Discussiones Mathematicae
Differential Inclusions, Control and Optimization 36 (2016) 181–206
doi:10.7151/dmdico.1186

A GENERAL CLASS OF MCKEAN-VLASOV STOCHASTIC EVOLUTION EQUATIONS DRIVEN BY BROWNIAN MOTION AND LÈVY PROCESS AND CONTROLLED BY LÈVY MEASURE

N.U. Ahmed

University of Ottawa, Canada

e-mail: ahmed@site.uottawa.ca

Abstract

In this paper we consider McKean-Vlasov stochastic evolution equations on Hilbert spaces driven by Brownian motion and Lèvy process and controlled by Lèvy measures. We prove existence and uniqueness of solutions and regularity properties thereof. We consider weak topology on the space of bounded Leŕy measures on infinite dimensional Hilbert space and prove continuous dependence of solutions with respect to the Leŕy measure. Then considering a certain class of Leŕy measures on infinite as well as finite dimensional Hilbert spaces, as relaxed controls, we prove existence of optimal controls for Bolza problem and some simple mass transport problems.

Keywords: McKean-Vlasov stochastic differential equation, Hilbert spaces, existence of optimal controls.

2010 Mathematics Subject Classification: 49J27, 60H15, 93E20.

1. Introduction

A general class of stochastic evolution equations was introduced by McKean in his seminal paper [16] published in 1966 which covers, as a special class, the class of standard stochastic evolution equations well known to many workers in the field. This general class of stochastic differential equations has many interesting applications as seen in the work of Dawson [12] and Dawson and Gartner [13]. Recently this class of models have been extended to second order evolution equations [15]. We have extensively used these mathematical models in the study of optimal controls [1, 2, 3, 4, 5, 6]. In the reference [5] we considered controlled McKean-Vlasov stochastic differential equation including Poisson jump process on finite dimensional spaces and proved existence of optimal relaxed controls and necessary conditions of optimality. In a recent paper [1] we considered general McKean-Vlasov equations on Hilbert spaces with relaxed controls and proved existence of optimal controls and presented necessary conditions of optimality. In this paper we consider an infinite dimensional McKean-Vlasov stochastic evolution equation driven both by Brownian motion and Lévy measure or Poisson random measure. We prove existence and uniqueness of mild solutions and their regularity properties. We present a result on continuous dependence of solutions with respect to the Lévy measure, more precisely the intensity measure. Here we use another Lévy measure as a control measure defined on infinite as well as finite dimensional Hilbert spaces and prove existence of optimal controls. To the best of our knowledge, there is no literature we know that treats this problem with the Lévy measure as the control.

The paper is organized as follows. In Section 2, we present the mathematical model of the system followed by some mathematical framework in Section 3. In Section 4, after basic assumptions are introduced, we prove the existence and uniqueness of mild solutions and their regularity properties. In Section 5, we prove continuous dependence of solution on the Lévy measure. In Section 6, we present existence of optimal controls.

2. System model

Let X and H denote a pair of real separable Hilbert spaces and $\{\Omega, \mathcal{F}, \mathcal{F}_t, t \in I, P\}$ a complete filtered probability space with $\mathcal{F}_t \subset \mathcal{F}$ being a family of right continuous non decreasing complete sub-sigma algebras of the sigma algebra $\mathcal{F}$ and $I \equiv [0,T]$, $T < \infty$ with λ denoting the Lebesgue measure. Let $W \equiv \{W(t), t \in I\}$, denote an H-Wiener process with covariance operator $\mathcal{R}$ in the sense that for any $h \in H$, $(W(t), h)$ is a real valued Brownian motion on I with mean zero and variance $\mathbf{E}(W(t), h)^2 = t(\mathcal{R}h, h)$. If the operator $\mathcal{R} = I_H$, the identity operator in H, we say that W is a cylindrical Brownian motion or cylindrical Wiener process; and if $\mathcal{R}$ is nuclear we have the H-valued Wiener process. Let $N \equiv \{0, 1, 2, 3, \dots\}$, denote the set of natural numbers, $X_0 \equiv X \setminus \{0\}$ and $\mathcal{B}(X_0)$ the sigma algebra of Borel subsets of X_0 whose closures do not contain the point $\{0\}$, and $\mathcal{M}_B^s(X_0)$ the space of countably additive finite signed measures equipped with the topology induced by the total variation norm and $\mathcal{M}_B^+(X_0) \subset \mathcal{M}_B^s(X_0)$ the class of nonnegative countably additive finite Borel measures defined on the sigma algebra $\mathcal{B}(X_0)$. Consider the product sigma algebra $\mathcal{B}(I) \times \mathcal{B}(X_0)$ and let $p : \mathcal{B}(I) \times \mathcal{B}(X_0) \longrightarrow N$ denote the Poisson random measure, that is, for each $J \in \mathcal{B}(I)$ and $\Gamma \in \mathcal{B}(X_0)$, $p(J \times \Gamma)$ is a Poisson random variable in the sense that for any $n \in N$,

$$P\{p(J \times \Gamma) = n\} = e^{-\lambda(J)\times\Lambda(\Gamma)}((\lambda(J)\Lambda(\Gamma))^n/n! \tag{1}$$

where $\Lambda \in \mathcal{M}_B^+(X_0)$. We refer to the measure Λ as the Lévy measure since this is the jump part of the full Lévy process. The random measure p is also known as the counting measure. The expression (1) gives the probability of occurrence of exactly n jumps of intensity in the range Γ over the time interval J. Let q denote the centered Poisson random measure defined by

$$q(J \times \Gamma) \equiv p(J \times \Gamma) - E\{p(J \times \Gamma)\} = p(J \times \Gamma) - \lambda(J)\Lambda(\Gamma).$$

Clearly, the compensated measure q has mean zero and variance $\lambda(J)\Lambda(\Gamma)$. The measure q is countably additive, that is, for every sequence of disjoint measurable sets $\{J_i \times \Gamma_j\} \in \mathcal{B}(I) \times \mathcal{B}(X_0)$ we have

$$q\bigg(\bigcup_{i,j=1}^{\infty}(J_i \times \Gamma_j)\bigg) = \sum_{i,j=1}^{\infty} q(J_i \times \Gamma_j), P - a.s.$$

Considering the random process $q([0,t]\times\Gamma) \equiv q(t,\Gamma)$ it is easy to see that $q(t,\Gamma)$ is a square integrable $\mathcal{F}_t$ cadlag (right continuous with left limits) martingale. For a fixed $t \geq 0$, $q(t,\cdot)$ is a random measure defined on $\mathcal{B}(X_0)$. It is known [9, Theorem 2.15, p. 254] that corresponding to every choice of a measure $\nu \in \mathcal{M}_B^+(X_0)$ there exists a (possibly unique) compensated Poisson random measure q_ν. Later in the sequel, we have to deal with a class of compensated Poisson random measures. For this we can choose any bounded set $M_0 \subset \mathcal{M}_B^+(X_0)$.

Now we are prepared to introduce the system considered in this paper. It is governed by the following McKean-Vlasov evolution equation on the Hilbert space X driven by the H-Brownian motion W and the centered Poisson random measure q_Λ with the intensity measure $\Lambda \in \mathcal{M}_B^+(X_0)$:

$$\begin{aligned} dx &= Axdt + f(t,x,\mu)dt + \sigma(t,x,\mu)dW \\ &\quad + \int_{X\setminus\{0\}} g(t,x,\mu,v)q_\Lambda(dt \times dv), \quad x(0) = x_0, \\ \text{and} \qquad \mu(t) &= \mathcal{P}(x(t)),\ t \in I \equiv [0,T], \end{aligned} \tag{2}$$

where A is the infinitesimal generator a C_0-semigroup $S(t)$, $t \in I$, on X and f is a Borel measurable map from $I\times X\times\mathcal{M}_1(X)$ to X and σ is also a Borel measurable map from $I \times X \times \mathcal{M}_1(X)$ to $\mathcal{L}(H,X)$, the space of bounded linear operators from H to X, and g is also a Borel measurable map from $I \times X \times \mathcal{M}_1(X) \times X$ to X and q_Λ is the centered Poisson random measure with the intensity measure $\Lambda \in \mathcal{M}_B^+(X_0)$ and x_0 is the initial state. In reference to the Borel measurability mentioned above, we note that the (separable) Hilbert space X is equipped the natural sigma algebra generated by closed or open sets; the space of bounded linear operators $\mathcal{L}(H, X)$ may be assumed to be equipped with the strong operator topology and the corresponding Borel algebra generated by closed or open sets. In the case of the space of probability measures $\mathcal{M}_1(X)$, on the Hilbert space X, one may assume the sigma algebra generated by the Prohorov metric (which induces a topology equivalent to the usual weak topology). As usual we assume that the random elements $\{x_0, W, q_\Lambda\}$ are stochastically mutually independent.

We have denoted the probability law of any stochastic process $\{\zeta(t), t \geq 0\}$ by $\mathcal{P}(\zeta(t)), t \geq 0$. The drift f, the Lévy kernel g and the diffusion σ are not only dependent on the current state $x(t)$ but also its probability law $\mu(t) \equiv \mathcal{P}(x(t))$, the measure induced by the X-valued random variable $x(t)$. In case both X and H are finite dimensional, this class of models arise naturally in finance where the objective functional is of mean-variance type maximizing terminal wealth while minimizing variance. Also such models are known to arise in biological sciences, in particular, population process.

