EFFECTIVE ENERGY INTEGRAL FUNCTIONALS FOR THIN FILMS WITH THREE DIMENSIONAL BENDING MOMENT IN THE ORLICZ-SOBOLEV SPACE SETTING

Włodzimierz Laskowski

School of Mathematics
West Pomeranian University of Technology
Al. Piastów 48, 70–311 Szczecin, Poland
e-mail: wlaskowski@zut.edu.pl

AND

Hong Thai Nguyen

Institute of Mathematics
Szczecin University
ul. Wielkopolska 15, 70–451 Szczecin, Poland
e-mail: nguyenhtaimathuspl@yahoo.com

Abstract

In this paper we consider an elastic thin film $\omega \subset \mathbb{R}^2$ with the bending moment depending also on the third thickness variable. The effective energy functional defined on the Orlicz-Sobolev space over $\omega$ is described by $\Gamma$-convergence and 3D-2D dimension reduction techniques. Then we prove the existence of minimizers of the film energy functional. These results are proved in the case when the energy density function has the growth prescribed by an Orlicz convex function $M$. Here $M$ is assumed to be non-power-growth-type and to satisfy the conditions $\Delta_2$ and $\nabla_2$.

Keywords: $\Gamma$-convergence, 3D-2D dimension reduction, quasiconvex relaxation, minimizers of variational integral functionals, thin films, elastic membranes, effective energy integral functional, bulk and surface energy, equilibrium states of the film, non-power-growth-type bulk energy density, reflexive Orlicz and Orlicz-Sobolev spaces.

1. Introduction

We consider an elastic thin film as a bounded open subset $\omega \subset \mathbb{R}^2$ with Lipschitz boundary. The set $\Omega_\varepsilon := \omega \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \subset \mathbb{R}^3$ for a small thickness $\varepsilon$ is considered as an elastic cylinder approximate to the film $\omega$. A three-dimensional deformation $U : \Omega_\varepsilon \to \mathbb{R}^3$ defined on the thin cylinder $\Omega_\varepsilon$ has the re-scaled elastic total energy represented by the difference of the re-scaled bulk and re-scaled surface energies

$$E_\varepsilon(U) = \frac{1}{\varepsilon^\alpha} \int_{\Omega_\varepsilon} W(DU) dx - \frac{1}{\varepsilon^\beta} Q_\varepsilon(U),$$

where $W : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is so-called the energy density function satisfying the growth and coercivity conditions

$$\frac{1}{C}(M(\|F\|) - 1) \leq W(F) \leq C(1 + M(\|F\|)) \quad (\forall F \in \mathbb{R}^{3 \times 3})$$

for some $C > 0$. Here $M : \mathbb{R} \to [0, \infty)$ is some Orlicz convex $N$-function.

The purpose of this type of research is to investigate, as the thickness $\varepsilon$ goes to zero, the $\Gamma$-convergence limit of the sequence of these re-scaled energies and to understand the behavior of minimizers subject to appropriate boundary conditions.

The values of exponents $\alpha$ and $\beta$ in the definition (1) are important. It turns out that when $\alpha = 1$, $\beta = 0$ and $M(t) = |t|^p$ ($1 < p < \infty$) the form of the functional (1) leads through the use of $\Gamma$-convergence to the nonlinear membrane theory in the reflexive Sobolev spaces $W^{1,p}$ [26, 27]. It is important to note that the papers [26, 27] by H. Le Dret and A. Raoult published in 1993-1995 contain the first precise convergence results for the re-scaled energy functionals in the nonlinear theory of thin membranes through the use of $\Gamma$-convergence. For the case $\alpha = 1$, $\beta = 1$ and $M(t) = |t|^p$ ($1 < p < \infty$), one has to deal with the additional two and three dimensional bending moment in the nonlinear membrane theory in the reflexive Sobolev spaces $W^{1,p}$, cf. [4] and [5] published in 2004 and 2009, respectively.

Many results through the use of $\Gamma$-convergence were established for other values of the exponents $\alpha$ and $\beta$ in the reflexive Sobolev spaces $W^{1,p}$ (see, e.g., [16, 31]).

We assume that $M$ is of the non-power-growth-type and satisfies the conditions $\Delta_2$ and $\nabla_2$ (that is equivalent to the reflexivity of Orlicz and Orlicz-Sobolev spaces generated by $M$).

In our previous papers (see [24, 25]) we consider the case $\alpha = 1$, $\beta = 0$ and the case $\alpha = 1$, $\beta = 1$ with the 2-dimensional bending moment in the Orlicz-Sobolev space setting $W^{1,M}$. 
In Theorem 1, the effective energy functional for the thin film $\omega$ with the 3-dimensional bending moment is obtained, by $\Gamma$-convergence and 3D-2D dimension reduction techniques applied to the sequence of the re-scaled total energy integral functionals of the elastic cylinders $\Omega_\varepsilon$ as $\varepsilon \to 0$. In Corollary 2, the existence of minimizers of the energy functional for the thin film is established by showing that some sequence of re-scaled minimizers weakly converges in an appropriate Orlicz-Sobolev space to a minimizer of the film energy functional.

In Section 5, we give the proofs of Theorem 1 and Corollary 2. For these proofs we apply also results: for Orlicz convex functions [20, Proposition 4], for the Orlicz-Sobolev spaces [22, Theorem 5, Theorem 7] (cf. [10]), [17, Proposition 2.1], for differentiability properties of the Orlicz-Sobolev functions [3, Lemma 3.1, Lemma 3.2], for the sub-differential operator in Orlicz spaces [35, Lemma 1], for quasiconvex integral functionals and quasiconvexification in the Orlicz-Sobolev space setting [13], for the $L^M$-version [34, Homogenization Theorem 7.1, Remark p. 121] for the Riemann-Lebesgue Lemma.

Examples of Orlicz $N$-functions $M$ with $M \in \Delta_2 \cap \nabla_2$ are $M(t) = |t|^p(\log(1+|t|))^q$, where $p > 1$ if $q > 1$ or $M(t) = |t|^p(\log(1+|t|))^{q_1} \cdot (\log(\log(1+|t|)))^{q_2}$, where $p > 1$ if $q_1, q_2 > 1$. Many other examples of $M$ with $M \in \Delta_2 \cap \nabla_2$ can be found in [23, Theorem 7.1, pp. 58–59] and [28, 29]. Furthermore, the assumption $M \in \Delta_2 \cap \nabla_2$ is indispensable in the regularity study of minimizers of multiple variational integrals with the $M$-growth on Orlicz-Sobolev spaces (see discussions and references for many other concrete examples in [12, 7]).

2. Some terminology and notation

>From now on, unless stated to the contrary, $M: \mathbb{R} \to [0, \infty)$ is assumed to be a non-power-growth-type Orlicz $N$-function (i.e., even convex function satisfying $\lim_{t \to 0} \frac{M(t)}{t^p} = 0$ and $\lim_{t \to +\infty} \frac{M(t)}{t^p} = +\infty$).

We assume $M \in \Delta_2 \cap \nabla_2$. Here the condition $M \in \Delta_2$ means that $M(2t) \leq c M(t)$ (for $t \geq t_0$) for some $t_0 \in [0, \infty)$ and $c \in (0, \infty)$. The condition $\Delta_2$ implies that $M(2t) \leq a(2) + b(2) M(t)$ for all $t \geq 0$, where $b(2) \in (0, +\infty)$ and $a(2) \in [0, +\infty)$. The condition $M \in \nabla_2$ means that $\exists l > 1, \exists t_\ast \in [0, \infty)$ such that $M(t) \leq \frac{l}{2} M(t)$ for all $t \geq t_\ast$.

