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MODIFICATIONS OF CSÁKÁNY'S THEOREM

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Abstract

Varieties whose algebras have no idempotent element were characterized by B. Csákány by the property that no proper subalgebra of an algebra of such a variety is a congruence class. We simplify this result for permutable varieties and we give a local version of the theorem for varieties with nullary operations.

Keywords: congruence class, idempotent element, permutable variety, Mal'cev condition.

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1 Introduction

B. Csákány [2] proved the following statement:

Proposition. *For a variety $\mathcal{V}$, the following conditions are equivalent:*

(i) *None of algebras in $\mathcal{V}$ having at least two elements have idempotent elements;*

(ii) *No algebra $\mathcal{A} \in \mathcal{V}$ has a proper subalgebra whose carrier is a class of some $\theta \in Con\ \mathcal{A}$;*

(iii) *There exist $n \in N$, ternary terms $p_1, \ldots, p_n$, and unary terms $f_1, \ldots, f_n$, $g_1, \ldots, g_n$ such that the identities*

$$x = p(f_1(x), x, y),$$

$$p_i(g_i(x), x, y) = p_{i+1}(f_{i+1}(x), x, y), \quad for \quad i = 1, \ldots, n-1,$$

$$y = p_n(g_n(x), x, y)$$

hold in $\mathcal{V}$.

It was recognized by J. Kollár [3] that each of the equivalent conditions of the Proposition is equivalent to

(iv) For all $\mathcal{A} \in \mathcal{V}$, the greatest congruence ι_A on $\mathcal{A}$ is a compact element of $Con\ \mathcal{A}$.

Analyzing the proof of Proposition, we find out that these conditions are also equivalent to

(v) $$\theta(F_{\mathcal{V}}(x) \times F_{\mathcal{V}}(x)) = F_{\mathcal{V}}(x,y) \times F_{\mathcal{V}}(x,y) \quad \text{in} \quad Con\ F_{\mathcal{V}}(x,y)\ ,$$

where $F_{\mathcal{V}}(x_1, \ldots, x_n)$ denotes the free algebra of $\mathcal{V}$ generated by the set $\{x_1, \ldots, x_n\}$ of free generators, and $\theta(M \times M)$ denotes the least congruence containing the set $M \times M$.

From this point of view, (v) can be modified by several ways. We can consider a variety $\mathcal{V}$ with constants (i.e. nullary operations) and we can omit a free variable on the left-hand side of (v) to obtain

(vi) $\theta(F_{\mathcal{V}}(\emptyset) \times F_{\mathcal{V}}(\emptyset)) = F_{\mathcal{V}}(x,y) \times F_{\mathcal{V}}(x,y) \quad \text{in} \quad Con\ F_{\mathcal{V}}(x,y)$.

This condition used in [1] it is equivalent to the property that ι_A is generated by the set of nullary operations for each $\mathcal{A} \in \mathcal{V}$.
Further, we can also

(a) simplify Csákány's original result for permutable varieties;

(b) omit one free variable in both sides of (v) to obtain

(vii) $\theta(F_{\mathcal{V}}(\emptyset) \times F_{\mathcal{V}}(\emptyset)) = F_{\mathcal{V}}(x) \times F_{\mathcal{V}}(x) \quad \text{in} \quad Con\ F_{\mathcal{V}}(x)$.

In the second case we obtain a local version of Csákány's theorem. These modifications are treated in this paper.

2 Results

Theorem 1. *Let $\mathcal{V}$ be a permutable variety. The following conditions are equivalent:*

(i) *None of algebras in $\mathcal{V}$ having at least two elements have idempotent elements;*

(ii) *No algebra $\mathcal{A} \in \mathcal{V}$ has a proper subalgebra whose carrier is a class of some $\theta \in Con\ \mathcal{A}$;*

(iii) *There exist $n \in N$ and a $(2+n)$-ary term p and unary terms $v_1, \ldots, v_n, w_1, \ldots, w_n$ such that the identities*

$$x = p(x, y, v_1(x), \ldots, v_n(x)),$$

$$y = p(x, y, w_1(x), \ldots, w_n(x))$$

hold in $\mathcal{V}$.