3. Mathematical framework

Let $\mathcal{B}(X)$ denote the Borel σ-algebra generated by closed (or open) subsets of the Hilbert space X and $\mathcal{M}_1(X)$ is the space of probability measures on $\mathcal{B}(X)$ carrying the usual topology of weak convergence. Let $C(X)$ denote the space of continuous functions on X. We use the notation $(\mu, \varphi) \equiv \mu(\varphi) \equiv \int_X \varphi(x)\mu(dx)$ whenever this integral makes sense. Throughout this paper we let γ denote the continuous function $\gamma(x) \equiv 1 + |x|$, $x \in X$, and introduce the Banach space

$$C_\rho(X) \quad = \quad \Big\{\varphi \in C(X) : ||\varphi||_{C_\rho(X)} \equiv \sup_{x \in X} \frac{|\varphi(x)|}{\gamma^2(x)} + \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|} < \infty\Big\}.$$

For $p \geq 1$, let $\mathcal{M}^s_{\gamma^p}(X)$ denote the Banach space of signed measures m on X satisfying $||m||_{\gamma^p} \equiv \big(\int_X \gamma^p(x)|m|(dx)\big)^{1/p} < \infty$, where $|m| = m^+ + m^-$ denotes the total variation of the signed measure m, with $m = m^+ - m^-$ being the Jordan decomposition of m. Let $\mathcal{M}_{\gamma^2}(X) = \mathcal{M}^s_{\gamma^2}(X) \cap \mathcal{M}_1(X)$ denote the class of probability measures possessing second moments. We put on $\mathcal{M}_{\gamma^2}(X)$ a topology induced by the following metric:

$$\rho(\mu, \nu) = \sup\big\{(\mu - \nu)(\varphi) \equiv (\varphi, \mu - \nu) : \varphi \in C_\rho(X) \;\text{ and }\; ||\varphi||_{C_\rho(X)} \leq 1\big\}.$$

Then $(\mathcal{M}_{\gamma^2}(X), \rho) \equiv \mathcal{M}_{2,\rho}(X)$ forms a complete metric space. Note that this is a closed bounded subset of the closed unit ball of the linear metric space $\mathcal{M}^s_{2,\rho}(X) \equiv (\mathcal{M}^s_{\gamma^2}, \rho)$. Define $I \equiv [0, T]$ with $T < \infty$. We denote by $B_\infty(I, \mathcal{M}_{2,\rho}(X))$ the complete metric space of bounded (measurable) functions from I to $\mathcal{M}_{2,\rho}(X)$ with the metric topology:

$$D(\mu,\nu) = \sup\left\{\rho(\mu(t),\nu(t)),\ t \in I\right\}$$

for any $\mu,\nu \in B_\infty(I,\mathcal{M}_{2,\rho}(X))$. By measurability here we mean that, for any $\varphi \in C_\rho(X)$, the function $t \longrightarrow \mu(t)(\varphi) \equiv \int_X \varphi(\xi)\mu(t)(d\xi)$ is Borel measurable. From now on all stochastic processes considered in this paper are assumed to be based on the complete filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_{t\geq 0},P)$ with $\mathcal{F}_T \subseteq \mathcal{F}$. For convenience of notation we denote the space $L_2((\Omega,\mathcal{F},P),X)$ by $L_2(\Omega,X)$ and let $B_\infty(I,L_2(\Omega,X))$ denote the Banach space of $\mathcal{F}$-measurable random processes defined on I and taking values from $L_2(\Omega,X)$ satisfying the condition $\sup_{t\in I} E|x(t)|_X^2 < \infty$. Let $\Xi \equiv B_\infty^a(I,X)$ denote the closed subspace of $B_\infty(I,L_2(\Omega,X))$ consisting of $\mathcal{F}_t$-adapted (progressively measurable) X-valued random processes $\{x = \{x(t) : t \in I \equiv [0,T]\}\}$ which is furnished with the norm topology, $|x|_\Xi = (\sup_{t\in I} E|x(t)|^2)^{1/2}$. Clearly Ξ is a Banach space with respect to this norm topology. We denote by $L_2^{\mathcal{F}_T}(\Omega,X)$ the space of $\mathcal{F}_T$ measurable X valued random variables having finite second moments. Similarly, we use $L_2^{\mathcal{F}}(I,X) \equiv L_2^{\mathcal{F}}(I\times\Omega,X)$ to denote the Banach space of $\mathcal{F}_t$-adapted X-valued norm-square integrable random processes defined on I. Let $\mathcal{L}_\mathcal{R}(H,X)$ denote the completion of the space of linear operators from H to X with respect to the inner product $< K,L >\equiv Tr(K\mathcal{R}L^*)$ and norm $|K|_\mathcal{R} \equiv \sqrt{Tr(K\mathcal{R}K^*)}$. Clearly this is a Hilbert space. In the sequel we also need the Hilbert space $L_2^{\mathcal{F}}(I,\mathcal{L}_\mathcal{R}(H,X))$ which consists of $\mathcal{F}_t$-adapted $\mathcal{L}_\mathcal{R}(H,X)$ valued random processes having finite square integrable norms in the sense that for any $K \in L_2^{\mathcal{F}}(I,\mathcal{L}_\mathcal{R}(H,X))$ we have $\mathbf{E}\int_I |K|_\mathcal{R}^2 dt < \infty$.

4. Basic assumptions and existence of solutions

Now we are prepared to introduce the basic assumptions. In order to study control problems involving the system (1) we must now state the basic properties of the drift and the diffusion operators $\{f,\sigma\}$ including the semigroup generator.

Basic Assumptions

(A1): The operator A is the infinitesimal generator of a C_0-semigroup $S(t), t \geq 0$, on the Hilbert space X satisfying

$$\sup\left\{\parallel S(t) \parallel_{\mathcal{L}(X)},\ t \in I\right\} \leq M < \infty.$$

(A2): The function $f : I\times X\times\mathcal{M}_1(X) \longrightarrow X$ is measurable in the first argument and continuous with respect to the rest of the arguments. Further, there exists a constant $K \neq 0$ such that

$$|f(t,x,\mu)|_X^2 \leq K^2\{1 + |x|_X^2 + |\mu|_{\mathcal{M}_{\gamma^2}}^2\},\ \forall\ x \in X,\ \mu \in \mathcal{M}_{2,\rho}(X)$$
$$|f(t,x_1,\mu_1) - f(t,x_2,\mu_2)|_X^2 \leq K^2\{|x_1 - x_2|_X^2 + \rho^2(\mu_1,\mu_2)\},$$

for all $x_1, x_2 \in X, \mu_1, \mu_2 \in \mathcal{M}_{2,\rho}(X)$ uniformly with respect to $t \in I$.

(A3): The incremental covariance of the Brownian motion W is denoted by $\mathcal{R} \in \mathcal{L}_s^+(H)$ (symmetric, positive). The diffusion coefficient is an operator valued function $\sigma : I \times X \times \mathcal{M}_1(X) \longrightarrow \mathcal{L}(H,X)$ which is Borel measurable in the first argument and continuous with respect to the rest of the variables and there exists a constant $K_{\mathcal{R}} \neq 0$ such that

$$|\sigma(t,x,\mu)|_{\mathcal{R}}^2 \leq K_{\mathcal{R}}^2\{1 + |x|_X^2 + |\mu|_{\mathcal{M}_{\gamma^2}}^2\},\ \forall\ x \in X, \mu \in \mathcal{M}_{\gamma^2}$$
$$|\sigma(t,x_1,\mu_1) - \sigma(t,x_2,\mu_2)|_{\mathcal{R}}^2 \leq K_{\mathcal{R}}^2\{|x_1 - x_2|_X^2 + \rho^2(\mu_1,\mu_2)\}$$

for all $x_1, x_2 \in X$ and $\mu_1, \mu_2 \in \mathcal{M}_{2,\rho}(X)$ uniformly with respect to $t \in I$, where $|\sigma|_{\mathcal{R}}^2 = tr(\sigma\mathcal{R}\sigma^*)$.

(A4): For any given Lévy measure $\Lambda \in \mathcal{M}_B^+(X_0)$, the Lévy kernel

$$g : I \times X \times \mathcal{M}_{\gamma^2}(X) \times X \longrightarrow X$$

is measurable in the first argument and continuous with respect to the rest of the variables and there exists a constant $K_\Lambda \neq 0$ such that

$$\int_{X_0} |g(t,x,\mu,v)|_X^2\ \Lambda(dv) \leq K_\Lambda^2\{1 + |x|_X^2 + |\mu|_{\mathcal{M}_{\gamma^2}}^2\},\ \forall\ x \in X, \mu \in \mathcal{M}_{\gamma^2}$$
$$\int_{X_0} |g(t,x_1,\mu_1,v) - \sigma(t,x_2,\mu_2,v)|_X^2\ \Lambda(dv) \leq K_\Lambda^2\{|x_1 - x_2|_X^2 + \rho^2(\mu_1,\mu_2)\}$$

for all $x_1, x_2 \in X$ and $\mu_1, \mu_2 \in M_{2,\rho}(X)$ uniformly with respect to $t \in I$.

To prove the existence of solution of the stochastic evolution equation (1) we will need certain intermediate results. First, we fix a $\nu \in B_\infty(I, \mathcal{M}_{2,\rho}(X))$ and consider the following system

$$\begin{aligned} dx &= Axdt + f(t,x,\nu)dt + \sigma(t,x,\nu)dW \\ &+ \int_{X_0} g(t,x,\nu,v)q_\Lambda(dt \times dv),\ x(0) = x_0,\ t \in I \equiv [0,T]. \end{aligned} \tag{3}$$

We prove that this equation has a unique mild solution $x \in \Xi \equiv B_\infty^a(I,X)$.

Lemma 4.1. *Consider the system* (3) *and let* q_Λ *denote the compensated Poisson random measure corresponding to the Lévy measure* $\Lambda \in \mathcal{M}_B^+(X_0)$ *and* $W \equiv \{W(t), t \geq 0\}$ *an* H*-Brownian motion with incremental covariance* (*operator*) $\mathcal{R} \in \mathcal{L}_1^+(H)$, *and suppose the assumptions* (A1)–(A4) *hold. Then, for every* $\mathcal{F}_0$ *measurable* X *valued random variable* $x_0 \in L_2^{\mathcal{F}_0}(\Omega, X)$, *and* $\nu \in B_\infty(I, \mathcal{M}_{2,\rho}(X))$, *the stochastic evolution equation* (3) *has a unique mild solution* $x^* = x_\nu \in \Xi$ *in the sense that it satisfies the following stochastic integral equation:*

$$
\begin{aligned}
x_\nu(t) \equiv S(t)x_0 &+ \int_0^t S(t-\tau) f(\tau, x_\nu(\tau), \nu(\tau)) d\tau \\
&+ \int_0^t S(t-\tau)\sigma(\tau, x_\nu(\tau), \nu(\tau)) dW(\tau) \\
&+ \int_0^t \int_{X_0} S(t-\tau) g(\tau, x_\nu(\tau), \nu(\tau) q_\Lambda(d\tau \times dv) \quad t \in I.
\end{aligned} \tag{4}
$$

Further the solution has no discontinuities of the second kind.