Let $M^*$ be the complementary (conjugate) Orlicz $N$-function of $M$ defined by $M^*(\tau) := \sup\{t\tau - M(t) : t \in \mathbb{R}\}$. It is known that the condition $M \in \nabla_2$ is equivalent to the condition $M^* \in \Delta_2$.

Denote by $|v|$ the Euclidean norm of $v$ and by $(u, v)$ the scalar product. Given an open bounded subset $G \subset \mathbb{R}^N$ with Lipschitz (e.g., $C^2$-smooth) boundary $\partial G$ equipped with the $(N - 1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$. Denote by $L^M(G, \mathbb{R}^m)$ the Orlicz space of all (equivalent classes of) measurable functions...
$u : G \to \mathbb{R}^m$ equipped with the Luxemburg norm

$$\|u\|_{L^M(G;\mathbb{R}^m)} := \inf \left\{ \lambda > 0 : \int_{\Omega} M(|u(x)|/\lambda) dx \leq 1 \right\}.$$ 

It is known that $M \in \Delta_2 \cap \nabla_2$ is equivalent to the reflexivity of $L^M(G;\mathbb{R}^m)$.

Recall that the Orlicz-Sobolev space $W^{1,M}(G;\mathbb{R}^3)$ is defined as the Banach space of $\mathbb{R}^3$-valued functions $u$ of $L^M(G;\mathbb{R}^3)$ with the Sobolev-Schwartz distributional derivative $Du \in L^M(G;\mathbb{R}^{3 \times N})$ equipped with the norm

$$\|u\|_{W^{1,M}(G;\mathbb{R}^3)} := \|u\|_{L^M(G;\mathbb{R}^3)} + \|Du\|_{L^M(G;\mathbb{R}^{3 \times N})} < \infty.$$ 

The subspace $W^{1,M}_{0}(G;\mathbb{R}^3)$ is defined as the closure in $\|\cdot\|_{W^{1,M}(G;\mathbb{R}^3)}$-norm of the set $C_0^\infty(G;\mathbb{R}^3)$ of $C^\infty$-smooth $\mathbb{R}^3$-valued functions with compact support in $G$. Since $\partial G$ is Lipschitz and $M, M^* \in \Delta_2$, by [15, Theorems 2.1, 2.3] (cf. [21]) there exists the bounded linear trace operator

$$\text{Tr} : W^{1,M}(G;\mathbb{R}^3) \to L^M(\partial G;\mathbb{R}^3)$$

such that: (i) $\text{Tr}(u) = u_{|\partial G}$ ($\forall u \in C^\infty(G)$) and (ii) $u \in W^{1,M}_{0}(G;\mathbb{R}^3)$ if and only if $\text{Tr}(u) = 0$. So, for the simplicity of notation we will write "$u(x) = \varphi(x)$ on $A$" for $u \in W^{1,M}(G;\mathbb{R}^3)$ and $\varphi \in L^M(\partial G;\mathbb{R}^3)$ and $A \subset \partial G$ if $\text{Tr}(u)(x) = \varphi(x)$ for almost every $x \in A$. Due to this reason, we also denote by "$u$ on $A$" for "$\text{Tr}(u)$ on $A$", etc.

Let $(X, \|\cdot\|_{W^{1,M}(G;\mathbb{R}^3)})$ be normed subspace of $W^{1,M}(G;\mathbb{R}^3)$. By [2, Proof of Theorem 3.9] and [19, Proof of Lemma 2.2] for every $\Lambda \in X^*$ there exist elements $h_0, h_1, \ldots, h_N \in L^{M^*}(G;\mathbb{R}^3)$ such that

$$\Lambda(u) = \int_G (h_0, u) dx + \sum_{i=1}^N \int_G \left( h_i, \frac{\partial u}{\partial x_i} \right) dx \quad (u \in X).$$

(3)

Conversely, every functional $\Lambda$ defined by (3) in the case $h_0, h_1, \ldots, h_N \in L^{M^*}(G;\mathbb{R}^3)$, is an element of $X^*$.

3. Setup

Define $I := (-\frac{1}{2}, \frac{1}{2})$, $\Omega := \omega \times I$, $S_\pm := \omega \times \{ \pm \frac{1}{2} \}$, $\Gamma := \partial \omega \times I$, and for each $\varepsilon > 0$, $S^\varepsilon := \omega \times \{ \pm \frac{\varepsilon}{2} \}$, $\Gamma^\varepsilon := \partial \omega \times \varepsilon I$. Greek indexes will be used to distinguish the first two components of a vector, for instance $(x_\alpha)$ and $(x_\alpha, x_3)$, designates $(x_1, x_2)$ and $(x_1, x_2, x_3)$, respectively. We denote by $\mathbb{R}^{3 \times 3}$ and $\mathbb{R}^{3 \times 2}$ the vector spaces of respectively $3 \times 3$ and $3 \times 2$ real-valued matrices. Given $\bar{F} \in \mathbb{R}^{3 \times 2}$ and $b \in \mathbb{R}^3$, denote by $(\bar{F} | b)$ the $3 \times 3$ matrix whose first two columns are those
of $\bar{F}$ and the last column is $b$. By the analogous way, set $e_\alpha := (e_1|e_2) \in \mathbb{R}^{3\times 2}$ where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. Set $D_\alpha U := (\frac{\partial U}{\partial \epsilon_1}, \frac{\partial U}{\partial \epsilon_2})$, $D_3 U := \frac{\partial U}{\partial \epsilon_3}$, $DU := (D_\alpha U|D_3 U)$ for an $\mathbb{R}^3$-valued function $U$. Denote by $C, \tilde{C}$ generic positive constants that may vary from line to line.

Let $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$ be a continuous function satisfying the $M$-growth-type and coercivity conditions:

$$\frac{1}{C}(M(|F|) - 1) \leq W(F) \leq C(1 + M(|F|)) \quad (\forall F \in \mathbb{R}^{3\times 3})$$

for some $C \in (0, \infty)$.

Set $\tilde{\Psi}_\varepsilon := \{U \in W^{1,M}(\Omega; \mathbb{R}^3) : U(\bar{x}) = \bar{x} \text{ on } \Gamma_\varepsilon\}$.

We consider the variational integral functional $\tilde{J}_\varepsilon : \tilde{\Psi}_\varepsilon \to \mathbb{R}$, where $\tilde{J}_\varepsilon(U)$ (the re-scaled total energy of the elastic cylinder $\Omega_\varepsilon$ under a deformation $U : \Omega_\varepsilon \to \mathbb{R}^3$) is represented by the difference of the re-scaled bulk and re-scaled surface energies:

$$\tilde{J}_\varepsilon(U) := \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} W(DU) d\bar{x} - \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} (f_\varepsilon, U) d\bar{x} - \frac{1}{\varepsilon} \int_{\partial \Omega_\varepsilon^+} (\varepsilon g_0 + g, U) d\mathcal{H}^2$$

$$+ \frac{1}{\varepsilon} \int_{\partial \Omega_\varepsilon^-} (\varepsilon g_0 - g, U) d\mathcal{H}^2.$$ (5)

Here, $f_\varepsilon := f(\bar{x}, \frac{\bar{x}}{\varepsilon})$, $f \in L^{M^*}(\Omega; \mathbb{R}^3)$, $g_0^+, g \in L^{M^*}(\omega; \mathbb{R}^3)$ and $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure in $\mathbb{R}^3$. Set $\tilde{\Psi}_0 := \{\bar{u} \in W^{1,M}(\omega; \mathbb{R}^3) : \bar{u}(x_\alpha) = (x_\alpha, 0) \text{ on } \partial \omega\}$.

