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THE POSITIVE AND GENERALIZED DISCRIMINATORS DON'T EXIST[1]

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Abstract

In this paper it is proved that there does not exist a function for the language of positive and generalized conditional terms that behaves the same as the discriminator for the language of conditional terms.

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The concepts of conditional term, conditional term function and conditionally rational equivalence were introduced in [1], [2]. For a survey of the results on those concepts see [4]. In particular, we have proved that the semigroups of inner isomorphisms of such algebras forms set of invariants of the relation of conditionally rational equivalence on the class of all finite or uniformely locally finite algebras. By an "inner isomorphisms" of an algebra we mean an isomorphism between some subalgebras of this algebra. In [5], the concept of a positive conditional term, a positive conditional term fuction and a positive conditionally rational equivalence were introduced and it is proved there that the semigroups of inner homomorphisms of all finite or uniformly locally finite algebras play the role of invariants of the relation of positive conditionally rational equivalence on those algebras. We repeat the definitions from [5].

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Definition 1. A *positive condition* of a signature σ is a finite set of conjunctions of equations, that is, a formula of the form

$$P(\overline{x}) \equiv [(t_1^1(\overline{x}) = t_1^2(\overline{x}))\& \cdots \&(t_n^1(\overline{x}) = t_n^2(\overline{x}))],$$

where $t_j^i(\overline{x})$'s are terms of the signature σ.

Definition 2. The notion of a *positive conditional term* for an algebra $\mathcal{A}$ can be determined by induction on complexity of terms:

a) any variable or a constant of signature σ is a positive conditional term;

b) if $t_1(\overline{x}), \dots, t_n(\overline{x})$ are positive conditional terms for the algebra $\mathcal{A} = (A;\sigma)$ and $f(x_1, \dots, x_n)$ is a function from σ then $f(t_1(\overline{x}), \dots, t_n(\overline{x}))$ is a positive conditional term for the algebra $\mathcal{A}$;

c) if $t_1(\overline{x}), \dots, t_k(\overline{x})$ are positive conditional terms for the algebra $\mathcal{A} = (A;\sigma)$ and $\{P_1(\overline{x}), \dots, P_k(\overline{x})\}$ is a system of positive conditions of the signature σ such that

$$\mathcal{A} \models \forall \overline{x} \left(\bigvee_{i=1}^{k} P_i(\overline{x}) \right),$$

and

$$\mathcal{A} \models \forall \overline{x} \, (P_i(\overline{x}) \,\&\, P_j(\overline{x}) \to (t_i(\overline{x}) = t_j(\overline{x}))),$$

for any $i, j \leq k$, then

$$t(\overline{x}) = \begin{cases} t_1(\overline{x}), & \text{if} \quad P_1(\overline{x}), \\ \vdots & \\ t_k(\overline{x}), & \text{if} \quad P_k(\overline{x}), \end{cases}$$

is a positive conditional term for the algebra $\mathcal{A}$.

For any positive conditional term $t(\overline{x})$ of the algebra $\mathcal{A} = (A;\sigma)$, a corresponding *positive conditional term function* can be constructed by induction.

A positive conditional term function $f(t_1(\overline{x}), \dots, t_n(\overline{x}))$ is the composition of a function f and of a set of positive conditional term functions which correspond to positive conditional terms $t_1, \dots, t_n$. In the case when a positive conditional term $t(\overline{x})$ is constructed by the rule c) of Definition 2, then for any elements $\overline{a}, b$ from $\mathcal{A}$ the equality $t(\overline{a}) = b$ holds on $\mathcal{A}$ if and only if $\mathcal{A} \models P_i(\overline{a})\&(t_i(\overline{a}) = b)$ holds for some $i \leq n$.

Let $PCT(\mathcal{A})$ ($T(\mathcal{A})$, $CT(\mathcal{A})$, respectivly) be the set of all positive conditional term (term, conditional term, respectively) functions of an algebra $\mathcal{A}$.