Proof. First we show that for every given $\nu \in B_\infty(I, \mathcal{M}_{2,\rho}(X))$, the solution of the integral equation (4), if one exists, has an a-priori bound. Clearly, for the given $\nu \in B_\infty(I, \mathcal{M}_{2,\rho}(X))$, there exists a finite positive number b such that $\parallel \nu \parallel_{B_\infty(I, M_{2,\rho})(X)} \equiv \sup\{\parallel \nu(t) \parallel_{\gamma^2}, t \in I\} \leq b$. Then using equation (4) and computing the expected value of the square of the norm of $x_\nu(t)$ one can easily obtain the following inequality,

$$
\mathbf{E}|x_\nu(t)|_X^2 \leq C_1 + C_2 \int_0^t \mathbf{E}|x_\nu(s)|_X^2 ds, \tag{5}
$$

where

$$
\begin{aligned}
C_1 &\equiv 8M^2 \left\{ \mathbf{E}|x_0|_X^2 + (TK^2 + K_R^2 + K_\Lambda^2) \int_0^T (1 + |\nu(s)|_{\gamma^2}^2) ds \right\} \\
C_2 &\equiv 8M^2 (TK^2 + K_R^2 + K_\Lambda^2).
\end{aligned}
$$

It follows from Gronwall inequality applied to (5) that

$$
\sup \left\{ \mathbf{E}|x_\nu(t)|_X^2, \ t \in I \right\} \leq C_1 \exp\{C_2 T\}. \tag{6}
$$

This shows that if the integral equation (4) has a solution x_ν it must belong to $\Xi \equiv B_\infty^a(I, X)$. Next we show that under the assumptions (A1)–(A4), the integral equation has a unique solution $x_\nu \in \Xi$. For the fixed ν, define the operator F_ν by

$$
\begin{aligned}
(F_\nu x)(t) \equiv S(t)x_0 &+ \int_0^t S(t-\tau)f(\tau, x(\tau), \nu(\tau))d\tau \\
&+ \int_0^t S(t-\tau)\sigma(\tau, x(\tau), \nu(\tau))dW(\tau) \\
&+ \int_0^t \int_{X_0} S(t-\tau)g(\tau, x(\tau), \nu(\tau), v)q_\Lambda(d\tau \times dv) \ t \in I.
\end{aligned} \tag{7}
$$

It is clear from the expression (7) that $F_\nu x$ is $\mathcal{F}_t$-adapted whenever x is. Thus it follows from the a priori bound (6) that for any $x \in \Xi$, $F_\nu x \in \Xi$. We prove that it has a unique fixed point in Ξ. For any pair of $x, y \in \Xi$, it follows from the Lipschitz properties of f, σ, g given in the assumptions (A2)–(A4) that

$$
\mathbf{E}|(F_\nu x)(t) - (F_\nu y)(t)|_X^2 \leq \beta \int_0^t \mathbf{E}|x(s) - y(s)|_X^2 ds, \ t \in I \equiv [0, T], \tag{8}
$$

where

$$
\beta = 4M^2 \left\{ K^2 T + K_{\mathcal{R}}^2 + K_\Lambda^2 \right\}.
$$

Since this inequality holds for any $x, y \in \Xi$, and we know that $F_\nu x, F_\nu y \in \Xi$, it follows from (8) that

$$
\mathbf{E}|(F_\nu^2 x)(t) - (F_\nu^2 y)(t)|_X^2 \leq \beta \int_0^t \mathbf{E}|(F_\nu x)(s) - (F_\nu y)(s)|_X^2 ds, \ t \in I \equiv [0, T], \tag{9}
$$

where $F_\nu^2 \equiv F_\nu \circ F_\nu$ is the 2-fold composition of the operator F_ν. Substituting the inequality (8) into the inequality (9) and interchanging the order of integration it follows from Fubini's theorem that

$$
\mathbf{E}|(F_\nu^2 x)(t) - (F_\nu^2 y)(t)|_X^2 \leq \beta^2 \int_0^t (t-s)\mathbf{E}|x(s) - y(s)|_X^2 ds, \ t \in I \equiv [0, T]. \tag{10}
$$

Iterating this process n times, the reader can easily verify that

$$
\mathbf{E}|(F_\nu^n x)(t) - (F_\nu^n y)(t)|_X^2 \leq \beta^n \int_0^t \big((t-s)^{n-1}/(n-1)!\big)\mathbf{E}|x(s) - y(s)|_X^2 ds, \tag{11}
$$

for all $t \in I$. Finally, from this inequality we obtain

$$
\sup_{t \in I} \left\{ \mathbf{E}|(F_\nu^n x)(t) - (F_\nu^n y)(t)|_X^2 \right\} \leq \big((\beta T)^n/n!\big) \sup_{t \in I} \mathbf{E}|x(t) - y(t)|_X^2. \tag{12}
$$

In other words, we have arrived at the following inequality

$$
\parallel F_\nu^n x - F_\nu^n y \parallel_\Xi \leq \alpha_n \parallel x - y \parallel_\Xi, \tag{13}
$$

where $\alpha_n \equiv \sqrt{(\beta T)^n/n!)}$. For n sufficiently large, $\alpha_n < 1$ and therefore F_ν^n is a contraction on the Banach space Ξ and it follows from Banach fixed point theorem that F_ν^n has a unique fixed point $x^* \in \Xi$. That is, $F_\nu^n x^* = x^*$. Hence

$$\| F_\nu x^* - x^* \|_\Xi = \| F_\nu F_\nu^n x^* - F_\nu^n x^* \|_\Xi \leq \alpha_n \| F_\nu x^* - x^* \|_\Xi . \tag{14}$$

This implies $(1-\alpha_n) \| F_\nu x^* - x^* \|_\Xi \leq 0$. Since $0 < \alpha_n < 1$, this inequality holds if and only $F_\nu x^* = x^*$. Thus x^* is also the unique fixed point of F_ν. We denote this solution by $x_\nu \equiv x^*$. This proves the existence of a unique solution of the integral equation (4) which, by definition, is the mild solution of equation (3). Since the Levý process p_ν (and hence q_ν) has discontinuities only of the first kind, the solution x_ν can not have discontinuities of the second kind. This completes the proof. ■

Now we are prepared to consider the question of existence of solution of the McKean-Vlasov jump evolution equation (2). By a solution of this equation, we mean the mild solution given by the solution of the following integral equation

$$\begin{aligned} x(t) = S(t)x_0 &+ \int_0^t S(t-s)f(s,x(s),\mu(s))ds \\ &+ \int_0^t S(t-s)\sigma(s,x(s),\mu(s))dW(s) \\ &+ \int_0^t \int_{X_0} S(t-s)g(s,x(s),\mu(s),v)q_\Lambda(ds\times dv), t \in I, \end{aligned} \tag{15}$$

with $\mu(t) = \mathcal{P}(x(t))$, $t \in I$.

Theorem 4.2. *Consider the system* (2) *and suppose the assumptions of Lemma* 4.1 *hold. Then the system* (2) *has a unique mild solution $x \in \Xi$ satisfying the integral equation* (15) *with probability law $\mu \in B_\infty(I, \mathcal{M}_{2,\rho}(X))$ such that $\mathcal{P}(x(t)) = \mu(t)$ for all $t \in I$.*

Proof. For any given $\nu \in B_\infty(I, \mathcal{M}_{2,\rho}(X))$, consider the evolution equation (3). By Lemma (4.1), we know that it has a unique mild solution $x_\nu \in \Xi$. Define the operator $\Phi : B_\infty(I, \mathcal{M}_{2,\rho}(X)) \longrightarrow B_\infty(I, \mathcal{M}_{2,\rho}(X))$ taking values

$$\Phi(\nu)(t) \equiv \mathcal{P}(x_\nu(t)),\ t \in I.$$

Clearly, if the operator Φ has a fixed point in $B_\infty(I, \mathcal{M}_{2,\rho}(X))$, that is $\Phi(\mu) = \mu$, then equation (2) has a unique mild solution and conversely, if equation (2) has a mild solution $x \in \Xi$, then $\mathcal{P}(x(t)) = \mu(t)$, $t \in I$, and μ is the fixed point of the operator Φ. Thus it suffices to prove that Φ has a unique fixed point $B_\infty(I, \mathcal{M}_{2,\rho}(X))$. For any fixed but arbitrary $\mathcal{F}_0$-measurable initial condition $x_0 \in L_2(\Omega, X)$, consider the evolution equation (3) corresponding to $\nu = \varrho$ and $\nu = \vartheta$

separately where $\varrho, \vartheta \in B_\infty(I, \mathcal{M}_{2,\rho}(X))$. By Lemma 4.1, equation (3) has unique mild solutions $x_\varrho, x_\vartheta \in \Xi$ corresponding to ϱ and ϑ, respectively. Clearly, these are solutions of the following integral equations

$$
\begin{aligned}
x_\varrho(t) \equiv S(t)x_0 &+ \int_0^t S(t-\tau)f(\tau, x_\varrho(\tau), \varrho(\tau))d\tau \\
&+ \int_0^t S(t-\tau)\sigma(\tau, x_\varrho(\tau), \varrho(\tau))dW(\tau) \\
&+ \int_0^t \int_{X_0} S(t-\tau)g(\tau, x_\varrho, \varrho(\tau), v)q_\Lambda(d\tau \times dv) \quad t \in I.
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
x_\vartheta(t) \equiv S(t)x_0 &+ \textstyle\int_0^t S(t-\tau)f(\tau, x_\vartheta(\tau), \vartheta(\tau))d\tau \\
&+ \int_0^t S(t-\tau)\sigma(\tau, x_\vartheta(\tau), \vartheta(\tau))dW(\tau) \\
&+ \int_0^t \int_{X_0} S(t-\tau)g(\tau, x_\vartheta, \vartheta(\tau), v)q_\Lambda(d\tau \times dv) \quad t \in I.
\end{aligned} \tag{17}
$$