Let $J_0 : \tilde{\Psi}_0 \times L^M(\Omega; \mathbb{R}^3) \to \mathbb{R}$ be defined by

$$J_0(\bar{u}, \bar{b}) := \int_\omega Q_\infty W(D_\alpha u(x_\alpha)|b(x_\alpha, \cdot)) dx_\alpha - P_0(\bar{u}, \bar{b}),$$

where

$$Q_\infty W(F, \bar{b}) := \sup_{k \in \mathbb{N}} Q_k W(F, \bar{b}) \quad (F \in \mathbb{R}^{3\times 2}, \bar{b} \in L^M(I; \mathbb{R}^3))$$

and

$$Q_k W(F, \bar{b}) := \inf \left\{ \int_Q W(F + D_\alpha \varphi|\lambda D_3 \varphi) dx : \lambda > 0, \varphi \in W^{1,M}(Q; \mathbb{R}^3), \right.$$

$$\varphi(\cdot, x_3) \text{ is } Q \text{ periodic } L^1 \text{ a.e. } x_3 \in I, \left. \int_Q \lambda D_3 \varphi \theta_3 dx - \int_I \bar{b}_3 dx_3 \right\} < \frac{1}{k} (i = 1, \ldots, k) \right\}.$$
for a fixed countable dense family \( \{ \theta_i \}_{i \in \mathbb{N}} \subset L^{M^*}(I; \mathbb{R}^3) \) (here, we assume that \( M \in \Delta_2 \cap \nabla_2 \)) and

\[
P_0(\bar{u}, b) := \int_\omega (\bar{f}, \bar{u}) dx_\alpha + \int_\omega (g_0^+ - g_0^- \bar{u}) dx_\alpha + \int_\Omega (g, b) dx,
\]

with \( \bar{f}(x) := \int_I f(x, x_3) dx_3 \). By (see [23, p. 81]), we may choose \( \{ \theta_i \}_{i \in \mathbb{N}} \) from \( L^\infty(I; \mathbb{R}^3) \). Later on, Proposition 12 shows that \( Q_\infty W \) and \( Q_k W \) are continuous.

4. The formulation of main results

Let \( \mathcal{Z} \) be the space of membrane deformations defined by

\[
\mathcal{Z} = \{ z \in W^{1, M}(\Omega; \mathbb{R}^3) : D_3 z = 0, z(x) = (x_\alpha, 0) \text{ on } \Gamma \}.
\]

Observe that \( \mathcal{Z} \) is canonically isomorphic to \( \mathcal{V}_0 \) [30, Theorem 1.1.3/1]. Let \( \bar{z} \) denote the element of \( \mathcal{V}_0 \) that is associated with \( z \in \mathcal{Z} \) through this isomorphism:

\[
z(x_\alpha, x_3) = \bar{z}(x_\alpha) \text{ a.e.}
\]

Since we want to identify the sequence convergence with the thickness of our domain tending to zero, for simplicity we assume this thickness parameter \( \varepsilon \) takes its values in a sequence \( \varepsilon_n \to 0 \).

**Theorem 1.** Let \( \tilde{J}_\varepsilon \) be defined in (14) and \( J_0 \) be defined in (6). Assume \( M \in \Delta_2 \cap \nabla_2 \). Assume that the continuous function \( W : \mathbb{R}^{3 \times 3} \to \mathbb{R} \) satisfies the hypothesis (4). Let \( \{ U_\varepsilon \} \in \mathcal{W}_\varepsilon \). For each \( \varepsilon > 0 \) and \( \tilde{x} = (\tilde{x}_\alpha, \tilde{x}_3) \in \Omega_\varepsilon \) we associate \( x = (x_\alpha, x_3) := (\tilde{x}_\alpha, \frac{1}{\varepsilon} \tilde{x}_3) \in \Omega \) and we set \( z_\varepsilon(x_\alpha, x_3) := U_\varepsilon(\tilde{x}_\alpha, \tilde{x}_3) \).

Then the sequence \( \tilde{J}_\varepsilon \) converges to \( J_0 \) in the following sense:

(i) (lower bound) if \( z_\varepsilon \rightharpoonup z \) weakly in \( W^{1, M}(\Omega; \mathbb{R}^3) \), \( \| z_\varepsilon \|_{W^{1, M}(\Omega; \mathbb{R}^3)} < +\infty \) and \( z \in \mathcal{Z} \) with \( z(x_\alpha, x_3) = \bar{z}(x_\alpha) \) through the isomorphism (10) and \( \frac{1}{\varepsilon} D_3 z_\varepsilon \rightharpoonup b \) weakly in \( L^M(\Omega; \mathbb{R}^3) \) and \( \| \frac{1}{\varepsilon} D_3 z_\varepsilon \|_{L^M(\Omega; \mathbb{R}^3)} < +\infty \), then

\[
\lim_{\varepsilon \to 0} \inf \tilde{J}_\varepsilon(U_\varepsilon) \geq J_0(\bar{z}, b);
\]

(ii) (upper bound) for every pair \( (\bar{z}, b) \in \mathcal{V}_0 \times L^M(\Omega; \mathbb{R}^3) \), there exists a sequence \( U_\varepsilon \in W^{1, M}(\Omega; \mathbb{R}^3) \) such that \( z_\varepsilon \rightharpoonup z \) weakly in \( W^{1, M}(\Omega; \mathbb{R}^3) \), \( \| z_\varepsilon \|_{W^{1, M}(\Omega; \mathbb{R}^3)} < +\infty \) and \( z \in \mathcal{Z} \) with \( z(x_\alpha, x_3) = \bar{z}(x_\alpha) \) through the isomorphism (10) and \( \frac{1}{\varepsilon} D_3 z_\varepsilon \rightharpoonup b \) weakly in \( L^M(\Omega; \mathbb{R}^3) \) and \( \| \frac{1}{\varepsilon} D_3 z_\varepsilon \|_{L^M(\Omega; \mathbb{R}^3)} < +\infty \) and

\[
\lim_{\varepsilon \to 0} \tilde{J}_\varepsilon(U_\varepsilon) = J_0(\bar{z}, b).
\]
Consider the asymptotic behavior of $U_\varepsilon \in \tilde{\Psi}_\varepsilon$ such that
\begin{equation}
\tilde{J}_\varepsilon(U_\varepsilon) \leq \inf_{U \in \tilde{\Psi}_\varepsilon} \tilde{J}_\varepsilon(U) + \gamma(\varepsilon),
\end{equation}
where $\gamma$ is a positive function such that $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$.

