TOPOLOGICAL PROPERTIES OF SOME SPACES OF CONTINUOUS OPERATORS

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Abstract

Let $X$ be a completely regular Hausdorff space, $E$ and $F$ be Banach spaces. Let $C_b(X,E)$ be the space of all $E$-valued bounded continuous functions on $X$, equipped with the strict topology $\beta$. We study topological properties of the space $L_\beta(C_b(X,E), F)$ of all $(\beta, \|\cdot\|_F)$-continuous linear operators from $C_b(X,E)$ to $F$, equipped with the topology $\tau_s$ of simple convergence. If $X$ is a locally compact paracompact space (resp. a P-space), we characterize $\tau_s$-compact subsets of $L_\beta(C_b(X,E), F)$ in terms of properties of the corresponding sets of the representing operator-valued Borel measures. It is shown that the space $(L_\beta(C_b(X,E), F), \tau_s)$ is sequentially complete if $X$ is a locally compact paracompact space.

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1. INTRODUCTION AND TERMINOLOGY

Throughout the paper let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be (real or complex) Banach spaces, and let $E'$ and $F'$ denote the Banach duals of $E$ and $F$, respectively. By $B_{F'}$ and $B_E$ we denote the closed unit ball in $F'$ and $E$, respectively. By $\mathcal{L}(E, F)$ we denote the space of all bounded linear operators $E$ to $F$. Given a locally convex space $(Z, \xi)$ by $(Z, \xi)'$ or $Z_\xi'$ we denote its topological dual. We denote by $\sigma(Z, Z_\xi')$ the weak topology on $Z$ with respect to a dual pair $(Z, Z_\xi')$.

Assume that $(X, T)$ is a completely regular Hausdorff space. Let $\mathcal{B}o$ stand for the $\sigma$-algebra of Borel sets in $X$. By $\mathcal{K}$ (resp. $\mathcal{F}$) we denote the family of all compact (resp. finite) sets in $X$. 

Let $C_b(X, E)$ stand for the space of all bounded continuous functions $f : X \to E$. By $\tau_u$ we will denote the topology on $C_b(X, E)$ of the uniform norm $\|\cdot\|$.

The strict topology $\beta$ (denoted also by $\beta_o$ and $\beta_l$) can be characterized as the finest locally convex topology on $C_b(X, E)$ which coincides with the compact-open topology $\tau_c$ on $\tau_u$-bounded subsets of $C_b(X, E)$ (see [6, 9, 11, 12]). This means that $(C_b(X, E), \beta)$ is a generalized DF-space (see [15], [17, Corollary]) (equivalently, $\beta$ coincides with the mixed topology $\gamma[\tau_u, \tau_c]$ in the sense of Wiweger (see [4, 19] for more details)). Then $\beta$ is weaker than $\tau_u$, and $\beta$ coincides with $\tau_u$ if and only if $X$ is compact (see [3, Theorem 2.3]).

By $\mathcal{L}_\beta(C_b(X, E), F)$ we will denote the family of all $(\beta, \|\cdot\|_F)$-continuous linear operators $T : C_b(X, E) \to F$. The topology $\tau_s$ of simple convergence in $\mathcal{L}_\beta(C_b(X, E), F)$ is defined by the family of seminorms $\{p_f : f \in C_b(X, E)\}$, where $p_f(T) = \|T(f)\|_F$ for $T \in \mathcal{L}_\beta(C_b(X, E), F)$.

In this paper we study topological properties of the space $(\mathcal{L}_\beta(C_b(X, E), F), \tau_s)$. We characterize $\tau_s$-compact sets in $\mathcal{L}_\beta(C_b(X, E), F)$ in terms of the properties of the corresponding sets of the representing operator-valued Borel measures whenever $X$ is a locally compact paracompact space (resp. $X$ is a P-space) (see Theorem 3.4 below). It is shown that the space $(\mathcal{L}_\beta(C_b(X, E), F), \tau_s)$ is sequentially complete if $X$ is a locally compact paracompact space (see Theorem 4.2 below).