Subtracting equation (17) from equation (16) and following similar steps as in the proof of Lemma 4.1, the reader can easily verify that for each $t \in I$,

$$
\mathbf{E}|x_\varrho(t) - x_\vartheta(t)|_X^2 \le L\bigg(\int_0^t \{\mathbf{E}|x_\varrho(s) - x_\vartheta(s)|_X^2 + \rho^2(\varrho(s), \vartheta(s))\}ds\bigg), \tag{18}
$$

where

$$
L \equiv 4M^2\left(K^2T + K_R^2 + K_\Lambda^2\right).
$$

Clearly, it follows from the above inequality that, for any $\tau \in I \equiv [0, T)$,

$$
\begin{aligned}
&\sup_{0\le t\le\tau} \mathbf{E}|x_\varrho(t) - x_\vartheta(t)|_X^2 \\
&\le (L\tau)\bigg\{\sup_{0\le t\le\tau} \mathbf{E}|x_\varrho(t) - x_\vartheta(t)|_X^2 + \sup_{0\le t\le\tau} \rho^2(\varrho(t).\vartheta(t))\bigg\}.
\end{aligned} \tag{19}
$$

Using the inequality (19) and choosing $\tau = t_1 \in (0, T)$, sufficiently small, so that $Lt_1 < (1/3)$, and allowing possible jump at the point t_1, we arrive at the following inequality

$$
\sup_{0\le t\le t_1+} \mathbf{E}|x_\varrho(t) - x_\vartheta(t)|_X^2 \le (1/2) \sup_{0\le t\le t_1+} \rho^2(\varrho(t), \vartheta(t)). \tag{20}
$$

Recall that by definition of the operator Φ, we have $(\Phi\varrho)(t) = \mathcal{P}(x_\varrho(t))$ and $(\Phi\vartheta)(t) = \mathcal{P}(x_\vartheta(t))$ for $t \in I$. Computing the distance between the two measures

$(\Phi\varrho)(t)$ and $(\Phi\vartheta)(t)$, it follows from the definition of the metric ρ that

$$
\begin{aligned}
\rho((\Phi\varrho)(t),(\Phi\vartheta)(t)) &= \sup\big\{< \varphi, (\Phi\varrho)(t)-(\Phi\vartheta)(t) >: \varphi \in C_\rho, \parallel \varphi \parallel_{C_\rho} \leq 1\big\} \\
&= \sup\big\{\mathbf{E}[\varphi(x_\varrho(t))-\varphi(x_\vartheta(t))] : \varphi \in C_\rho, \parallel \varphi \parallel_{C_\rho} \leq 1\big\} \qquad (21)\\
&\leq \mathbf{E}|x_\varrho(t)-x_\vartheta(t)|_X,\ \forall\ t \in I.
\end{aligned}
$$

Clearly, it follows from the above inequalities that

$$
\sup_{0\leq t\leq t_1+} \rho^2((\Phi\varrho)(t),(\Phi\vartheta)(t)) \leq \sup_{0\leq t\leq t_1+} \mathbf{E}|x_\varrho(t)-x_\vartheta(t)|_X^2. \qquad (22)
$$

Using the inequalities (20) and (22) we obtain

$$
\sup_{0\leq t\leq t_1+} \rho^2((\Phi\varrho)(t),(\Phi\vartheta)(t)) \leq (1/2) \sup_{0\leq t\leq t_1+} \rho^2(\varrho(t),\vartheta(t)),
$$

from which we arrive at the following inequality

$$
\sup_{0\leq t\leq t_1+} \rho((\Phi\varrho)(t),(\Phi\vartheta)(t)) \leq (1/\sqrt{2}) \sup_{0\leq t\leq t_1+} \rho(\varrho(t),\vartheta(t)). \qquad (23)
$$

This shows that Φ is a contraction on the restriction $B_\infty([0,t_1],\mathcal{M}_{2,\rho}(X))$ of the metric space $B_\infty([0,T],\mathcal{M}_{2,\rho}(X))$ and hence by Banach fixed point theorem, it has a unique fixed point, say, $\mu^1 \in B_\infty([0,t_1],\mathcal{M}_{2,\rho}(X))$, that is, $(\Phi\mu^1)(t) = \mu^1(t)$, $t \in [0,t_1]$ with $x^1(t), t \in [0,t_1]$, being the corresponding trajectory in $B_\infty^a([0,t_1],X)$. Next, starting with the initial state $x(t_1) = x^1(t_1+)$ and choosing $t_2 \in (t_1,T]$ such that $L(t_2-t_1) \leq (1/3)$ and carrying out similar computations, we arrive at the following inequality,

$$
\sup_{t_1\leq t\leq t_2+} \rho((\Phi\varrho)(t),(\Phi\vartheta)(t)) \leq (1/\sqrt{2}) \sup_{t_1\leq t\leq t_2+} \rho(\varrho(t),\vartheta(t)). \qquad (24)
$$

Thus Φ, restricted to the metric space $B_\infty([t_1,t_2],\mathcal{M}_{2,\rho}(X))$, is again a contraction and hence it has a unique fixed point $\mu^2 \in B_\infty([t_1,t_2],\mathcal{M}_{2,\rho}(X))$, with the associated path process $x^2 \in B_\infty([t_1,t_2],X)$, giving $\Phi\mu^2 = \mu^2$ for $t \in [t_1,t_2]$ with $\mu^2(t_1) = \mu^1(t_1+)$ and $x^2(t_1) = x^1(t_1+) \in L_2^{\mathcal{F}_{t_1}}(\Omega,X)$ (because $\mathcal{F}_t$ is right continuous). Again, choosing $t_3 \in (t_2,T]$ so that $L(t_3-t_2) \leq (1/3)$, and initial state $x(t_2) = x^2(t_2+)$ and $\mu(t_2) = \mu^2(t_2+)$ and carrying out similar computations one obtains the following inequality

$$
\sup_{t_2\leq t\leq t_3+} \rho((\Phi\varrho)(t),(\Phi\vartheta)(t)) \leq (1/\sqrt{2}) \sup_{t_2\leq t\leq t_3+} \rho(\varrho(t),\vartheta(t))
$$

which is similar to the inequality (24). Since T is finite, continuing this process step by step, we can cover the whole interval $I \equiv [0,T]$ in a finite number of steps (smallest integer $n \geq 3LT$) and by concatenation of the above solutions we obtain a unique measure valued function $\mu \in B_\infty(I, \mathcal{M}_{2,\rho}(X))$ along with a path process $x \in \Xi = B_\infty^a(I, X)$ such that $\mu(t) = \mathcal{P}(x(t)), t \in I$. This proves that Φ has a unique fixed point in $B_\infty(I, \mathcal{M}_{2,\rho}(X))$. Hence we conclude that the McKean-Vlasov evolution equation (2) has a unique mild solution $x \in \Xi$ with probability law $\mu \in B_\infty(I, \mathcal{M}_{2,\rho}(X))$. This completes the proof. ■

Remark 4.3. Note that the system (2) is driven by the compensated Poisson random measure q_Λ corresponding to a fixed Lévy measure $\Lambda \in \mathcal{M}_B^+(X_0)$. We want to consider a family of Lévy measures $M_0 \subset \mathcal{M}_B^+(X_0)$.

Corollary 4.4. *Let M_0 be a closed bounded subset of $\mathcal{M}_B^+(X_0)$ and consider the family of evolution equations* (2) *with the compensated Lévy measure q_Λ for $\Lambda \in M_0$. Suppose the assumptions of Theorem* 4.2 *hold with the assumption* (*A*4) *replaced by the following assumption:*

(A4)* *The Lévy kernel $g : I \times X \times \mathcal{M}_{\gamma^2} \times X \longrightarrow X$ is measurable in the first argument and continuous with respect to the rest of the variables and there exists a finite number $K_0 \neq 0$, independent of $\{x, y, \mu, \nu\}$, such that*

$$\sup_{\Lambda \in M_0} \int_{X_0} |g(t,x,\mu,v)|_X^2 \ \Lambda(dv) \leq K_0^2 \left\{1 + |x|_X^2 + |\mu|_{\mathcal{M}_{\gamma^2}}^2 \right\},$$
$$\sup_{\Lambda \in M_0} \int_{X_0} |g(t,x,\mu,v) - g(t,y,\nu,v)|_X^2 \ \Lambda(dv) \leq K_0^2 \left\{|x-y|_X^2 + \rho^2(\mu,\nu)\right\}$$

Then the solution set $\mathcal{S} \equiv \{x(\Lambda), \Lambda \in M_0\}$ is a bounded subset of Ξ and the corresponding set of measure valued functions lies in a bounded subset of $B_\infty(I, \mathcal{M}_{2,\rho}(X))$.

Proof. Using equation (15) and taking the expectation of the norm-square one arrives at a similar inequality as given by the expression (5) with minor modification as follows:

$$\mathbf{E}|x(\Lambda)(t)|_X^2 \leq \tilde{C}_1 + \tilde{C}_2 \int_0^t \mathbf{E}|x(\Lambda)(s)|_X^2 ds, \tag{25}$$

where

$$\tilde{C}_1 \equiv 8M^2\big\{\mathbf{E}|x_0|_X^2 + 2T(TK^2 + K_R^2 + K_0^2)\big\}, \quad \tilde{C}_2 \equiv 24M^2(TK^2 + K_R^2 + K_0^2).$$

The inequality (25) holds uniformly with respect to $\Lambda \in M_0$. Clearly, it follows from Gronwall inequality applied to (25) that

$$\sup\big\{\| \ x(\Lambda) \ \|_{B_\infty^a(I,X)}, \Lambda \in M_0\big\} \leq \sqrt{\tilde{C}_1} \exp(1/2)\tilde{C}_2 T \equiv b < \infty.$$

Thus the solution set $\mathcal{S}$ is a bounded subset of Ξ. Using the definition of the metric space $\mathcal{M}_{\gamma^2}(X)$ one can verify that

$$\begin{aligned}&\sup\{\parallel \mu(\Lambda)(t) \parallel^2_{\mathcal{M}_{\gamma^2}(X)},\ t \in I,\ \Lambda \in M_0\}\\ &\leq 2\big(1 + \sup\{\mathbf{E}|x(\Lambda)(t)|^2_X,\ t \in I,\ \Lambda \in M_0\}\big) \leq 2(1 + b^2) < \infty.\end{aligned}$$