**Corollary 2** (The minimization problem). Assume that $U_\varepsilon \in \tilde{\Psi}_\varepsilon$ satisfies (11). Let the functions $M$, $W$ and $z_\varepsilon$, $\bar{z}$ be such as in Theorem 1. Then:

(i) the sequence $(z_\varepsilon, \frac{1}{\varepsilon} D_3 z_\varepsilon)$ is relatively weakly compact in $W^{1,M}(\Omega; \mathbb{R}^3) \times L^M(\Omega; \mathbb{R}^3)$;

(ii) the set $C_{\text{film}}$ of cluster points of the sequence $(z_\varepsilon, \frac{1}{\varepsilon} D_3 z_\varepsilon)$ in the weak topology is a non-empty subset of $Z \times L^M(\Omega; \mathbb{R}^3)$;

(iii) any point $(z_\infty, b)$ of $C_{\text{film}}$ can be identified with $(\bar{z}_\infty, b) \in \overline{\Psi}_0 \times L^M(\Omega; \mathbb{R}^3)$ by the $3D$-$2D$ dimension reduction isomorphism (10) and $(\bar{z}_\infty, \bar{b})$ is a solution of the minimization problem
\begin{equation}
\inf_{\bar{u} \in \overline{\Psi}_0} \left\{ J_0(\bar{u}, b) : b \in L^M(\Omega; \mathbb{R}^3) \right\}.
\end{equation}

5. The Proofs of Theorem 1 and Corollary 2

We will reformulate Theorem 1 and Corollary 2 by the use of the following equivalent functionals $\tilde{J}_\varepsilon^*$ and $J_0^*$ (see the re-formulation in Theorem 3 and Corollary 4). Define
\begin{equation}
(12) \quad u_{0,\varepsilon}(x) := (x_\alpha, \varepsilon x_3), \quad u_{0,0}(x) := (x_\alpha, 0).
\end{equation}

Notice that after the change of variables as in Theorem 1 with the association
\begin{equation}
(13) \quad x = (x_\alpha, x_3) := \left( \tilde{x}_\alpha, \frac{1}{\varepsilon} \tilde{x}_3 \right), \quad u(x_\alpha, x_3) := U(\tilde{x}_\alpha, \tilde{x}_3),
\end{equation}

and by the Fubini Theorem the re-scaled energy $\tilde{J}_\varepsilon(U)$ in (14) can be rewritten in the equivalent form
\begin{equation}
\begin{aligned}
J_\varepsilon(u) &= \int_{\Omega} W \left( D_\alpha u \left| \frac{1}{\varepsilon} D_3 u \right| \right) dx - \int_{\Omega} (f, u) dx \\
&\quad - \int_{S^+} (g_0^+, u) d\mathcal{H}^2 + \int_{S^-} (g_0^-, u) d\mathcal{H}^2 - \int_{\omega} \left( g, \frac{u^+ - u^-}{\varepsilon} \right) dx_\alpha \\
&= \int_{\Omega} W \left( D_\alpha u \left| \frac{1}{\varepsilon} D_3 u \right| \right) dx - \int_{\Omega} (f, u) dx
\end{aligned}
\end{equation}
\[- \int_{S^+} (g_0^+, u) dH^2 + \int_{S^-} (g_0^-, u) dH^2 - \int_\omega \left( g, \frac{1}{\varepsilon} \int_\Gamma D_3 udx_3 \right) dx_\alpha \]

\[= \int_\Omega W \left( D_\alpha u \left| \frac{1}{\varepsilon} D_3 u \right. \right) dx - \int_\Omega (f, u) dx \]

\[- \int_{S^+} (g_0^+, u) dH^2 + \int_{S^-} (g_0^-, u) dH^2 - \int_\Omega \left( g, \frac{1}{\varepsilon} D_3 u \right) dx,\]

where \( u^\pm(x_\alpha) := \text{Tr}_{S^\pm}(u)(x_\alpha) \) and \( u \) is an element of

\[\Psi_\varepsilon := \{ u \in W^{1,1}_M(\Omega; \mathbb{R}^3) : u(x) = u_{0,\varepsilon}(x) \text{ on } \Gamma \} .\]

In order to individualize this new sequence \( \frac{1}{\varepsilon} D_3 u \) and since the direct consideration of \( J_\varepsilon \) would imply the study involving the weak topology of the Orlicz-Sobolev space \( W^{1,1}_M(\Omega; \mathbb{R}^3) \) which is non-metrizable on unbounded sets, then it is needed to consider the new functional \( \bar{J}_\varepsilon : W^{1,1}_M(\Omega; \mathbb{R}^3) \times L^M(\Omega; \mathbb{R}^3) \to \mathbb{R} \cup \{ +\infty \} \) defined by

\[\bar{J}_\varepsilon(u, b) := \begin{cases} \int_\Omega W \left( e_\alpha + D_\alpha u \left| \frac{1}{\varepsilon} D_3 u \right. \right) dx - P_\varepsilon(u) & \text{if } \frac{1}{\varepsilon} D_3 u = b \\
+\infty & \text{and } u \in \Psi_\varepsilon \\
& \text{otherwise,} \end{cases}\]

where

\[P_\varepsilon(u) := \int_\Omega (f, u) dx - \int_{S^+} (g_0^+, u) dH^2 + \int_{S^-} (g_0^-, u) dH^2 + \int_\Omega \left( g, \frac{1}{\varepsilon} D_3 u \right) dx .\]

Observe that the re-scaled displacement \( v = u - u_{0,\varepsilon} \) belongs to the set

\[V = W^{1,1}_M(\Gamma; \mathbb{R}^3) := \{ v \in W^{1,1}_M(\Omega; \mathbb{R}^3) : v(x) = 0 \text{ on } \Gamma \} \]

and

\[J_\varepsilon(v + u_{0,\varepsilon}) = \int_\Omega W \left( e_\alpha + D_\alpha v \left| e_3 + \frac{1}{\varepsilon} D_3 v \right. \right) dx - \int_\Omega (f, v + u_{0,\varepsilon}) dx \]

\[- \int_{S^+} (g_0^+, v + u_{0,\varepsilon}) dH^2 + \int_{S^-} (g_0^-, v + u_{0,\varepsilon}) dH^2 - \int_\Omega \left( g, \frac{1}{\varepsilon} D_3 v + e_3 \right) dx.\]

Define \( \bar{J}^*_\varepsilon : W^{1,1}_M(\Omega; \mathbb{R}^3) \times L^M(\Omega; \mathbb{R}^3) \to \mathbb{R} \cup \{ +\infty \} \) by

\[\bar{J}^*_\varepsilon(v, b) := \begin{cases} \int_\Omega W \left( e_\alpha + D_\alpha v \left| \frac{1}{\varepsilon} D_3 v + e_3 \right. \right) dx - P_\varepsilon(v + u_{0,\varepsilon}) & \text{if } \frac{1}{\varepsilon} D_3 v + e_3 = b \\
+\infty & \text{and } v \in V \\
& \text{otherwise} .\end{cases}\]
Let $V$ be the space of membrane displacements defined by

$$V = \{ v \in W^{1,M}(\Omega; \mathbb{R}^3) : D_3v = 0, v(x) = 0 \text{ on } \Gamma \} \subset V.$$ 