For any algebra $\mathcal{A}$, we have the following equality

$$CT(\mathcal{A}) = T(\mathcal{A}^d),$$

where $\mathcal{A}^d$ is the expansion of the algebra $\mathcal{A}$ obtained by adding to the signature of $\mathcal{A}$ a ternary function symbol $d(x, y, z)$ which is interpreted in $\mathcal{A}^d$ as the ternary discriminator: i.e. for any $a, b, c \in \mathcal{A}$

$$d(a, b, c) = \begin{cases} a, & \text{if} \quad a \neq b, \\ c, & \text{if} \quad a = b. \end{cases}$$

It is natural to ask whether there exists a function analogous to the discriminator (a positive discriminator) for positive conditional term functions. That is, does there exist an operation $p(x_1, \ldots, x_n)$ defined on the universe of the algebra $\mathcal{A}$ such that the equation

$$PCT(\mathcal{A}) = T(\mathcal{A}^p)$$

is true (where $\mathcal{A}^p$ is the expansion of the algebra $\mathcal{A}$ obtained by adding the function p to the signature of $\mathcal{A}$)? We prove that the answer to this question is negative. We construct a uniformly locally finite algebra $\mathcal{A}$ such that for any function $p(x_1, \ldots, x_n)$ on $\mathcal{A}$, we have

$$PCT(\mathcal{A}) \neq T(\mathcal{A}^p).$$

We will prove an analogous result for n-conditional terms which are defined in [3].

Let $\sigma = \{f_n, h_n |\ n \in \omega\}$, where f_i, h_i are unary operation symbols. Let $A_n = \{< n, 0 >, < n, 1 >\}$, $A = \bigcup_{n \in \omega} A_n$ and $A^{(m)} = \bigcup_{n \geq m} A_n$.

We define the functions of signature σ on A as follows

$$f_i(< n, m >) = \begin{cases} < n, m + 1 \,(mod\, 2) >, & \text{if} \quad n > i, \\ < n, m >, & \text{if} \quad n \leq i; \end{cases}$$

$$h_i(< n, m >) = \begin{cases} < n, m >, & \text{if} \quad n > i, \\ < n, m + 1 \,(mod\, 2) >, & \text{if} \quad n \leq i. \end{cases}$$

Let $\mathcal{A} = (A; \sigma)$. Then $\mathcal{A}^{(n)} = (A^{(n)}; \sigma)$ is a subalgebra of the algebra $\mathcal{A}$ for each $n \in \omega$. The algebra $\mathcal{A}$ is uniformly locally finite.

We prove that there does not exist any positive discriminator for the algebra $\mathcal{A}$, in the sense described above.
Indeed, assume, for contadiction, that there is an operation $p(x_1, \ldots, x_n)$ on the set A such that

$$PCT(\mathcal{A}) = T(\mathcal{A}^p),$$

Then there must exist a positive conditional term $t(x_1, \ldots, x_m)$ of $\mathcal{A}$ that defines the function p on the set A. The definition of t contains only finitely many of the operation symbols from σ. Let $r \in \omega$ be such that the operation symbols f_i, h_i (for $i \geq r$) are not contained in the definition of the term t.

The positive conditional term t has a normal form, that is, t has the form

$$t(x_1, \ldots, x_m) = \begin{cases} t_1(x_1, \ldots, x_m), & \text{if} \quad P_1(x_1, \ldots, x_m), \\ \vdots & \\ t_q(x_1, \ldots, x_m), & \text{if} \quad P_q(x_1, \ldots, x_m), \end{cases}$$

where $P_j(x_1, \ldots, x_m)$ are positive conditions for the algebra $\mathcal{A}$ and $t_j(x_1, \ldots, x_m)$ are terms of signature σ.

Since the symbols f_j, h_j are not contained in the positive conditions $P_k(x_1, \ldots, x_m)$ for $r \leq j$ and since, for $r > j$, $< n, m >\in A^{(r)}$, we have the equations

$$\begin{aligned} &h_j(< n, m >) =< n, m >, \\ &f_j(< n, m >) = f(< n, m >) =< n, m + 1\,(mod\,2) > \end{aligned}$$

(where f is the function defined by the rule

$$f(< n, m >) =< n, m + 1\,(mod\,2) > \quad \text{for any} \quad < n, m >\in A).$$

On the subalgebra $\mathcal{A}^{(r)}$ of the algebra $\mathcal{A}$, any positive condition $P_s(x_1, \ldots, x_m)$ (for $s \leq q$) is equivalent to some finite system of equations of the form

$$\begin{aligned} f^k(x_i) &= f^l(x_i), \quad \text{for} \quad i \leq m, \\ \text{or} \quad f^k(x_i) &= f^l(x_j), \quad \text{for} \quad j \neq i \quad \text{and} \quad i, j \leq m, \end{aligned}$$

where $f^k(x)$ denotes the k^{th} iteration of the function f, and k and l are arbitrary. Since $f^2(a) = a$ for any $a \in A$, we must have $k, l \leq 1$.

