 7$ . E.g.  $p = 7$  and  $q = 13$ . Let  $G_0$  be a  $(q, p)$ -group of order  $q^3p$  with extraspecial Sylow  $q$ -subgroup. In this case  $o(q) \pmod{p} = 2$ , so such a group exists. Then  $Q_0 = G'_0 \in \text{Syl}_q(G_0)$ ,

and  $G_0$  has a character  $\chi_0$  of degree  $q$  which is irreducible and faithful. So  $\alpha$ ) is not satisfied for  $\chi_0$ , as  $\chi_0(1) = 2p - 1$ . Since  $\chi_0$  is faithful  $\gamma$ ) holds. As all proper subgroups of  $G_0$ , except for  $Q_0$ , are abelian, and  $\chi_{Q_0}$  is irreducible, for  $\chi_0$   $\beta$ ) also holds. (i) is not true for  $G_0$ . But  $p$  is not a Fermat prime either, as then  $p = 2^{2^k} + 1$  would hold and  $q = 2p - 1 = 2^{2^k+1} + 1 \equiv 0 \pmod{3}$ ) so (ii) cannot hold, either.

**Remark 42.** This theorem extends a well-known result of N. Itô ([8], see also [7]).

It can be deduced from our statement if we replace  $2p - 2$  by  $p - 1$  in  $\alpha$ ) and we assume also  $\gamma$ ). Then  $\beta$ ) is automatically satisfied. If  $P$  is not normal in  $G$ , then  $\chi(1) = p - 1$  and  $\chi \in \text{Irr}(G)$  also holds.

On the other hand the assumption  $\beta$ ) is vital for our proof as if case (i) holds, then, by a theorem of N. Itô (see e.g. [7]),  $(\chi(1), p) = 1$  holds for every irreducible character  $\chi \in \text{Irr}(G)$ .

**Remark 43.** The conditions of our Theorem 2.4 however do not guarantee that if  $P$  is not normal in  $G$ , then  $\chi$  should be irreducible. Let us take  $p = 2^{2^k} + 1$  to be a Fermat-prime. Let  $G$  be a  $(2, p)$ -group of order  $2^{2^{k+1}+1}p$  with extraspecial Sylow 2-subgroup and Sylow  $p$ -subgroup  $P$  of order  $p$ . Then  $G$  has a faithful irreducible character  $\chi$  of degree  $p - 1$ . Let us choose a character  $\sigma \in \text{Char}(G)$  with  $p - 1 < \sigma(1) \leq 2p - 2$ , and  $(\sigma, \chi) = 1$  and all other constituents of  $\sigma$  are chosen to be linear. Then  $\sigma$  satisfies  $\alpha$ ),  $\beta$ ) and  $\gamma$ ), but  $\sigma$  is not irreducible.

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### References

- [1] J.L. Alperin, *Centralizers of abelian normal subgroups of  $p$ -groups*, J. Algebra **1** (1964), 110–113.
- [2] K. Corrádi, *On certain properties of centralizers hereditary to the factor group*, Publ. Math. (Debrecen) **37** (1990), 203–206.
- [3] K. Corrádi and E. Horváth, *Steps towards an elementary proof of Frobenius' theorem*, Comm. Algebra **24** (1996), 2285–2292.

- [4] K. Corrádi and E. Horváth, *Normal  $\pi$ -complement theorems*, Arch. Math. (Basel) **71** (1998), 262–269.
- [5] D. Gorenstein, “*Finite groups*”, Chelsea Publ. Comp., New York 1980.
- [6] B. Huppert, “*Endliche Gruppen*”, Springer-Verlag, Berlin 1967.
- [7] I.M. Isaacs, “*Character theory of finite groups*”, Dover Publ., Inc., New York 1994.
- [8] N. Itô, *On a theorem of H.F. Blichfeldt*, Nagoya Math. J. **5** (1953), 75–77.

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