Similarly as in (9)–(10), $V$ is canonically isomorphic to $W_0^{1,M}(\omega; \mathbb{R}^3)$ [30, Theorem 1.1.3/1]. Let $\bar{v}$ denote the element of $W_0^{1,M}(\omega; \mathbb{R}^3)$ that is associated with $v \in V$ through this isomorphism:

$$v(x_\alpha, x_3) = \bar{v}(x_\alpha) \text{ a.e.}$$

Analogously for $v \in V$ and $b \in L^M(\Omega; \mathbb{R}^3)$ define the functional

$$J_0^\ast(v, b) := \int_\omega Q_{\infty} W(e_\alpha + D_\alpha \bar{v}(x_\alpha)|b(x_\alpha, \cdot) - e_3|dx_\alpha - P_0(\bar{v} + u_{0,0}, b + e_3).$$

In this notion we have for $U_\varepsilon \in \tilde{\Psi}_\varepsilon$

$$\tilde{J}_\varepsilon(U_\varepsilon) = J_\varepsilon(u_\varepsilon) = J_\varepsilon(v_\varepsilon + u_{0,\varepsilon}),$$

where $u_\varepsilon \in \Psi_\varepsilon$, $v_\varepsilon \in V$ with $u_\varepsilon = v_\varepsilon + u_{0,\varepsilon}$ and

$$J_0(\bar{z}, b) = J_0^\ast(v, b) \quad (v \in V, \bar{z} = \bar{v} + u_{0,0} \in \overline{V}_0).$$

Recall [9] that a sequence of functions $I_\varepsilon$ from a topological space $X$ to $\mathbb{R}$ is said to $\Gamma$-converge to $I_0$ for the topology of $X$ if the following conditions are satisfied for all $x \in X$:

$$\begin{cases} \forall x_\varepsilon \to x, & I_0(x) \leq \liminf I_\varepsilon(x_\varepsilon), \\ \exists y_\varepsilon \to y, & I_\varepsilon(y_\varepsilon) \to I_0(y). \end{cases}$$

**Theorem 3.** Let $\tilde{J}_\varepsilon^\ast$ be defined in (16) and $J_0^\ast$ be defined in (19). Assume $M \in \Delta_2 \cap \nabla_2$. Suppose that the continuous function $W: \mathbb{R}^{3 \times 3} \to \mathbb{R}$ satisfies the hypothesis (4). Then the sequence $\tilde{J}_\varepsilon^\ast$ $\Gamma$-converges to $J_0^\ast$ in the weak topology of $W^{1,M}(\Omega; \mathbb{R}^3) \times L^M(\Omega; \mathbb{R}^3)$, as $\varepsilon \to 0$.

Consider the asymptotic behavior of $u_\varepsilon \in \Psi_\varepsilon$ such that

$$J_\varepsilon(u_\varepsilon) \leq \inf_{u \in \Psi_\varepsilon} J_\varepsilon(u) + \gamma(\varepsilon),$$

where $\gamma$ is a positive function such that $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$.

**Corollary 4** (The minimization problem). Assume that $u_\varepsilon \in \Psi_\varepsilon$ satisfies (21). Let the functions $M$ and $W$ be such as in Theorem 3. Then:

(i) the sequence $(u_\varepsilon, \varepsilon D_3u_\varepsilon)$ is relatively weakly compact in $W^{1,M}(\Omega; \mathbb{R}^3) \times L^M(\Omega; \mathbb{R}^3)$.
(ii) the set $\mathcal{C}_{\text{film}}$ of cluster points of the sequence $(u_\varepsilon, \frac{1}{\varepsilon} D_3 u_\varepsilon)$ in the weak topology is a non-empty subset of $Z \times L^M(\Omega; \mathbb{R}^3)$;

(iii) any point $(u_\infty, b)$ of $\mathcal{C}_{\text{film}}$ can be identified with $(\bar{u}_\infty, b) \in \overline{\Psi}_0 \times L^M(\Omega; \mathbb{R}^3)$ by the $3D$-$2D$ dimension reduction isomorphism (10) and $(\bar{u}_\infty, b)$ is a solution of the minimization problem

$$\inf_{\bar{u} \in \overline{\Psi}_0} \left\{ J_0(\bar{u}, b) : b \in L^M(\Omega; \mathbb{R}^3) \right\}.$$  

We start the proofs of Theorem 3 and Corollary 4, with Lemmas 5–6.

We consider the following condition (22):

$$\exists i(M) \in [1, \infty), \exists c \in (0, \infty) \text{ such that } M(at) \leq c a^{i(M)} M(t) \quad (\forall t \geq 0, \forall a \leq 1).$$

The condition (22) is equivalent to the condition

$$\exists i(M) \in [1, \infty), \exists c \in (0, \infty) \text{ such that } \frac{1}{c} b^{i(M)} M(s) \leq M(bs) \quad (\forall s \geq 0, \forall b \geq 1).$$

Lemma 5 is a re-formulation of a part of [20, Proposition 4] (see the explanation in our previous paper [24, Lemma 4.3, pp. 592–593]).

**Lemma 5.** Assume the dual Orlicz $N$-function $M^*$ satisfies the condition $\Delta^g_2$, i.e., $M^*(2\tau) \leq K M^*(\tau)$ for all $\tau \in [0, \infty)$ and for some $K \in (0, \infty)$. Then $M$ satisfies the condition (22) for some $i(M) \in (1, \infty)$.

**Lemma 6** (compactness lemma). Let $M$ and $W$ be such as in Theorem 3. Let $v_\varepsilon \in W^{1,M}(\Omega; \mathbb{R}^3)$ and $b_\varepsilon \in L^M(\Omega; \mathbb{R}^3)$ be a sequence such that

$$\sup_{\varepsilon \in (0,1)} J^*_\varepsilon(v_\varepsilon, b_\varepsilon) \leq d < +\infty.$$  

Then there exists $\tilde{d}_1 > 0$ and $\tilde{d}_2 > 0$ such that:

(1)

$$\sup_{\varepsilon \in (0,1)} \|v_\varepsilon\|_{W^{1,M}(\Omega; \mathbb{R}^3)} \leq \tilde{d}_1 < +\infty$$

and

(2)

$$\sup_{\varepsilon \in (0,1)} \left\| \frac{1}{\varepsilon} D_3 v_\varepsilon \right\|_{L^M(\Omega; \mathbb{R}^3)} \leq \tilde{d}_2 < +\infty$$

and the sequence $(v_\varepsilon, \frac{1}{\varepsilon} D_3 v_\varepsilon)$ is relatively weakly compact in $W^{1,M}(\Omega; \mathbb{R}^3) \times L^M(\Omega; \mathbb{R}^3)$;
(ii) the set of cluster points of the sequence \((v_\varepsilon, \frac{1}{\varepsilon} D_3 v_\varepsilon)\) in the weak topology of 
\(W^{1,M}(\Omega; \mathbb{R}^3) \times L^M(\Omega; \mathbb{R}^3)\) is a non-empty subset of \(\mathcal{W} \times L^M(\Omega; \mathbb{R}^3)\).