Because $\mathcal{A}$, $\mathcal{A}^{(r)} \models \forall\overline{x}(\bigvee_{i=1}^{q} P_i(\overline{x}))$, there must exist an $s \leq q$ such that the positive condition $P_s(x_1, \ldots, x_m)$ is valid for the elements $x_1 =< r, 0 >$, $x_2 =< r+1, 0 >, \ldots, x_m =< r+m-1, 0 >$. Then the positive condition $P_s(x_1, \ldots, x_m)$ does not include any of the equation

$$f^k(x_i) = f^l(x_j)$$

and it includes only the equations of the form

$$f^k(x_i) = f^l(x_i).$$

Therefore, from the definition of the function f and from the fact that $\mathcal{A}^{(r)} \models \exists x_1, \ldots, x_m \ P_s(x_1, \ldots, x_m)$, it follows that the positive condition $P_s(x_1, \ldots, x_m)$ is valid for any elements from $\mathcal{A}^{(r)}$.

Thus, the positive conditional term function $p(x_1, \ldots, x_m)$, which is defined on the algebra $\mathcal{A}^{(r)}$ by the positive conditional term $t(x_1, \ldots, x_m)$, coincides with the term $t_s(x_1, \ldots, x_m)$ (from the normal form of t). If we consider this term as a function with variables $x_1, \ldots, x_m$, then $t_s(x_1, \ldots, x_m)$ coincides with the term function of the form $f_0(x_i)$ or x_i, for some $i \leq m$. Then

$$T(\mathcal{A}^{(r)}) = T((\mathcal{A}^{(r)})^p).$$

For an algebra $\mathcal{A}$, consider the positive conditional term function $t'(x)$ defined on A as follows:

$$t'(x) = \begin{cases} x, & \text{if } \ f_r(x) = x, \\ f_r(x), & \text{if } \ f_{r+1}(x) = x \ \& \ h_r(x) = x, \\ x, & \text{if } \ h_{r+1}(x) = x. \end{cases}$$

The function $t'(x)$ does not coincide with any term function of the algebra $\mathcal{A}^{(r)}$ and for any function $p(x_1, \ldots, x_m)$ which is defined on the set A, we have the inequality

$$PCT(\mathcal{A}) \neq T(\mathcal{A}^p)$$

i.e. the positive discriminator for the algebra $\mathcal{A}$ does not exist. We have proved the following theorem.

Theorem 1. *There exists a uniformly locally finite universal algebra for which the positive discriminator does not exist.* ■

In [3], we defined the concept of an *n-conditionl term* for each $n \in \omega \cup \{\omega\}$. The definition is analogous to those of conditional and positive conditional term. The role of the n-condition is played by any elementary formula with n blocs of quantifiers, if $n \in \omega$, and by any elementary formula, if $n = \omega$. In the paper [3] we give the relationship between n-conditionally rational equivalence of arbitrary elementary classes of universal algebras with the isomorphic categories of elementary embeddings on those classes. It is natural to ask whether there exist n-conditional discriminators, that is, are there functions $\varphi_n^{\mathcal{A}}(x_1, \ldots, x_{m_n})$ such that the equation

$$C_n T(\mathcal{A}) = T(\mathcal{A}^{\varphi_n^{\mathcal{A}}})$$

is valid for any universal algebra $\mathcal{A}$? Here $C_n T(\mathcal{A})$ is the collection of all n-conditionally term functions of an algebra $\mathcal{A}$, and $C_0 T(\mathcal{A}) = CT(\mathcal{A}) = T(\mathcal{A}^d)$. We give the negative answer to this question for all $n > 0$.