This completes the proof. ■

5. Continuous dependence of solutions on Lev́y measure

In this section we wish to study the question of continuous dependence of solutions on the Poisson random measure. Consider the product sigma algebra $\mathcal{B}(I) \times \mathcal{B}(X_0)$ where $\mathcal{B}(X_0)$ denotes the class of Borel subsets of X whose closures do not contain the element $\{0\}$. Let $\mathcal{M}^+_B(X_0)$ denote the class of finite countably additive positive Borel measures on $\mathcal{B}(X_0)$ and M_0 a bounded subset of $\mathcal{M}^+_B(X_0)$. It is well known [9, Theorem 2.15, p. 254] that, given any $\Lambda \in \mathcal{M}^+_B(X_0)$, there exists a probability space on which there exists a Poisson random measure with mean measure $\lambda \times \Lambda$ where λ denotes the Lebesgue measure. In view of this result we may assume that for all $\Lambda \in M_0$, there exists a common (and complete filtered) probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$, so that for every measure $\Lambda \in M_0$, there exists a Poisson random measure p_Λ defined on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$ with mean measure $\lambda \times \Lambda$. The corresponding compensated Poisson random measure is denoted by $q_\Lambda \equiv p_\Lambda - \lambda \times \Lambda$. Consider the set $E \equiv I \times X_0$ with the product sigma algebra $\mathcal{E} \equiv \mathcal{B}(I) \times \mathcal{B}(X_0)$ and the corresponding measurable space $(E, \mathcal{E})$. On this measurable space we shall consider a whole family of measures like $\{\lambda \times \Lambda, \Lambda \in M_0\}$ with λ being the Lebesgue measure (fixed) and Λ the Lévy measure. In the literature, the measure $\lambda \times \Lambda$ is called Lev́y measure. But since λ is fixed we prefer to call Λ the Le'vy measure. Define the sets

$$P_0 \equiv \{p_\Lambda, \Lambda \in M_0\} \text{ and } Q_0 \equiv \{q_\Lambda = p_\Lambda - \lambda \times \Lambda : \Lambda \in M_0\},$$

where P_0 denotes the class of Poisson random measures and Q_0 the corresponding set of compensated Poisson random measures.

We are interested in the question of continuous dependence of solution of the integral equation (15) with respect to the Lev́y measure Λ on M_0.

Theorem 5.1. *Consider the integral equation* (15) *driven by the compensated Poisson random measure q_Λ for any $\Lambda \in M_0$ and suppose the assumptions of Theorem* 4.2, *including* (A1)–(A3) *and* (A4)* *hold. Then the solution $x : M_0 \longrightarrow B^a_\infty(I, X)$ is continuous in the sense that whenever $\Lambda_n \xrightarrow{w} \Lambda_o$ in M_0, $x^n \xrightarrow{s} x^o$ in $\Xi \equiv B^a_\infty(I, X)$.*

Proof. Let $\{x^n, x^o\} \in \Xi \equiv B^a_\infty(I, X)$ denote the solutions of the integral equation (15) corresponding to the sequence of Leẃy measures $\{\Lambda_n, \Lambda_o\}$ and suppose $\Lambda_n \xrightarrow{w} \Lambda_o$. We show that $x^n \xrightarrow{s} x^o$ in Ξ. Since Λ_n converges weakly to Λ_o, it is easy to verify that Λ_n converges to Λ_o set wise on all continuity sets, that is, for all sets $D \in \mathcal{B}(X_0)$ for which $\Lambda_o(\partial D) = 0$. Let $\{q_n, q_o\} \in Q_0$ be the corresponding sequence of compensated Poisson random measures. Then the convergence of Λ_n to Λ_o on continuity sets implies that q_n converges to q_o set wise in probability which is stated as $q_n \xrightarrow{p} q_o$ set wise. Clearly, there exists a subsequence of the sequence $\{q_n\}$, relabeled as the original sequence, such that q_n converges to q_o set wise P-a.s. We show that the corresponding subsequence of solutions of the integral equation (15) converges strongly to $x^o \in \Xi \equiv B^a_\infty(I, X)$. It follows from Theorem 4.2 that $\{x^n, x^o\} \in \Xi$ are the unique solutions satisfying the following integral equations,

$$\begin{aligned} x^n(t) = S(t)x_0 &+ \int_0^t S(t-s)f(s, x^n(s), \mu^n(s))ds \\ &+ \int_0^t S(t-s)\sigma(s, x^n(s), \mu^n(s))dW(s) \\ &+ \int_0^t \int_{X_0} S(t-s)g(s, x^n(s), \mu^n(s), v)q_n(ds \times dv), \ t \in I, \end{aligned} \tag{26}$$

with $\mu^n(t) = \mathcal{P}(x^n(t))$, $t \in I$.

$$\begin{aligned} x^o(t) = S(t)x_0 &+ \int_0^t S(t-s)f(s, x^o(s), \mu^o(s))ds \\ &+ \int_0^t S(t-s)\sigma(s, x^o(s), \mu^o(s))dW(s) \\ &+ \int_0^t \int_{X_0} S(t-s)g(s, x^o(s), \mu^o(s), v)q_o(ds \times dv), \ t \in I, \end{aligned} \tag{27}$$

with $\mu^o(t) = \mathcal{P}(x^o(t))$, $t \in I$,

with $\{\mu^n, \mu^o\} \in B_\infty(I, \mathcal{M}_{2,\rho}(X))$ being the corresponding probability laws. Subtracting equation (26) from equation (27) and computing the expected value of the norm-square and using the assumption (A1), the Lipschitz conditions (A2), (A3) and (A4)*, we obtain

$$\begin{aligned} \mathbf{E}|x^o(t) - x^n(t)|_X^2 \le C_1 &\Big(\int_0^t \{\mathbf{E}|x^o(s) - x^n(s)|_X^2 + \rho^2(\mu^o(s), \mu^n(s))\}ds \Big) \\ &+ \mathbf{E}|e_n(t)|_X^2, \ t \in I, \end{aligned} \tag{28}$$

where

$$C_1 \equiv 8M^2(K^2T + K_R^2 + K_0^2)$$

and the process e_n is given by

$$e_n(t) \equiv \int_0^t \int_{X_0} S(t-s)g(s, x^o(s), \mu^o(s), v)(q_o - q_n)(ds \times dv), \ t \in I. \tag{29}$$

Now recalling that

$$\rho^2(\mu^o(s), \mu^n(s)) \leq \mathbf{E}|x^o(s) - x^n(s)|_X^2, \ s \in I,$$

it follows from (28) that

$$\mathbf{E}|x^o(t) - x^n(t)|_X^2 \leq 2C_1 \left(\int_0^t \mathbf{E}|x^o(s) - x^n(s)|_X^2 \ ds \right) + \mathbf{E}|e_n(t)|_X^2, \ t \in I. \tag{30}$$

Hence, by virtue of Gronwall inequality, we obtain

$$\mathbf{E}|x^o(t) - x^n(t)|_X^2 \leq \mathbf{E}|e_n(t)|_X^2 + 2C_1 \exp(C_1 T) \int_0^t \mathbf{E}|e_n(s)|_X^2 \ ds, \ t \in I. \tag{31}$$

We verify that $\mathbf{E}|e_n(t)|_X^2 \longrightarrow 0$ uniformly on the interval I. Clearly, e_n is given by the difference $z_o - z_n$ where

$$z_o(t) \equiv \int_0^t \int_{X_0} S(t-s)g(s, x^o(s), \mu^o(s), v) q_o(ds \times dv), \ t \in I. \tag{32}$$

$$z_n(t) \equiv \int_0^t \int_{X_0} S(t-s)g(s, x^o(s), \mu^o(s), v) q_n(ds \times dv), \ t \in I. \tag{33}$$

It follows from the basic theory of integration with respect to compensated Poisson random measures that

$$\mathbf{E}|z_o(t)|_X^2 \equiv \mathbf{E} \int_0^t \int_{X_0} |S(t-s)g(s, x^o(s), \mu^o(s), v)|_X^2 \Lambda_o(dv) ds, \ t \in I. \tag{34}$$

$$\mathbf{E}|z_n(t)|_X^2 \equiv \mathbf{E} \int_0^t \int_{X_0} |S(t-s)g(s, x^o(s), \mu^o(s), v)|_X^2 \Lambda_n(dv) ds, \ t \in I, \tag{35}$$

where Λ_o and Λ_n are the intensity measures corresponding to the compensated Poisson random measures q_o and q_n, respectively. Using the assumption (A4)* and the definition of the norm for the Banach space $\mathcal{M}_{\gamma^2}(X)$ and the expressions (34) and (35) one can easily verify that the following estimates hold uniformly with respect to $t \in I$,

$$\mathbf{E}|z_n(t)|_X^2 \leq 3M^2 K_0^2 \int_0^t \left(1 + \mathbf{E}|x^o(s)|_X^2\right) ds \leq 3M^2 K_0^2 T(1 + b^2),$$

$$\mathbf{E}|z_o(t)|_X^2 \leq 3M^2 K_0^2 L \int_0^t \left(1 + \mathbf{E}|x^o(s)|_X^2\right) ds \leq 3M^2 K_0^2 T(1 + b^2).$$

As $q_n \longrightarrow q_o$ set wise P-a.s, it is clear from (32) and (33) that

$$z_n(t) \xrightarrow{s} z_o(t) \text{ in } X,\ P - a.s,\ \forall\, t \in I. \tag{36}$$

Further, since $\Lambda_n \xrightarrow{w} \Lambda_o$ and the integrand

$$v \longrightarrow |S(t-s)g(s, x^o(s), \mu^o(s), v)|_X^2$$

is continuous and bounded P-a.s for almost all $s \in [0, t]$, it follows from (34) and (35) that

$$\mathbf{E}|z_n(t)|_X^2 \longrightarrow \mathbf{E}|z_o(t)|_X^2,\ \forall\, t \in I. \tag{37}$$

It is well known in classical analysis involving $L_p(\Omega)$ $(1 < p < \infty)$ spaces, that convergence of a sequence almost everywhere (or in measure) to a limit and convergence of the norm of the sequence to the norm of the limit imply convergence in the norm. Thus it follows from (36)–(37) that

$$\mathbf{E}|z_n(t) - z_o(t)|_X^2 \longrightarrow 0,\ \forall\, t \in I. \tag{38}$$

This proves that

$$\lim_{n\to\infty} \mathbf{E}|e_n(t)|_X^2 \equiv \lim_{n\to\infty} \mathbf{E}|z_o(t) - z_n(t)|_X^2 = 0, \forall\, t \in I. \tag{39}$$

Let us define

$$\varphi_n(t) \equiv \mathbf{E}|x^o(t) - x^n(t)|_X^2,\quad \eta_n(t) \equiv \mathbf{E}|e_n(t)|_X^2,\ t \in I,$$

and rewrite the inequality (31) as follows,

$$\varphi_n(t) \le \eta_n(t) + 2C_1 \exp(C_1 T) \int_0^t \eta_n(s)\, ds,\ t \in I. \tag{31*}$$

From (39) we have $\eta_n(t) \to 0$ for all (not for almost all) $t \in I$. Further, it follows from the estimates following the expression (35) that it is bounded from above as follows

$$0 \le \eta_n(t) \le 12 M^2 K_0^2 L T (1 + b^2),\ \forall\, t \in I.$$

Hence, it follows from Lebesgue bounded convergence theorem that

$$\lim_{n\to\infty} \int_0^t \eta_n(s) ds = 0,\ \forall\, t \in I.$$

Thus it follows from the inequality $(31)^*$ that $\varphi_n(t) \longrightarrow 0$ uniformly on I, and therefore, by letting $n \to \infty$, it follows from the inequality (31) that $x^n \xrightarrow{s} x^o$ in Ξ. This completes the proof. ∎

As an immediate consequence of the above theorem we have the following corollary.