**Proof.** We divide the proof into Steps 6.1–6.2, where in Step 6.2 we assume additionally \(M^* \in \Delta^2\).

**Step 6.1.** By (24) and (16) for \(J^*_v\), \(v_\varepsilon \in \mathcal{W}\) for all \(\varepsilon > 0\). Denote \(u_\varepsilon = v_\varepsilon + u_{0,\varepsilon}\). We claim that
\[
\int_{\Omega} M\left(\left|\frac{D_2 u_\varepsilon}{\varepsilon}\right| \left|\frac{D_3 u_\varepsilon}{\varepsilon}\right|\right) dx \leq C_1 + C_1 \left(\|f\|_{C^1(\Omega; \mathbb{R}^3)} + \|g\|_{L^M(\Omega; \mathbb{R}^3)}\right)
\]
(27)
\[
+ C_1 \left(\|g\|_{L^M(\omega; \mathbb{R}^3)} + \|g\|_{L^M(\partial \Omega; \mathbb{R}^3)}\right) \|D u_\varepsilon\|_{C^1(\Omega; \mathbb{R}^3)}
\]
for some \(C_1 \in (0, +\infty)\) and for all \(\varepsilon \in (0, 1)\). Here \(\|\cdot\|_{C^1} := \|N^+ + N^-\), where \(N^+\) (resp., \(N^-\)) denotes the operator norm of the linear trace operator \(\text{Tr} : W^{1,M}(\Omega; \mathbb{R}^3) \to L^M(S^+; \mathbb{R}^3)\) (resp., \(\text{Tr} : W^{1,M}(\Omega; \mathbb{R}^3) \to L^M(S^-; \mathbb{R}^3)\)).

For this, by the coercivity condition (4) together with (24) and Fubini Theorem, we infer that
\[
\frac{1}{C} \left(\int_{\Omega} M\left(\left|\frac{D_2 u_\varepsilon}{\varepsilon}\right| \left|\frac{D_3 u_\varepsilon}{\varepsilon}\right|\right)^\alpha dx \leq d + \int (f, u_\varepsilon) dx + \int S_+(g_0^+, u_\varepsilon) d\mathcal{H}^2 + \int S_-(g_0^-, u_\varepsilon) d\mathcal{H}^2 + \int (g_\varepsilon, \frac{1}{\varepsilon} D_3 u_\varepsilon) dx.
\]
By the generalized Hölder inequality (see, e.g., [33, Theorems 13.13, 13.11], [23]), we deduce that
\[
\frac{1}{C} \left(\int_{\Omega} M\left(\left|\frac{D_2 u_\varepsilon}{\varepsilon}\right| \left|\frac{D_3 u_\varepsilon}{\varepsilon}\right|\right) dx \leq d + 2\|f\|_{L^M(\Omega; \mathbb{R}^3)} \|u_\varepsilon\|_{C^1(\Omega; \mathbb{R}^3)}
\]
\[
+ 2 \left(\|g_0^+\|_{L^M(S^+; \mathbb{R}^3)} \|u_\varepsilon^+\|_{L^M(S^+; \mathbb{R}^3)} + \|g_0^-\|_{L^M(S^-; \mathbb{R}^3)} \|u_\varepsilon^-\|_{L^M(S^-; \mathbb{R}^3)}\right)
\]
(28)
\[
+ 2 \left(\|g_\varepsilon\|_{L^M(\omega; \mathbb{R}^3)} + \|g_0^\varepsilon\|_{L^M(\partial \Omega; \mathbb{R}^3)}\right) \|\text{Tr} (u_\varepsilon)\|_{L^M(\Omega; \mathbb{R}^3)}
\]
\[
+ \|D u_\varepsilon\|_{C^1(\Omega; \mathbb{R}^3)} + 2\|g\|_{L^M(\omega; \mathbb{R}^3)} \|\frac{1}{\varepsilon} D_3 u_\varepsilon\|_{L^M(\Omega; \mathbb{R}^3)}.
\]
By the $W^{1,M}$-generalization (see [22, Theorem 5 and 7] together with [10, Theorem 3.9], [18, Lemma 4.14], [17, Proposition 2.1]) for the Poincaré-Sobolev-type inequality (see [32, Theorem 3.6.4]), there exists $C \in (0, \infty)$ such that

$$\|u_\varepsilon\|_{L^M(\Omega; \mathbb{R}^3)} \leq \tilde{C} \left( \|Du_\varepsilon\|_{L^M(\Omega; \mathbb{R}^{3 \times 3})} + \int_\Gamma |u_\varepsilon| d\mathcal{H}^2 \right)$$

$$= \tilde{C} \left( \|Du_\varepsilon\|_{L^M(\Omega; \mathbb{R}^{3 \times 3})} + \int_\Gamma |u_{0,\varepsilon}| d\mathcal{H}^2 \right)$$

$$\leq \tilde{C} \left( \|Du_\varepsilon\|_{L^M(\Omega; \mathbb{R}^{3 \times 3})} + \mathcal{H}^2(\Gamma) \sup_{x \in \Omega} |x| \right) < \infty \quad (\forall \varepsilon \in (0, 1)).$$

Then (28)–(29) imply (27).

**Step 6.2.** By the additional assumption $M^* \in \Delta_2^{glob}$, we may apply Lemma 5, and so $M$ satisfies the condition (22) for some $i(M) \in (1, \infty)$.

We claim that

$$\|Du_\varepsilon\|_{L^M(\Omega; \mathbb{R}^{3 \times 3})} \leq C_2 < \infty \quad (\forall \varepsilon \in (0, 1)), \tag{30}$$

$$\|u_\varepsilon\|_{L^M(\Omega; \mathbb{R}^3)} \leq C_3 < \infty \quad (\forall \varepsilon \in (0, 1)), \tag{31}$$

$$\left\| \frac{1}{\varepsilon} D_3 u_\varepsilon \right\|_{L^M(\Omega; \mathbb{R}^3)} \leq C_4 < \infty \quad (\forall \varepsilon \in (0, 1)), \tag{32}$$

$$\int_\Omega M \left( \left| \left( \frac{D_\alpha u_\varepsilon}{\varepsilon} \right) \right| \right) dx \leq C_5 < \infty \quad (\forall \varepsilon \in (0, 1)) \tag{33}$$

for some $C_2, C_3, C_4, C_5$.

For this, by (27) we infer that

$$\frac{1}{1 + \|Du_\varepsilon\|_{L^M(\Omega; \mathbb{R}^{3 \times 3})} + \|\frac{1}{\varepsilon} D_3 u_\varepsilon\|_{L^M(\Omega; \mathbb{R}^3)}} \int_\Omega M \left( \left| \left( \frac{D_\alpha u_\varepsilon}{\varepsilon} \right) \right| \right) dx$$

$$\leq C_6 < \infty \tag{34}$$

for all $\varepsilon \in (0, 1)$ and for some $C_6$.

Consider the case when $\|Du_\varepsilon\|_{L^M(\Omega; \mathbb{R}^{3 \times 3})}/2 \geq 1 > 0$ and $\|\frac{1}{\varepsilon} D_3 u_\varepsilon\|_{L^M(\Omega; \mathbb{R}^3)}/2 \geq 1 > 0$. Since $0 < \frac{\|Du_\varepsilon\|_{L^M(\Omega; \mathbb{R}^{3 \times 3})}}{2} < \|Du_\varepsilon\|_{L^M(\Omega; \mathbb{R}^{3 \times 3})}$ and $0 < \frac{\|\frac{1}{\varepsilon} D_3 u_\varepsilon\|_{L^M(\Omega; \mathbb{R}^3)}}{2} < \|\frac{1}{\varepsilon} D_3 u_\varepsilon\|_{L^M(\Omega; \mathbb{R}^3)}.$
\[ \| \frac{1}{\varepsilon} D_3 u_\varepsilon \|_{L^M(\Omega;\mathbb{R}^3)} \] by the definition of the Luxemburg norm and by (22), we deduce that

\[ 1 < \int\limits_\Omega M \left( \frac{|D u_\varepsilon|}{\|D u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3 \times \mathbb{R}^3)}/2} \right) dx \]

\[ \leq \left( \frac{2}{\|D u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3 \times \mathbb{R}^3)}} \right)^{i(M)} \int\limits_\Omega M(|D u_\varepsilon|) dx \quad (\forall \varepsilon \in (0, 1)) \]

and

\[ 1 < \int\limits_\Omega M \left( \frac{1}{\varepsilon} D_3 u_\varepsilon \right) dx \]

\[ \leq \left( \frac{2}{\|1/\varepsilon D_3 u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)}} \right)^{i(M)} \int\limits_\Omega M\left( \frac{1}{\varepsilon} D_3 u_\varepsilon \right) dx \quad (\forall \varepsilon \in (0, 1)). \]