2. INTEGRAL REPRESENTATION OF OPERATORS ON $C_b(X, E)$

Recall that a countably additive measure scalar measure $\nu$ on $\mathcal{B}o$ is said to be a Radon measure if its variation $|\nu| : \mathcal{B}o \to \mathbb{R}_+$ is regular, i.e., for each $A \in \mathcal{B}o$,

$$|\nu|(A) = \sup \{|\nu|(K) : K \in \mathcal{K}, K \subset A\}, \quad |\nu|(A) = \inf \{|\nu|(O) : O \in \mathcal{T}, O \supset A\}.$$ 

By $M(X)$ we denote the space of all Radon measures.

Let $M(X, E')$ denote the space of all countably additive measures $\mu : \mathcal{B}o \to E'$ of bounded variation ($|\mu|(X) < \infty$) such that for each $x \in E$, $\mu_x \in M(X)$, where $\mu_x(A) := \mu(A)[x]$ for $A \in \mathcal{B}o$. Then $|\mu| \in M(X)$ (see [10, Lemma 2.3]).

It is known that for $\mu \in M(X, E')$, every $f \in C_b(X, E)$ is $\mu$-integrable in the Riemann-Stieltjes sense (see [7, Definition 2], [14, Definition 2.2]).

The following characterization of $\beta$-continuous linear functionals on $C_b(X, E)$ will be of importance (see [14, § 2]).

**Theorem 2.1.** For a linear functional $\Phi$ on $C_b(X, E)$ the following statements are equivalent:

(i) $\Phi$ is $\beta$-continuous.
(ii) There exists a unique \( \mu \in M(X, E') \) such that

\[
\Phi(f) = \Phi_\mu(f) = \int_X f \, d\mu \quad \text{for } f \in C_b(X, E).
\]

Moreover, \( \|\Phi_\mu\| = |\mu|(X) \).

The following result will be useful (see [12, Lemma 2]).

**Lemma 2.2.** For a subset \( M \) of \( M(X, E') \) the following statements are equivalent:

(i) \( \sup_{\mu \in M} |\mu|(X) < \infty \) and \( M \) is uniformly tight, that is, for each \( \varepsilon > 0 \) there exists \( K \in \mathcal{K} \) such that \( \sup_{\mu \in M} |\mu|(X \setminus K) \leq \varepsilon \).

(ii) The family \( \{\Phi_\mu : \mu \in M\} \) in \( C_b(X, E')_\beta \) is \( \beta \)-equicontinuous.

Let \( i_F : F \to F'' \) denote the canonical embedding, i.e., \( i_F(y)(y') = y'(y) \) for \( y \in F, y' \in F' \). Moreover, let \( j_F : i_F(F) \to F \) stand for the left inverse of \( i_F \), that is, \( j_F \circ i_F = id_F \).

Assume that \( T : C_b(X, E) \to F \) is a \((\beta, \|\cdot\|_F)\)-continuous linear operator. Then according to [14, Theorem 3.1] there exists a unique measure \( m_T : \mathcal{B}_0 \to L(E, F'') \) (called the representing measure of \( T \)) such that the following statements hold:

(2.1) For every \( y' \in F' \), \( (m_T)_{y'} \in M(X, E') \), where

\[
(m_T)_{y'}(A)(x) := (m_T(A)(x))(y') \quad \text{for } A \in \mathcal{B}_0, x \in E.
\]

(2.2) The mapping \( F' \ni y' \mapsto (m_T)_{y'} \in M(X, E') \) is \((\sigma(F', F), \sigma(M, E'), C_b(X, E))\)-continuous.

(2.3) \( \tilde{m}_T(X) < \infty \) and for every \( \varepsilon > 0 \) there exists \( K \in \mathcal{K} \) such that \( \tilde{m}_T(X \setminus K) \leq \varepsilon \) (here \( \tilde{m}_T(A) \) stands for the semivariation of \( m_T \) on \( A \in \mathcal{B}_0 \)).

(2.4) \( \|T\| = \tilde{m}_T(X) \).

(2.5) Every \( f \in C_b(X, E) \) is \( m \)-integrable in the Riemann-Stjeltjes sense and

\[
\int_X f \, dm \in i_F(F) \quad \text{(here } \int_X f \, dm \text{ denotes the Riemann-Stjeltjes integral)}
\]

and \( T(f) = j_F(\int_X f \, dm) \).