Corollary 5.2. *Under the assumptions of Theorem* 5.1, *the probability laws* $\{\mu(t),\, t \in I\}$ *associated with the solution* $\{(\mu(t), x(t)) = (\mathcal{P}(x(t)), x(t)),\, t \in I\}$ *of the integral equation* (15) *mapping*

$$\mu : M_0 \longrightarrow B_\infty(I, \mathcal{M}_{2,\rho}(X))$$

is continuous with respect to the weak topology on M_0 *and the metric topology* D *on* $B_\infty(I, \mathcal{M}_{2,\rho}(X))$.

Proof. It follows from Theorem 5.1 that as $\Lambda_n \xrightarrow{w} \Lambda_o$ in M_0, the corresponding sequence of solutions $\{x^n\}$ of the integral equation (15) converges strongly in Ξ, that is, $x^n \xrightarrow{s} x^o$ in Ξ. Since

$$\rho(\mu^n(t), \mu^o(t)) \leq \mathbf{E}|x^n(t) - x^o(t)|_X \leq \left(\mathbf{E}|x^n(t) - x^o(t)|_X^2\right)^{1/2},\ \forall\, t \in I,$$

it follows from the definition of the metric D that

$$D(\mu^n, \mu^o) \equiv \sup\{\rho(\mu^n(t), \mu^o(t), t \in I\} \leq \| x^n - x^o \|_\Xi\ .$$

Hence it follows from the above inequality that the corresponding measure valued functions $\{\mu^n\}$ also converge to μ^o in the metric topology D as stated. ∎

6. Optimal control

In this section we want to study the question of existence of optimal controls from certain class of Poisson random measures, more precisely compensated Poisson random measures. Effectively this presents a class of stochastic relaxed controls which are Poisson random measures (or Lèvy process) with Lèvy measures being the choice variables. In the first subsection we consider control measures on an infinite dimensional Hilbert space and in the second subsection we consider measures on a finite dimensional space and in the third subsection we consider measures on a finite subset of X_0. We believe that the last two classes of control measures are much easier to construct for practical applications.

6.1. Control measures on infinite dimensional space

It is well known that Poisson random process taking values from a finite set of points can be generated using standard algorithms on computer. For example, in our case, if one chooses a finite number of distinct points $\{\zeta_1, \zeta_2, \ldots, \zeta_m\}$ from the Hilbert space X and take $X_0 \equiv \{\zeta_1, \zeta_2, \ldots, \zeta_m\}$ one can develop an algorithm to generate a Poisson process with values in X_0. In this case the Levý measure Λ is given by the set of positive numbers $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ where $\lambda_i \equiv \Lambda(\{\zeta_i\})$ denotes the average number of jumps of size ζ_i per unit time (or equivalently the frequency of jumps of size $\{\zeta_i\}$). Thus it makes sense to choose Poisson random measure for control purposes. We consider this class in the last subsection.

First, we consider the infinite dimensional case. Thus, in general, the choice variable here is the Levý measure $\Lambda \in \mathcal{M}_B^+(X_0)$. So we consider the following optimization problem. The cost functional J is given by

$$J(\Lambda) = \mathbf{E}\left\{\int_0^T \ell(t, x_\Lambda(t), \mu_\Lambda(t))dt + \Psi_1(x_\Lambda(T-), \mu_\Lambda(T))\right\} + \Psi_2(\Lambda) \tag{40}$$

where $x_\Lambda \in \Xi$, with the associated law $\mu_\Lambda \in B_\infty(I, \mathcal{M}_{2,\rho}(X))$, is the mild solution of the evolution equation (2) corresponding to the compensated Poisson random measure q_Λ. The cost integrands $\{\ell, \Psi_1\}$ and Ψ_2 are defined below.

Let $M_{ad} \subset M_0$ denote the class of admissible Levý measures. The problem is to find a $\Lambda_o \in M_{ad}$ such that J attains its minimum on M_{ad} at Λ_o.

We will assume that M_{ad} is a weakly compact subset of $M_0 \subset \mathcal{M}_B^+(X_0)$. For complete characterization of weakly compact sets in the space of finite measures on separable Hilbert spaces see Merkle [17] and Gihman & Skorohod [18]. For convenience of the reader, we present the necessary and sufficient conditions for weak compactness. Let $\{e_i\}$ be a complete ortho-normal basis of the (separable) Hilbert space X and define, for any $\Lambda \in \mathcal{M}_B^+(X)$ and $c > 0$, the operator Q_c^Λ by

$$(Q_c^\Lambda \xi, \eta) \equiv \int_{\{x \in X : |x| > c\}} (\xi, x)(\eta, x)\Lambda(dx), \quad \xi, \eta \in X.$$

Clearly, the operator Q_c^Λ is a positive (not necessarily bounded) self adjoint operator on the Hilbert space X. In case $\Lambda \in \mathcal{M}_2(X_0) \equiv \{\Lambda \in \mathcal{M}_B^+(X_0) : \int_{X_0} |x|_X^2 \Lambda(x) < \infty\}$, the operator Q_c^Λ is also bounded.

Lemma 6.1. *The set $M_{ad} \subset M_0 \subset \mathcal{M}_B^+(X_0)$ is relatively weakly compact if and only if the following two conditions hold:*
(1) *for every $\varepsilon > 0$, there exists a constant $c \in (0, \infty)$, such that*

$$\sup_{\Lambda \in M_{ad}} \Lambda\{x \in X : |x|_X > c\} < \varepsilon$$

(2) *for every $c > 0$, $\lim_{n\to\infty} \sum_{i \geq n} (Q_c^\Lambda e_i, e_i) = 0$ uniformly with respect to $\Lambda \in M_{ad}$.*

For proof of this result see Gihman & Skorohod [18, Theorem 2, p. 377] and Merkle [17, Theorem 4, p. 255].

Using the above compactness result, we can prove the following result on existence of optimal control.

Theorem 6.2. *Consider the system* (2) *with the admissible set of Leṽy measures* M_{ad}*, a weakly compact subset of* M_0*. Suppose the assumptions of Theorem* 5.2 *hold and further,* ℓ *is a Borel measurable map from* $I \times X \times \mathcal{M}_{\gamma^2}(X)$ *to* R *and lower semicontinuous in* $(x, \mu) \in X \times \mathcal{M}_{\gamma^2}(X)$; Ψ_1 *is a real valued lower semicontinuous function on* $X \times \mathcal{M}_{\gamma^2}(X)$ *satisfying the following growth conditions,*

$$|\ell(t, x, \mu)| \leq \alpha_1(t) + \alpha_2\{|x|_X^2 + |\mu|^2_{\mathcal{M}_{\gamma^2}}\},\ \alpha_1 \in L_1^+(I), \alpha_2 > 0,$$

$$|\Psi_1(x, \mu)| \leq \beta_1 + \beta_2\{|x|_X^2 + |\mu|^2_{\mathcal{M}_{\gamma^2}}\},\ \beta_1, \beta_2 \geq 0.$$

The function $\Psi_2 : M_0 \longrightarrow R^+ \equiv [0, \infty]$ *is lower semicontinuous with respect to the relative weak topology on* M_0*. Then there exists an optimal Leṽy measure in* M_{ad} *minimizing the cost functional* J*.*

Proof. Since M_{ad} is weakly compact, it suffices to prove that J is weakly lower semicontinuous on it. Consider the sequence $\{\Lambda_n\} \in M_{ad}$ with the corresponding sequence of compensated Poisson random measures$\{q_n\}$, and let $\{x_n, \mu_n\} \in \Xi \times B_\infty(I, \mathcal{M}_{2,\rho}(X))$ be the corresponding sequence of mild solutions of the evolution equation (2). Since M_{ad} is weakly (sequentially) compact, there exists a subsequence of the sequence $\{\Lambda_n\}$, relabeled as $\{\Lambda_n\}$, and a $\Lambda_o \in M_{ad}$ such that $\Lambda_n \xrightarrow{w} \Lambda_o$. Let $\{x_o, \mu_o\} \in \Xi \times B_\infty(I, \mathcal{M}_{2,\rho}(X))$ denote the mild solution of equation (2) corresponding to Λ_o (more precisely) q_{Λ_o}. Then it follows from Theorem 5.1 and Corollary 5.2 that, as $\Lambda_n \xrightarrow{w} \Lambda_o$, we have

$$x_n \xrightarrow{s} x_o \ \text{ in } \ \Xi \tag{41}$$

$$\mu_n \xrightarrow{s} \mu_o \ \text{ in } \ B_\infty(I, \mathcal{M}_{2,\rho}(X)). \tag{42}$$

Let $L_2^a(I \times \Omega, X)$ denote the space of norm square integrable $\mathcal{F}_t$-adapted X valued random processes equipped with the standard norm topology. Since T is finite, it is easy to verify that the embeddings $B_\infty^a(I, X) \equiv \Xi \hookrightarrow L_2^a(I \times \Omega, X)$ and $B_\infty(I, \mathcal{M}_{2,\rho}(X)) \hookrightarrow L_2(I, \mathcal{M}_{2,\rho}(X))$ are continuous. Thus x_n also converges to x_o strongly in $L_2^a(I \times \Omega, X)$ and μ_n converges to μ_o strongly in $L_2(I, \mathcal{M}_{2,\rho}(X))$. Therefore, there exists a subsequence of the sequence $\{x_n, \mu_n\}$, again relabeled as the original sequence, such that

$$x_n(t) \xrightarrow{s} x_o(t) \ \text{ in } \ X, \ \ dt \times dP \ \ a.e \ \text{ in } \ I \times \Omega \tag{43}$$

$$\mu_n(t) \xrightarrow{w} \mu_o(t) \ \text{ in } \ \mathcal{M}_{2,\rho}(X), \ \ a.e \ \ t \in I. \tag{44}$$