Obviously

\[ \int\limits_\Omega M\left( \left| D u_\varepsilon \right| \right) dx + \int\limits_\Omega M\left( \left| 1/\varepsilon D_3 u_\varepsilon \right| \right) dx \]

\[ \leq 2 \int\limits_\Omega M\left( \left| D_\alpha u_\varepsilon \left| 1/\varepsilon D_3 u_\varepsilon \right| \right) dx \quad (\forall \varepsilon \in (0, 1)). \]

Therefore, (34), (35)–(36) and (37) implies

\[ A \left( \|D u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3 \times \mathbb{R}^3)}, \|1/\varepsilon D_3 u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} \right) \leq C_6 < \infty \]

whenever \( \|D u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3 \times \mathbb{R}^3)} \geq 2 \) and \( \|1/\varepsilon D_3 u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} \geq 2 \). Here

\[ A(s,t) := \frac{1}{2} \cdot \frac{s^{i(M)} + t^{i(M)}}{2^{i(M)}(1 + s + t)}. \]

Since \( i(M) > 1 \), \( A(s, 2) \rightarrow +\infty \) as \( s \rightarrow +\infty \) and \( A(2, t) \rightarrow +\infty \) as \( t \rightarrow +\infty \) and so there exists \( C_7, C_8 \in (0, \infty) \) such that \( A(s, 2) > C_6 \) \( (\forall s > C_7) \) and \( A(2, t) > C_6 \) \( (\forall t > C_8) \). Hence, (38) implies the claims (30) and (32), where \( C_2 = C_4 := \max\{C_7, C_8, 2\} \). By (29) and (27) we deduce the claims (31) and (33).

The remaining steps of the proof are analogous to Steps 4.3–4.6 in our previous paper [24]. The arguments of these Steps allow to consider also the general case of \( M^* \).
Remind that the quasiconvex envelope $Q_g: \mathbb{R}^{m \times n} \to \mathbb{R}$ of a continuous function $g: \mathbb{R}^{m \times n} \to \mathbb{R}$ is defined (see, e.g., [8, Theorem 6.9]) by

$$Q_g(E) := \inf \left\{ \frac{1}{\text{meas}(B)} \int_B g(E + D\varphi) \, dx : \varphi \in C_0^\infty(B; \mathbb{R}^m) \right\}$$

for all $E \in \mathbb{R}^{m \times n}$ where $B$ is the open unit ball of $\mathbb{R}^n$.

We recall that a sequence $w_n \in L^1(G; \mathbb{R}^d)$ is said $L^1$-equi-integrable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_E |w_n| \, dx < \varepsilon$ whenever $E \subset G$ with $\mathcal{L}^n(E) < \delta$, where $G \subset \mathbb{R}^N$.

The next lemma is the direct $W^{1,M}$-generalization in the case of $M \in \Delta_2 \cap \nabla_2$ of the Fonseca-Müller-Pedregal Decomposition Lemma in the Sobolev $W^{1,p}$-space [14].

**Lemma 7** (decomposition lemma). Assume $M \in \Delta_2 \cap \nabla_2$. Let $G$ be an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary. Let $w_n \in W^{1,M}(G; \mathbb{R}^d)$ be such that $w_n \rightharpoonup w_0$ weakly in $W^{1,M}(G; \mathbb{R}^d)$. Then there exists a subsequence of $w_n$ (not relabelled) and a sequence $z_n \in W^{1,M}(G; \mathbb{R}^d)$ such that $z_n \rightarrow w_0$ weakly in $W^{1,M}(G; \mathbb{R}^d)$, $z_n = w_0$ in a neighborhood of $\partial G$,

$$\mathcal{L}^N(\{ x \in G : w_n(x) \neq z_n(x), Dw_n(x) \neq Dz_n(x) \}) \rightarrow 0,$$

as $n \rightarrow +\infty$, and the sequence $M(|Dz_n|)$ is $L^1$-equi-integrable on $G$.

**Proposition 8.** Let $Q_\infty W$ be defined by (7) and let $W: \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ be a continuous function satisfying the hypothesis (4). Then

$$(39) \quad Q_\infty QW(\bar{F}|b) = Q_\infty W(\bar{F}|b),$$

where $QW$ denotes the quasiconvex envelope of $W$.

Proof of Proposition 8 is analogous to the proof of Proposition 2.6 in [5]. It is enough to apply $W^{1,M}$-generalization in [13, Theorem 3.1] for the Acerbi-Fusco weak l.s.c. $W^{1,p}$-theorem [1, Theorem II.5] and Decomposition Lemma 7.

Let $A(\omega)$ be a family of all open subsets of $\omega$. According to (15) define the functional $E_\varepsilon: W^{1,M}(\Omega; \mathbb{R}^3) \times L^M(\Omega; \mathbb{R}^3) \times A(\omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$E_\varepsilon(u, b, A) = \left\{ \begin{array}{ll} \int_{A \times I} W(D\alpha u |\frac{1}{\varepsilon}D_3u) \, dx & \text{if } \frac{1}{\varepsilon}D_3u = b \text{ and } u \in \Psi_\varepsilon \\ +\infty & \text{otherwise.} \end{array} \right.$$ 

Denote by $E_0: \mathcal{Z} \times L^M(\Omega; \mathbb{R}^3) \times A(\omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ the $\Gamma$- lower limit (see [9]) of $E_\varepsilon$, i.e.,
Lemma 11. \( E_0(u, b, A) := \inf \left\{ \liminf_{n \to +\infty} \int_{A \times I} W(D_\alpha u_n|\lambda_n D_3 u_n) \, dx : u_n \rightharpoonup u \text{ weakly in } W^{1, M}(A \times I; \mathbb{R}^3), \lambda_n D_3 u_n \rightharpoonup b \text{ weakly in } L^M(A \times I; \mathbb{R}^3) \right\} \),

where \( \lambda_n := (\varepsilon_n)^{-1} \). Later on, we say that \( u_n \to u \) in \( L^M_{\text{loc}}(A \times I; \mathbb{R}^3) \) if for any \( D \subset \subset A \) we have \( u_n \to u \) in \( L^M(D \times I; \mathbb{R}^3) \)-norm.

**Lemma 9.** Let the functions \( M \) and \( W \) be such as in Theorem 3 and \( E_0 \) be defined by (40). Then for any sequence \( \lambda_n \to +\infty \), there exists a subsequence \( \lambda_{n_k} \) such that for each \( (u, b) \in \mathcal{Z} \times L^M(\Omega; \mathbb{R}^3) \), the set function \( E_0(u, b, \cdot) \) is a trace of a Radon measure, absolutely continuous with respect to the 2-dimensional Lebesgue measure.