Proof. The equivalence of (i) and (ii) is proven by the Proposition. Prove (ii)⇒(iii): Set $\mathcal{A} = F_{\mathcal{V}}(x, y)$ and $\mathcal{B} = F_{\mathcal{V}}(x)$. Let $\theta = \theta(B \times B) \in Con\ \mathcal{A}$ (where B is the carrier of $\mathcal{B}$). Since $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, we have $\theta(B \times B) = \theta = \iota_A$ by (ii). However, $\mathcal{V}$ is permutable; thus $\theta(B \times B) = R(B \times B)$, the least reflexive and compatible relation on $\mathcal{A}$ containing the set $B \times B$. It follows $\iota_A = R(B \times B)$ which yields $\langle x, y\rangle \in R(B \times B)$. Hence, there exists a $(2+n)$-ary term p and elements $b_1, \ldots, b_n, b'_1, \ldots, b'_n \in B$ such that

$$x = p(x, y, b_1, \ldots, b_n) \text{ and } y = p(x, y, b'_1, \ldots, b'_n).$$

Since $b_i, b'_i \in F_{\mathcal{V}}(x)$, there are unary terms v_i, w_i with $b_i = v_i(x)$ and $b'_i = w_i(x)$, $i = 1, \ldots, n$.

For (iii)⇒(i) let $\mathcal{A} \in \mathcal{V}$ with $\mid A \mid > 1$ and suppose that $a \in A$ is an idempotent element. Let $b \in A \setminus \{a\}$. Then $v_i(a) = a = w_i(a)$ and, by (iii), $a = p(a, b, v_i(a), \ldots, v_n(a)) = p(a, b, a, \ldots, a) = p(a, b, w_1(a), \ldots, w_n(a)) = b$, a contradiction. ■

Example 1. For a variety $\mathcal{R}$ of rings with 1, we can set $n = 2$, $v_1(x) = 1 = w_2(x)$, $v_2(x) = 0 = w_1(x)$ and $p(x, y, a, b) = ax + by$. Hence, it follows that the reduct of a ring with 1, determined by the terms $0, 1, x - y + z$, and xy, generates a permutable variety with no idempotent elements.

In this section we consider only varieties $\mathcal{V}$ having a nullary operation which will be denoted by 0; it is usually called a *constant*. For $\mathcal{A} \in \mathcal{V}$, this constant will be denoted by 0_A. We do not restrict the number of nullary operations of $\mathcal{V}$ but this 0 will be considered to be fixed.

Theorem 2. *Let $\mathcal{V}$ be a variety with 0. The following conditions are equivalent:*

(i) *No $\mathcal{A} \in \mathcal{V}$ having at least two elements has 0_A as an idempotent element;*

(ii) *For each $\mathcal{A} \in \mathcal{V}$ and each $\theta \in Con\ \mathcal{A}$, $[0_a]_\theta$ is not a proper subalgebra of $\mathcal{A}$;*

(iii) *There exist $n \in N$, binary terms $q_1, \ldots, q_n$, and unary terms $v_1, \ldots, v_n$, $w_1, \ldots, w_n$ such that the identities*

$$x = q_1(x, v_1(0)),$$

$$q_1(x, w_i(0)) = q_{i+1}(x, v_{i+1}(0)), \quad i = 1, \dots n-1,$$
$$0 = q_n(x, w_n(0))$$

hold in $\mathcal{V}$.

Proof. (i)⇒(ii): Let $\mathcal{A} \in \mathcal{V}$, $\mid A \mid > 1$, $\theta \in Con\ \mathcal{A}$ and suppose $[0_A]_\theta \neq A$. Then $\mathcal{A}/\theta$ has at least two elements, and, of course, $0_{A/\theta} = [0_A]_\theta$. Since $0_{A/\theta}$ is not an idempotent of $\mathcal{A}/\theta \in \mathcal{V}$, $[0_A]_\theta$ cannot be a subalgebra of $\mathcal{A}$.