These follow from the fact that mean convergence implies convergence in measure which in turn implies the existence of a subsequence that converges almost everywhere. Since both ℓ and Ψ_1 are lower semicontinuous on $X \times \mathcal{M}_{\gamma^2}(X)$, it follows from (43) and (44) that

$$\ell(t, x_o(t), \mu_o(t)) \leq \underline{\lim}_{n} \ell(t, x_n(t), \mu_n(t)) \quad dt \times dP \ a.e, \tag{45}$$

$$\Psi_1(x_o(T-), \mu_o(T)) \leq \underline{\lim}_{n} \Psi_1(x_n(T), \mu_n(T)), \ dP \ a.e. \tag{46}$$

It follows from the assumptions on $\{\ell, \Psi_1\}$ and Corollary 4.4 asserting boundedness of the solution set that both ℓ and Ψ_1 are dominated by integrable processes. Thus by virtue of generalized Fatou's Lemma it follows from (45) and (46) that

$$\begin{aligned}
&\mathbf{E}\int_0^T \ell(t, x_o(t), \mu_o(t))dt \leq \underline{\lim}_{n} \mathbf{E}\int_0^T \ell(t, x_n(t), \mu_n(t))dt \\
&\mathbf{E}\Psi_1(x_o(T), \mu_o(T)) \leq \underline{\lim}_{n} \mathbf{E}\Psi_1(x_n(T), \mu_n(T)).
\end{aligned}$$

By assumption, Ψ_2 is a nonnegative weakly lower semicontinuous function on M_{ad} and therefore we have $\Psi_2(\Lambda_o) \leq \underline{\lim}_n \Psi_2(\Lambda_n)$. Since a finite sum of lower semicontinuous functions is a lower semicontinuous function, we conclude that J is weakly lower semicontinuous, that is, $J(\Lambda_o) \leq \underline{\lim}_n J(\Lambda_n)$. Since $M_{ad} \subset M_0$ is a bounded set and J weakly lower semicontinuous on it, $\inf_{\Lambda \in M_{ad}} J(\Lambda) > -\infty$. Hence it follows from weak compactness of the set M_{ad} that J attains its minimum on it. This completes the proof. ■

6.2. Control measures on finite dimensional spaces

As stated in the introduction of this section, Poisson random processes can be generated by use of standard computer algorithms developed specifically for applications. It is much easier to generate Poisson random measures on finite dimensional spaces as opposed to infinite dimensional Hilbert spaces. Thus, for convenience of applications, we consider the following model where the Poisson random measures used for control are defined on a finite dimensional space. This is a simple extension of the system (2)

$$\begin{aligned}
dx &= Axdt + f(t,x,\mu)dt + \sigma(t,x,\mu)dW + \int_{X_0} g(t,x,\mu,v)q_\Lambda(dt \times dv) \\
&\quad + \int_{R_0^n} h(t,x,\mu,\xi)q_\nu(dt \times d\xi), \quad x(0) = x_0, \qquad (47)\\
&\text{and} \quad \mu(t) = \mathcal{P}(x(t)), \ t \in I \equiv [0,T].
\end{aligned}$$

Here q_Λ is a fixed compensated Poisson random measure corresponding to a fixed finite Leév́y measure $\Lambda \in \mathcal{M}_B^+(X_0)$ while q_ν is another compensated Poisson random measure corresponding to a Leév́y measure $\nu \in \mathcal{M}_B^+(R_0^n)$, and this is considered as the control measure. In other words, the last integral in the above equation contains the control measure. We recall that the standard assumption on mutual independence of the random elements $\{x_0, W, q_\Lambda, q_\nu\}$ remains in force. Here the function h is assumed to be a Borel measurable map from $I \times X \times \mathcal{M}_{\gamma^2}(X) \times R^n$ to X satisfying similar properties as g. It is measurable in the first argument and continuous in the rest of the variables satisfying the following properties:

(A5): For any Lévy measure $\nu \in \mathcal{M}_B^+(R_0^n)$, the function $h : I \times X \times \mathcal{M}_{\gamma^2}(X) \times R^n \longrightarrow X$ is measurable in the first argument and continuous with respect to the rest of the variables and there exists a constant $L_\nu \neq 0$ such that

$$\int_{R_0^n} |h(t,x,\mu,\xi)|_X^2 \ \nu(d\xi) \le L_\nu^2 \left\{1 + |x|_X^2 + |\mu|_{\mathcal{M}_{\gamma^2}}^2\right\}, \ \forall\, x \in X, \mu \in \mathcal{M}_{\gamma^2}(X)$$
$$\int_{R_0^n} |h(t,x_1,\mu_1,\xi) - h(t,x_2,\mu_2,\xi)|_X^2 \ \nu(d\xi) \le L_\nu^2 \left\{|x_1 - x_2|_X^2 + \rho^2(\mu_1,\mu_2)\right\}$$

for all $x_1, x_2 \in X$ and $\mu_1, \mu_2 \in M_{2,\rho}(X)$ uniformly with respect to $t \in I$.

Consequently, under the additional assumption (A5), Theorem 4.2, asserting the existence and uniqueness of solution, holds also for the system (47). Again we choose a closed bounded subset $\Gamma \subset \mathcal{M}_B^+(R_0^n)$ and choose $Q_0 \equiv \{q_\nu, \nu \in \Gamma\}$ as the admissible set of controls (compensated Poisson random measures). The necessary and sufficient conditions for relative weak compactness of the set Γ in the finite dimensional case is just the condition (1) of Lemma 6.1. This condition guarantees tightness of the set Γ and hence, by virtue of well known Prohorov's theorem, the set Γ is weakly compact. Now we introduce an assumption similar to the one in Corollary 4.4.

(A5)*: Assumption (A5) holds and further, for all $x, y \in X$ and $\mu, \mu_1, \mu_2 \in \mathcal{M}_{\gamma^2}(X)$, there exists a finite positive number L_0 such that

$$\sup_{\nu\in\Gamma} \int_{R_0^n} |h(t,x,\mu,\xi)|_X^2 \ \nu(d\xi) \le L_0^2 \left\{1 + |x|_X^2 + |\mu|_{\mathcal{M}_{\gamma^2}}^2\right\},$$
$$\sup_{\nu\in\Gamma} \int_{R_0^n} |h(t,x,\mu_1,\xi) - h(t,y,\mu_2,\xi)|_X^2 \ \nu(d\xi) \le L_0^2 \left\{|x - y|_X^2 + \rho^2(\mu_1,\mu_2)\right\}.$$

Under the assumption (A5)* and the assumption that the admissible set Γ is a closed bounded subset of $\mathcal{M}_B^+(R_0^n)$, Corollary 4.4 holds. In other words, the solution set of the system (47) given by $\mathcal{S} \equiv \{x_\nu : \nu \in \Gamma\}$ is a bounded subset of Ξ. Similarly the set $\Pi \equiv \{\mu_\nu, \nu \in \Gamma\}$ is a bounded subset of $B_\infty(I, \mathcal{M}_{2,\rho}(X))$. Again, we can prove exactly in the same way as Theorem 5.2, that the map $\nu \longrightarrow x_\nu$ is continuous with respect to the weak topology on Γ and norm topology on Ξ. Similarly we have continuity of the map $\nu \longrightarrow \mu_\nu$ with respect to the weak topology on Γ and the metric topology D on $B_\infty(I, \mathcal{M}_{2,\rho}(X))$. We consider the system (47) as the control system with the Lević measure $\nu \in \Gamma$ as the control. The cost functional is given by

$$J(\nu) = \mathbf{E}\left\{\int_0^T \ell(t, x_\nu(t), \mu_\nu(t))dt + \Psi_1(x_\nu(T), \mu_\nu(T))\right\} + \Psi_2(\nu). \tag{48}$$

The objective is to find a $\nu^o \in \Gamma$ that minimizes the functional $J(\nu)$. In the following theorem we prove the existence of such an element.

Theorem 6.3. *For a fixed $\Lambda \in \mathcal{M}_B^+(X_0)$, consider the system* (47) *driven by the compensated Poisson random measure q_ν, with $\nu \in \Gamma$ being the control. The cost functional is given by* (48) *with $\{\ell, \Psi_1\}$ satisfying the properties as stated in Theorem* 6.2. *Suppose Γ is a weakly compact subset of $\mathcal{M}_B^+(R_0^n)$ and Ψ_2 is a weakly lower semicontinuous functional on Γ. Then there exists a $\nu^o \in \Gamma$ at which J given by* (48) *attains its minimum.*

Proof. The proof is similar to that of Theorem 6.2. ∎

6.3. Control measures supported on a finite set

In case the control measures (Lević measures $\{\nu\}$) are supported on a finite subset of X_0, the problem of constructing computer algorithms for generating Poisson random measures becomes much easier. Let $\mathcal{Z} = \{\zeta_1, \zeta_2, \ldots, \zeta_m\} \subset X_0, m \geq 2$, and choose the set $\mathcal{M}_B^+(\mathcal{Z})$ for the Lević measures. Any $\nu \in \mathcal{M}_B^+(\mathcal{Z})$ has the representation,

$$\nu = (\nu(\{\zeta_1\}), \nu(\{\zeta_2\}), \nu(\{\zeta_3\}), \ldots, \nu(\{\zeta_m\})) = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m),$$

where $\lambda_i \in [0, \infty)$ is the mean intensity of jump size $\{\zeta_i\}$, that is, the average number of jumps of size $\{\zeta_i\}$ per unit time or, equivalently, the frequency of jumps of size ζ_i. In this case the dynamic system (47) is given by the following stochastic differential equation,

$$\begin{aligned} dx &= Axdt + f(t,x,\mu)dt + \sigma(t,x,\mu)dW + \int_{X_0} g(t,x,\mu,v)q_\Lambda(dt \times dv) \\ &\quad + \int_{\mathcal{Z}} h(t,x,\mu,\xi)q_\nu(dt \times d\xi), \quad x(0) = x_0, \end{aligned} \tag{49}$$
$$\text{and} \quad \mu(t) = \mathcal{P}(x(t)), t \in I \equiv [0,T].$$

Again, the objective functional is given by the expression (48). Here, one possible choice for the cost of control is $\Psi_2(\nu) = \sum_{i=1}^m \lambda_i$. This is a measure of total variation of the Lévy measure ν and it is considered here as the control cost. The larger the frequency of jumps, the larger is the variation of the measure. The integral equation associated with the evolution equation (49) is given by

$$
\begin{aligned}
x(t) &= S(t)x_0 + \int_0^t S(t-s)f(s,x(s),\mu(s))ds + \int_0^t S(t-s)\sigma(s,x(s),\mu(s))dW \\
&+ \int_0^t \int_{X_0} S(t-s)g(s,x(s),\mu(s),v)q_\Lambda(ds \times dv) \qquad (50)\\
&+ \int_0^t \int_{\mathcal{Z}} S(t-s)h(s,x(s),\mu(s),\xi)q_\nu(ds \times d\xi), \quad x(0) = x_0,
\end{aligned}
$$

and $\mu(t) = \mathcal{P}(x(t)),\ t \in I \equiv [0,T]$.