(2.6) For every \( y' \in F' \),

\[
y'(T(f)) = \left( \int_X f \, dm_T \right)(y') = \int_X f \, d(m_T)_{y'} \quad \text{for } f \in C_b(X, E).
\]

Note that (see [5, §4, Proposition 5]),

(2.7) \( \tilde{m}_T(A) = \sup \{ |(m_T)_{y'}|(A) : y' \in B_{F'} \} \) for \( A \in \mathcal{B}_0 \).
Let \( B_{C_b(X,E)} := \{ f \in C_b(X,E) : \| f \| \leq 1 \} \).

We will need the following result.

**Lemma 2.3.** Assume that \( T : C_b(X,E) \to F \) be \((\beta, \| \cdot \|_F)\)-continuous linear operator and \( m_T \) is its representing measure. Then for \( y' \in F' \) and \( K \in \mathcal{K} \), we have:

(i) \( |(m_T)_{y'}|(X \setminus K) = \text{sup} \left\{ \left| \int_X f \, d(m_T)_{y'} \right| : f \in B_{C_b(X,E)} \text{ with } f \equiv 0 \text{ on } K \right\} \)

\[ = \text{sup} \left\{ |y'(T(f))| : f \in B_{C_b(X,E)} \text{ with } f \equiv 0 \text{ on } K \right\} \]

(ii) \( \tilde{m}_T(X \setminus K) = \text{sup} \left\{ \left\| \int_X f \, dm_T \right\|_{F''} : f \in B_{C_b(X,E)} \text{ with } f \equiv 0 \text{ on } K \right\} \)

\[ = \text{sup} \left\{ \|T(f)\|_F : f \in B_{C_b(X,E)} \text{ with } f \equiv 0 \text{ on } K \right\}. \]

**Proof.** (i) It follows from [14, Lemma 2.3] and (2.6).

(ii) Using (i), (2.7), (2.6) and (2.5), we get

\[
\tilde{m}_T(X \setminus K) = \text{sup} \left\{ \left\| \int_X f \, dm_T \right\|_{F''} : f \in B_{C_b(X,E)} \text{ with } f \equiv 0 \text{ on } K \right\}
\]

\[ = \text{sup} \left\{ \|T(f)\|_F : f \in B_{C_b(X,E)} \text{ with } f \equiv 0 \text{ on } K \right\}. \]

3. Relative compactness in \((L_\beta(C_b(X,E),F), \tau_8)\)

We start with the following characterization of \((\beta, \| \cdot \|_F)\)-equicontinuous subsets of \( L_\beta(C_b(X,E),F) \).

**Proposition 3.1.** For a subset \( A \) of \( L_\beta(C_b(X,E),F) \) the following statements are equivalent:

(i) \( A \) is \((\beta, \| \cdot \|_F)\)-equicontinuous.

(ii) \( \sup_{T \in A} \tilde{m}_T(X) < \infty \) and for every \( \varepsilon > 0 \) there exists \( K \in \mathcal{K} \) such that \( \sup_{T \in A} \tilde{m}_T(X \setminus K) \leq \varepsilon \).

(iii) \( \sup_{T \in A} \|T\| < \infty \) and for every \( \varepsilon > 0 \) there exists \( K \in \mathcal{K} \) such that \( \sup_{T \in A} \left\| \int_X f \, dm_T \right\|_{F''} \leq \varepsilon \) whenever \( f \in C_b(X,E), \|f\| \leq 1 \) with \( f \equiv 0 \) on \( K \).
Proof. (i)⇒(ii) Assume that $\mathcal{A}$ is $(\beta, \| \cdot \|_F)$-equicontinuous. This means that the set $\{ y' \circ T : T \in \mathcal{A}, y' \in B_{F'} \}$ is $\beta$-equicontinuous in $C_b(X, E)'_\beta$. Hence by (2.6), (2.7) and Lemma 2.2, we get
\[
\sup_{T \in \mathcal{A}} \tilde{m}_T(X) = \sup\{ |(m_T)_{y'}|(X) : T \in \mathcal{A}, y' \in B_{F'} \} < \infty,
\]
and for every $\varepsilon > 0$ there exists $K \in \mathcal{K}$ such that
\[
\sup_{T \in \mathcal{A}} \tilde{m}_T(X \setminus K) = \sup\{ |(m_T)_{y'}|(X \setminus K) : T \in \mathcal{A}, y' \in B_{F'} \} \leq \varepsilon.
\]

(ii)⇒(i) Assume that (ii) holds. Then by Lemma 2.2 and (2.6), (2.7), we obtain that the family $\{ y' \circ T : T \in \mathcal{A}, y' \in B_{F'} \}$ is $\beta$-equicontinuous in $C_b(X, E)'_\beta$, and it follows that $\mathcal{A}$ is $(\beta, \| \cdot \|_F)$-equicontinuous.