Let N_i (for $i \in \omega$) be a disjoint collection of copies of the set of natural numbers. For any $x \in N_i$, put $x' = x + 1$. Let $A = \bigcup_{i\in\omega} N_i$. For each $m \in \omega$, let f_m be a unary function symbol. Consider the signatures $\sigma_m =< f_1, \ldots, f_m >$. Let $\sigma = \bigcup_{m\in\omega} \sigma_m$. The function symbol f_m on the set A is interpreted as

$$f_m(x) = \begin{cases} x', & \text{if}\, x \in N_i \ \text{ and } \ i \le m, \\ x, & \text{if}\, x \in N_i \ \text{ and } \ i > m. \end{cases}$$

Let $\mathcal{A} =< A; \sigma >$. We prove that for any function $\varphi(x_1, \ldots, x_m)$ from $C_\omega T(\mathcal{A})$ the inclusion $C_1 T(\mathcal{A}) \subseteq T(\mathcal{A}^\varphi)$ does not hold. Therefore, for any function $\varphi(x_1, \ldots, x_m)$ defined on the set A, the equality

$$CT_n(\mathcal{A}) = T(\mathcal{A}^\varphi)$$

fails for all $n \geq 1$.

Let $\varphi(x_1, \ldots, x_m) \in C_\omega T(\mathcal{A})$. Then there exists an ω-conditional term $t(x_1, \ldots, x_m)$, which defines the function φ on A. The term $t(\overline{x})$ has a normal form, that is, there exists some complete system of ω-conditions $\{\Phi_1(\overline{x}), \ldots, \Phi_k(\overline{x})\}$ (where $\Phi_i(\overline{x})$ are elementary formulas) and terms $t_1(\overline{x}), \ldots, t_k(\overline{x})$ such that

$$t(\overline{x}) = \begin{cases} t_1(\overline{x}), & \text{if } \ \Phi_1(\overline{x}), \\ \vdots & \\ t_k(\overline{x}), & \text{if } \ \Phi_k(\overline{x}). \end{cases}$$

Let $p \in \omega$ be such that the formulas $\phi_i(\overline{x})$ and terms $t_j(\overline{x})$ are formulas and terms of the signature σ_p. Let $a_1, \ldots, a_m \in \bigcup_{i>p} N_i$, and take $D_{\overline{a}}(x_1, \ldots, x_m)$ to be the *diagram of the sequence* $\overline{a} =< a_1, \ldots, a_m >$, that is, the set of all equalities and inequalities in the variables $x_1, \ldots, x_m$ that are satisfied by $\overline{a}$ in $\mathcal{A}$. For any $a \in \bigcup_{i>p} N_i$ and $f \in \sigma_p$ the equation $f(a) = a$ is valid. Therefore, two m-termed sequence from $\bigcup_{i>p} N_i$ with the same diagram satisfy the same formulas $\Phi_j(x_1, \ldots, x_m)$ for each j, with $1 \leq j \leq k$.

Let $\{\Phi_1^{'}(\overline{x}), \ldots, \Phi_s^{'}(\overline{x})\}$ be a complete system for the diagrams of m-termed sequences from A. Then for any $1 \leq j \leq s$ there exists $i_j \in \{1, \ldots, m\}$ such that the function $\varphi(\overline{x})$ is defined on the set $\bigcup_{i>p} N_i$ by the conditional term

$$t'(\overline{x}) = \begin{cases} x_{i_1}, & \text{if } \Phi'_1(\overline{x}), \\ \vdots & \\ x_{i_s}, & \text{if } \Phi'_s(\overline{x}). \end{cases}$$

Then, for each unary term $q(x)$ of the signature $\sigma \cup \{\varphi(\overline{x})\}$, there exists a unary term $q'(x)$ of the signature σ such that

$$\mathcal{A} \models q(a) = q'(a)$$

for any $a \in N_{p+1}$. But such a term q' does not exist for the 1-conditional term $r(x)$ defined by:

$$r(x) = \begin{cases} f_{p+1}(x), & \text{if } \ (f_p(x) = x) \ \& \ (f_{p+1}(x) \neq x) \quad \& \ \forall y(f_{p+1}(y) \neq x), \\ x, & \text{if } \ (f_p(x) = x) \ \& \ (f_{p+1}(x) \neq x) \quad \& \ \exists y(f_{p+1}(y) = x), \\ x, & \text{if } \ \neg[(f_p(x) = x) \ \& \ (f_{p+1}(x) \neq x)]. \end{cases}$$

Thus, $r(x) \in C_1T(\mathcal{A})$ and $r(x) \notin T(\mathcal{A}^{\varphi})$. This completes the proof of the following theorem.

Theorem 2. *There exist a universal algebra that does not have an n-conditional discriminator for each n with $1 \leq n \leq \omega$.* ■

From Theorems 1 and 2 it follows that any finite set of functions on the basic sets of algebras from Theorems 1 and 2 cannot play the role of the corresponding discriminators.

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