Considering the norm of the last term of the above integral equation, one finds that

$$
\begin{aligned}
&\mathbf{E}\left| \int_{[0,t]\times\mathcal{Z}} S(t-s)h(s,x(s),\mu(s),\zeta)q_\nu(ds \times d\zeta)\right|_X^2 \\
&= \int_{[0,t]\times\mathcal{Z}} \mathbf{E}|S(t-s)h(s,x(s),\mu(s),\zeta)|_X^2\ \nu(d\zeta)\ ds \qquad (51)\\
&= \int_0^t \sum_{i=1}^m \lambda_i \mathbf{E}|S(t-s)h(s,x(s),\mu(s),\zeta_i)|_X^2\ ds.
\end{aligned}
$$

It is clear that any closed bounded subset $M_0 \subset \mathcal{M}_B^+(\mathcal{Z})$ is weakly compact. Assuming that the rest of the assumptions of Theorem 6.2 hold, it follows from Theorem 6.2 that there exists a $\nu^o \in M_0$ at which $J(\nu^o) \le J(\nu)$ for all $\nu \in M_0$.

6.4. Some simple target problems

Throughout this section we consider the system (2) or equivalently (15) with the Lévy measure Λ as the control belonging to the admissible set M_{ad}.

(P1): (Mass Transport Problem) Let $\mu_0 = \mathcal{P}(x_0)$ denote the initial probability measure with support C_0 a closed subset of X and let $C \subset X$ be another closed set with $C_0 \cap C = \emptyset$. The problem is to find a control in $M_{ad} \subset M_0$ that maximizes the functional

$$J(\Lambda) \equiv \mu_\Lambda(T)(C).$$

It follows from Corollary 5.2 that the map $\Lambda \longrightarrow \mu_\Lambda(T)(\cdot)$ from M_0 to $\mathcal{M}_{2,\rho}(X)$ is continuous with respect to the weak topology on M_0 and the metric topology on $\mathcal{M}_{2,\rho}(X)$. Since M_{ad} is weakly compact J attains its maximum on M_{ad}.

(P2): (Collision Avoidance Problem) Let $\mathcal{V}$ be an open set in X with $\mathcal{V} \cap C_0 = \emptyset$. The problem is to find a control in M_{ad} that minimizes the functional

$$J(\Lambda) = \int_0^T \mu_\Lambda(t)(\mathcal{V})dt.$$

It follows from Corollary 5.2 that the map

$$\mu : M_0 \longrightarrow B_\infty(I, \mathcal{M}_{2,\rho}(X))$$

is continuous with respect to the weak topology on M_0 and the metric topology D on $B_\infty(I, \mathcal{M}_{2,\rho}(X))$. Thus, as $\Lambda_n \xrightarrow{w} \Lambda_o$ in M_0, it follows from Fatou's Lemma that

$$\begin{aligned} \underline{\lim}\, J(\Lambda_n) &= \underline{\lim} \int_0^T \mu_{\Lambda_n}(t)(\mathcal{V})dt \\ &\leq \int_0^T \underline{\lim}\ \mu_{\Lambda_n}(t)(\mathcal{V})dt \ \leq \int_0^T \mu_{\Lambda_0}(t)(\mathcal{V})dt = J(\Lambda_0). \end{aligned}$$

Hence $\Lambda \longrightarrow J(\Lambda)$ is weakly lower semicontinuous on M_0. Since the admissible controls $M_{ad} \subset M_0$ is weakly compact, J attains its minimum on M_{ad} proving existence of optimal control.

(P3): (Compactness of Attainable Sets) For each $t > 0$, define the set

$$\mathcal{A}(t) \equiv \{m \in \mathcal{M}_{2,\rho}(X) : m = \mu_\Lambda(t), \text{ for some } \Lambda \in M_{ad}\}.$$

This is the set of measures at time $t \in [0, \infty)$ that the system (2) or the integral equation (15) with admissible controls M_{ad} can attain (produce). Under the assumptions of Theorem 5.1, the reader can easily verify that, for each $t \in I$, the attainable set $\mathcal{A}(t)$ is a compact subset of $\mathcal{M}_{2,\rho}(X)$ in the metric topology ρ. Let $\{\mu^d(t), t \in I\}$ be a desired measure valued function with values in $\mathcal{M}_{2,\rho}(X)$. The problem is to find a control $\Lambda \in M_{ad}$ that minimizes the functional

$$J(\Lambda) \equiv \int_0^T \rho(\mu_\Lambda(t), \mu^d(t))\lambda(dt)$$

where λ is any positive measure having finite total variation on I. One can verify that this functional is lower semicontinuous with respect to the weak topology on M_{ad} and the metric topology on $B_\infty(I, \mathcal{M}_{2,\rho}(X))$. Thus there exists an optimal control that minimizes this functional.

Remark 6.4. In our recent paper [1], we considered McKean-Vlasov equations on Hilbert spaces in the absence of Lévy process and proved existence of optimal relaxed controls and developed necessary conditions of optimality. It would be interesting to develop necessary conditions of optimality for the systems driven by Lévy measure as considered here. This remains an open problem.

Acknowledgement

This work was partially supported by the National Science and Engineering Research Council of Canada under grant no A7101.

References

[1] N.U. Ahmed, *Optimal control of general McKean-Vlasov stochastic evolution equations on Hilbert spaces and necessary conditions of optimality*, Discuss. Math. Diff. Incl. Control and Optim. **35** (2015) 165–195. doi:10.7151/dmdico.1171

[2] N.U. Ahmed, *Nonlinear diffusion governed by McKean-Vlasov equation on Hilbert space and optimal control*, SIAM J. Cotrol and Optim. **46** (2007) 356–378. doi:10.1137/050645944

[3] N.U. Ahmed and X. Ding, *A semilinear McKean-Vlasov stochastic evolution equation in Hilbert space*, Stochastic Process. Appl. **60** (1995) 65–85. doi:10.1016/0304-4149(95)00050-X

[4] N.U. Ahmed and X. Ding, *Controlled McKean-Vlasov equations*, Commun. Appl. Anal. **5** (2001) 183-206.

[5] N.U. Ahmed, C.D. Charalambous, *Stochastic minimum principle for partially observed systems subject to continuous and jump diffusion processes and drviven by relaxed controls*, SIAM J. Control and Optim. **51** (2013) 3235–3257. doi:10.1137/120885656

[6] N.U. Ahmed, *Systems governed by mean-field stochastic evolution equations on Hilbert spaces and their optimal control*, Dynamic Systems and Applications (DSA) **25** (2016) 61–88.

[7] N.U. Ahmed, Semigroup Theory with Applications to Systems and Control, Pitman Research Notes in Mathematics Series, Vol. 246, Longman Scientific and Technical (U.K., Co-published with John-Wiely & Sons, Inc. New York, 1991).

[8] N.U. Ahmed, *Stochastic evolution equations on Hilbert spaces with partially observed relaxed controls and their necessary conditions of optimality*, Discuss. Math. Diff. Incl., Control and Optim. **34** (2014) 105–129. doi:10.7151/dmdico.1153

[9] E. Cinlar, Probability and Stochastics, Graduate Text in Mathematics, Vol. 261 (Springer, 2011).

[10] G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions (Cambridge University Press, 1992). doi:10.1017/CBO9780511666223

[11] G. Da Prato and J. Zabczyk, Ergodicity for Infinite Dimensional Systems, London Math. Soc. Lecture Note Ser. 229 (Cambridge University Press, London, 1996). doi:10.1017/CBO9780511662829

[12] D.A. Dawson, *Critical dynamics and fluctuations for a mean-field model of cooperative behavior*, J. Stat. Phys. **31** (1983) 29–85. doi:10.1007/BF01010922

[13] D.A. Dawson and J. Gärtner, Large Deviations, Free Energy Functional and Quasi-Potential for a Mean Field Model of Interacting Diffusions, Mem. Amer. Math. Soc. 398 (Providence, RI, 1989).

[14] N. Dunford and J.T Schwartz, Linear Operators, Part 1 (Interscience Publishers, Inc., New York, 1958).

[15] N.I. Mahmudov and M.A. McKibben, *Abstract second order damped McKean-Vlasov stochastic evolution equations*, Stoch. Anal. Appl. **24** (2006) 303–328. doi:10.1080/07362990500522247

[16] H.P. McKean, *A class of Markov processes associated with nonlinear parabolic equations*, Proc. Natl. Acad. Sci. USA **56** (1966) 1907–1911. doi:10.1073/pnas.56.6.1907

[17] M.J. Merkle, *On weak convergence of measures on Hilbert spaces*, J. Multivariate Anal. **29** (1989) 252–259. doi:10.1016/0047-259X(89)90026-2

[18] I.I. Gihman and A.V. Skorohod, The Theory of Stochastic Processes, I (Springer-Verlag, New York, Heidelberg Berlin, 1974). doi:10.1007/978-3-642-61943-4

[19] Y. Shen and T.K. Siu, *The maximum principle for a jump-diffusion mean-field model and its application to the mean-variance problem*, Nonlinear Anal. **86** (2013) 58–73. doi:10.1016/j.na.2013.02.029

Received 30 November 2016