(ii)⇒(iii): Set $\mathcal{A} = \mathcal{F}_\mathcal{V}(x)$ and $\mathcal{B} = \mathcal{F}_\mathcal{V}(\emptyset)$. Let $\theta = \theta(B \times B)$ in $Con\ \mathcal{A}$. Since $0_A = 0_B \in B$, the class $[0_A]_\theta$ contains B. Hence, for every n-ary operation f of the type of $\mathcal{V}$, for every $c_1, \dots, c_n \in [0_A]_\theta$, and every $b_1, \dots, b_n \in B$ we have $\langle c_i, b_i \rangle \in \theta \quad (i = 1, \dots, n)$; thus, also $\langle f(c_1, \dots, c_n), f(b_1, \dots, b_n) \rangle \in \theta$. But $f(b_1, \dots, b_n) \in B \subseteq [0_A]_\theta$, whence $f(c_1, \dots, c_n) \in [0_A]_\theta$. Thus, $[0_A]_\theta$ is a subalgebra of $\mathcal{A}$. In account of (ii), $[0_A]_\theta = A$; thus, $\langle x, 0 \rangle \in \theta(B \times B)$. Hence, there exist $d_0, d_1, \dots, d_n \in A$ such that $d_0 = x$, $d_n = 0$ and $\langle d_{i-1}, d_i \rangle = \langle \varphi_i(b_i), \varphi_i(b_i') \rangle \quad (i = 1, \dots, n)$ for some $b_i, b_i' \in B$ and unary polynomials φ_i over $\mathcal{A}$. Thus $b_i = v_i(0)$, $b_i' = w_i(0)$ for some unary terms v_i, w_i. Of course, $\varphi_i(z) = q_i(x, z)$ for some binary terms $q_1, \dots, q_n$. The condition (iii) is evident.

(iii)⇒(i): Let $\mathcal{A} \in \mathcal{V}$, $\mid A \mid > 1$, $0_A \neq a \in A$. Suppose that 0_A is an idempotent of $\mathcal{A}$. Then $v_i(0_A) = 0_A = w_i(0_A)$ and
$a = q_1(a, v_1(0_A)) = q_1(a, 0_A) = q_1(a, w_1(0_A)) =$
$q_2(a, v_2(0_A)) = q_2(a, 0_A) = q_2(a, w_2(0_A)) = \dots = 0_A$,
a contradiction. ■

Example 2. For a variety $\mathcal{P}$ of pseudocomplemented semilattices, we set $n = 1$, $v_1(x) = x^*$, $w_1(x) = x$ and $q_1(x, y) = x \wedge y$. Then

$$q_1(x, v_1(0)) = x \wedge 0^* = x,$$

$$q_1(x, w_1(0)) = x \wedge 0 = 0.$$

Analogously as previously, we can simplify Theorem 2 for permutable varieties.

Theorem 3. *Let* $\mathcal{V}$ *be a permutable variety with* 0. *The following conditions are equivalent:*

(i) *No* $\mathcal{A} \in \mathcal{V}$ *consisting of at least two elements has* 0_A *as an idempotent element;*

(ii) *For each* $\mathcal{A} \in \mathcal{V}$ *and each* $\theta \in Con\ \mathcal{A}$, $[0_A]_\theta$ *is not a proper subalgebra of* $\mathcal{A}$*;*

(iii) *There exist* $n \in N$ *and a* $(1+n)$*-ary term* q *and unary terms* $v_1, \ldots, v_n$, $w_1, \ldots, w_n$ *such that the identities*

$$x = q(x, v_1(0), \ldots, v_n(0)),$$
$$0 = q(x, w_1(0), \ldots, w_n(0))$$

hold in $\mathcal{V}$.

Example 3. For the variety of Boolean algebras, we can set $n = 1$, $v_1(x) = x'$, $w_1(x) = x$ and $q(x, y) = x \wedge y$. Then $q(x, v_1(0)) = x \wedge 0' = x$, and $q(x, w_1(0)) = x \wedge 0 = 0$.

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