(ii)⇔(iii) It follows from Lemma 2.3.

In view of [16, Theorem 2] we have the following useful result.

Theorem 3.2. Let $\mathcal{A}$ be a $\tau_s$-compact subset of $L_\beta(C_b(X, E), F)$. Then the set $\{ y' \circ T : T \in \mathcal{A}, y' \in B_{F'} \}$ is a $\sigma(C_b(X, E)'_\beta, C_b(X, E))$-compact subset of $C_b(X, E)'_\beta$.

Assume that $X$ is a locally compact space. Then $\beta = \beta_\tau$ and $\beta$ is the topology defined by Buck [2] (see [6, p. 844]).

Recall that $X$ is a P-space if every $G_\delta$ set in $X$ is open (see [8]). Then every compact set in $X$ is finite and $\beta = \beta_\tau$ on $C_b(X)$ (see [18, Theorem 2.2]) and it follows that $\beta = \beta_\tau$ on $C_b(X, E)$.

Note that if $X$ is a locally compact paracompact space (resp. a P-space), then $(C_b(X, E), \beta)$ is a strongly Mackey space, that is, every relatively $\sigma(C_b(X, E)'_\beta, C_b(X, E))$-countably compact subset of $C_b(X, E)'_\beta$ is $\beta$-equicontinuous (see [11, Theorem 6.1], [12, theorem 5]).

Corollary 3.3. Assume that $X$ is a locally compact paracompact space (resp. a P-space). Let $\mathcal{A}$ be a $\tau_s$-compact subset of $L_\beta(C_b(X, E), F)$. Then $\mathcal{A}$ is $(\beta, \| \cdot \|_F)$-equicontinuous.

Proof. Since $(C_b(X, E), \beta)$ is a strongly Mackey space, by Theorem 3.2 $\{ y' \circ T : T \in \mathcal{A}, y' \in B_{F'} \}$ is a $\beta$-equicontinuous subset of $C_b(X, E)'_\beta$, and it follows that $\mathcal{A}$ is $(\beta, \| \cdot \|_F)$-equicontinuous.
Now we can state a characterization of $\tau_s$-compact sets in $L_\beta(C_b(X,E), F)$ in terms of the properties of the corresponding sets of representing operator-valued Borel measures.

**Theorem 3.4.** Assume that $X$ is a locally compact paracompact space (resp. a $P$-space). Then for a subset $A$ of $L_\beta(C_b(X,E), F)$, the following statements are equivalent:

(i) $A$ is relatively $\tau_s$-compact.

(ii) $A$ is $(\beta, \| \cdot \|_F)$-equicontinuous and for every $f \in C_b(X,E)$, the set $\{T(f) : T \in A\}$ is relatively compact in $F$.

(iii) The following statements hold:

(a) $\sup_{T \in A} \tilde{m}_T(X) < \infty$ and for every $\varepsilon > 0$ there exists $K \in \mathcal{K}$ (resp. $M \in \mathcal{F}$) such that $\sup_{T \in A} \tilde{m}_T(X \setminus K) \leq \varepsilon$ (resp. $\sup_{T \in A} \tilde{m}_T(X \setminus M) \leq \varepsilon$).

(b) For every $f \in C_b(X,E)$, the set $\{\int_X f \, dm_T : T \in A\}$ is relatively compact in $F''$.

(iv) The following statements hold:

(a) $\sup_{T \in A} \|T\| < \infty$ and for every $\varepsilon > 0$ there exists $K \in \mathcal{K}$ (resp. $M \in \mathcal{F}$) such that $\sup_{T \in A} \int_X f \, dm_T \|f''\|_F \leq \varepsilon$ whenever $f \in C_b(X,E)$, $\|f\| \leq 1$ and $f \equiv 0$ on $K$ (resp. $M$).

(b) For every $f \in C_b(X,E)$, the set $\{\int_X f \, dm_T : T \in A\}$ is relatively compact in $F''$.

**Proof.** (i)$\Rightarrow$(ii) Assume that (i) holds. Then by Corollary 3.3 the set $A$ is $(\beta, \| \cdot \|_F)$-equicontinuous. Clearly for each $f \in C_b(X,E)$, the set $\{T(f) : T \in A\}$ is relatively compact in $F$.

(ii)$\Rightarrow$(ii) It follows from [1, Chap. 3, §3.4, Corollary 1].

(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) It follows from Proposition 3.1.

4. **Sequential completeness of $(L_\beta(C_b(X,E), F), \tau_s)$**

It is known that if $X$ is a paracompact space, then $X$ is metacompact and normal. It follows that if $X$ is a locally compact paracompact space, then $\beta = \beta_r$ on $C_b(X,E)$ and the space $(C_b(X,E)_{\beta_r}, \sigma(C_b(X,E)_{\beta_r}, C_b(X,E)))$ is sequentially complete (see [13, Theorem 3]).

Now we can state a Banach-Steinhaus type theorem for $(\beta, \| \cdot \|_F)$-continuous operators $T : C_b(X,E) \to F$. 

**Theorem 4.1.** Assume that $X$ is a locally compact paracompact space. Let $T_k : C_b(X,E) \to F$ be a $(\beta, \| \cdot \|_F)$-continuous linear operator for $k \in \mathbb{N}$. Assume that $T(f) := \lim_k T_k(f)$ exists in $F$ for every $f \in C_b(X,E)$. Then $T$ is a $(\beta, \| \cdot \|_F)$-continuous linear operator and the set $\{T_k : k \in \mathbb{N}\}$ is $(\beta, \| \cdot \|_F)$-equicontinuous.

**Proof.** In view of the Banach-Steinhaus theorem $T : C_b(X,E) \to F$ is a bounded linear operator. Then for each $y' \in F'$, $(y' \circ T)(f) = \lim(y' \circ T_k)(f)$ for all $f \in C_b(X,E)$, where $y' \circ T_k \in C_b(X,E)'_\beta$ for $k \in \mathbb{N}$ and $y' \circ T \in C_b(X,E)'$. It follows that $(y' \circ T_k)$ is a $\sigma(C_b(X,E)'_\beta, C_b(X,E))$-Cauchy sequence in $C_b(X,E)'_\beta$. Note that under the assumptions on $X$, we have that $\beta = \beta_\tau$ on $C_b(X,E)$ and hence by [13, Theorem 3] the space $(C_b(X,E)'_\beta, \sigma(C_b(X,E)'_\beta, C_b(X,E)))$ is sequentially complete. Hence for each $y' \in F'$ there exists $\Phi_{y'} \in C_b(X,E)'_\beta$ such that $\Phi_{y'}(f) = \lim(y' \circ T_k)(f)$ for all $f \in C_b(X,E)$. Then $y' \circ T = \Phi_{y'} \in C_b(X,E)'_\beta$. Since $\beta$ is a Mackey topology, we derive that $T$ is $(\beta, \| \cdot \|_F)$-continuous. Thus $T_k \to T$ for $\tau_\beta$ in $L_\beta(C_b(X,E), F)$, so $\{T_k : k \in \mathbb{N}\} \cup \{T\}$ is a $\tau_\beta$-compact subset of $L_\beta(C_b(X,E), F)$. Hence by Corollary 3.3 the set $\{T_k : k \in \mathbb{N}\}$ is $(\beta, \| \cdot \|_F)$-equicontinuous. \hfill $\blacksquare$

As a consequence of theorem 4.1 we get:

**Corollary 4.2.** Assume that $X$ is a locally compact paracompact space. Then the space $(L_\beta(C_b(X,E), F), \tau_\beta)$ is sequentially complete.

**Proof.** Let $(T_k)$ be a $\tau_\beta$-Cauchy sequence in $L_\beta(C_b(X,E), F)$. Then for each $f \in C_b(X,E)$, $(T_k(f))$ is a Cauchy sequence in $F$, so $T(f) := \lim_k T_k(f)$ exists in $F$. By Theorem 4.1 $T$ is $(\beta, \| \cdot \|_F)$-continuous and $T_k \to T$ for $\tau_\beta$. \hfill $\blacksquare$

**References**


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