**Lemma 10.** Let the functions \( M \) and \( W \) be such as in Theorem 3. Let \( A \in \mathcal{A}(\omega), L \in \mathbb{R}, u \in \mathcal{Z} \) and consider a subsequence \( u_{n_k} \in W^{1, M}(A \times I; \mathbb{R}^3) \) and \( \lambda_{n_k} \in \mathbb{R} \) such that \( \lambda_{n_k} \to +\infty \), \( u_{n_k} \rightharpoonup u \) in \( L^M_{\text{loc}}(A \times I; \mathbb{R}^3) \)-norm, \( \lambda_{n_k} D_3 u_{n_k} \rightharpoonup b \) weakly in \( L^M(A \times I; \mathbb{R}^3) \) and

\[
\lim_{n \to +\infty} \int_{A \times I} W(D_\alpha u_n|\lambda_n D_3 u_n) \, dx = L.
\]

Then there exists a subsequence \( \lambda_{n_k} \) of \( \lambda_n \) and a sequence \( \tilde{u}_k \in W^{1, M}(A \times I; \mathbb{R}^3) \) such that \( \tilde{u}_k = u \) on \( \Theta_k(\partial A) \times I \) for some neighborhood \( \Theta_k(\partial A) \), \( \tilde{u}_k \rightharpoonup u \) in \( L^M_{\text{loc}}(A \times I; \mathbb{R}^3) \)-norm, \( \lambda_{n_k} D_3 \tilde{u}_k \rightharpoonup b \) weakly in \( L^M(A \times I; \mathbb{R}^3) \) and

\[
\limsup_{k \to +\infty} \int_{A \times I} W(D_\alpha \tilde{u}_k|\lambda_{n_k} D_3 \tilde{u}_k) \, dx \leq L.
\]

The proofs of Lemma 9 and Lemma 10 are analogous to the proofs of Lemma 2.1 and Lemma 2.2 in [4].

**Lemma 11.** The infimum in (40) for \( E_0 \) remains unchanged if we replace \( W \) by its quasiconvex envelope \( \mathcal{Q}W \).

Proof of Lemma 11 is analogous to the proof of Proposition 2.7 in [5]. It is enough to apply \( W^{1, M} \)-generalization in [13, Theorem 3.1] for the Acerbi-Fusco weak l.s.c. \( W^{1, p} \)-theorem [1, Theorem II.5] and the fact that embedding \( W^{1, M}(A \times I; \mathbb{R}^3) \hookrightarrow L^M_{\text{loc}}(A \times I; \mathbb{R}^3) \) is compact (see Donaldson-Trudinger [10, Theorem 3.9] together with Gossez [18, Proposition 4.3]).

Notice that by Proposition 8 and Lemma 11 we may assume without loss of generality that \( W \) is quasiconvex. Therefore by the hypothesis (4), \( M \in \Delta_2 \) together with Focardi [13, Proposition 3.2] \( W \) satisfies

\[
|W(\xi_1) - W(\xi_2)| \leq C(1 + h(1 + |\xi_1| + |\xi_2|))|\xi_1 - \xi_2|
\]
for some $C \in (0, +\infty)$ and for all $\xi_1, \xi_2 \in \mathbb{R}^{3 \times 3}$, where $h$ denotes the right derivative of $M$.

**Proposition 12.** Assume that a quasiconvex function $W : \mathbb{R}^{3 \times 3} \to \mathbb{R}$ satisfies the hypothesis (4) and $M \in \Delta_2 \cap \nabla_2$. Then the functions

$$
(\bar{F}, b) \mapsto Q_k W(\bar{F}|b) \in \mathbb{R}, \quad (\bar{F}, b) \mapsto Q_{\infty} W(\bar{F}|b) \in \mathbb{R}
$$

are continuous on $\mathbb{R}^{3 \times 2} \times L^M(I; \mathbb{R}^3)$.

**Proof.** Let $\lambda > 0$ and $k \in \mathbb{N}$ be fixed and define

$$
Q_k^\lambda W(\bar{F}, b) := \inf \left\{ \int_Q W(\bar{F} + D_\alpha \varphi |\lambda D_3 \varphi) dx : \varphi \in W^{1,M}(Q; \mathbb{R}^3), \varphi(\cdot, x_3) \right\}.
$$

Let $Q^\lambda_k W(\bar{F}, b) = Q^\lambda_{\infty} W(\bar{F}, b)$ be $Q'$ periodic $L^1$ a.e. $x_3 \in I$, $\int_Q \lambda D_3 \varphi_i dx - \int_I b \theta_i dx_3 < \frac{1}{k} (i = 1, \ldots, k)$.

Let $(\bar{F}, b), (\bar{F}', b') \in \mathbb{R}^{3 \times 2} \times L^M(I; \mathbb{R}^3)$. For any infimizing sequence $\{\psi_n\}$ in the definition of $Q_k^\lambda W(\bar{F}, b)$ consider the sequence $\psi_n := \varphi_n + \frac{\int_Q \lambda D_3 \psi_n dx}{\lambda}$. By (4) and by considering the function $\varphi := \frac{\int_Q \lambda D_3 \psi_n dx}{\lambda}$ in the definition (43), we obtain that

$$
-\frac{1}{C} L^3(Q) \leq Q^\lambda_k W(\bar{F}, b) \leq \int_I W(\bar{F}, b) dx_3.
$$

Hence we may assume that

$$
\int_Q W(\bar{F} + D_\alpha \varphi_n |\lambda D_3 \varphi_n) dx \leq Q^\lambda_k W(\bar{F}, b) + 1
$$

$$
\leq \int_I W(\bar{F}, b) dx_3 + 1.
$$

Since

$$
D_\alpha \psi_n = D_\alpha \varphi_n, \quad D_3 \psi_n = D_3 \varphi_n + \frac{b' - b}{\lambda},
$$

then

$$
\left| \int_Q \lambda D_3 \psi_n \theta_i dx - \int_I b' \theta_i dx_3 \right| = \left| \int_Q \lambda D_3 \varphi_n \theta_i dx - \int_I b \theta_i dx_3 \right| < \frac{1}{k} (i = 1, \ldots, k),
$$

which implies that $\{\psi_n\}$ is admissible for the definition of $Q^\lambda_k W(\bar{F}', b')$. Observe that $\int_Q M \left( \frac{|a(x_3)|}{\alpha} \right) dx = \int_I M \left( \frac{|a(x_3)|}{\alpha} \right) dx_3$ for $a \in L^M(I)$, and so $\|a\|_{L^M(Q)} = \|a\|_{L^M(I)}$ follows. By [13, Proposition 3.2] the quasiconvex function $W$ satisfies (41). Thus by the Hölder inequality in $L^M$-norm [23], we deduce that
By the coercivity condition (4), (44) implies that

\[
\int_Q W(\bar{F}' + D_\alpha \psi_n | \lambda D_3 \psi_n) dx - \int_Q W(\bar{F} + D_\alpha \varphi_n | \lambda D_3 \varphi_n) dx \leq C \int_Q (1 + h(1 + |(\bar{F}' + D_\alpha \psi_n | \lambda D_3 \psi_n)| + |(\bar{F} + D_\alpha \varphi_n | \lambda D_3 \varphi_n)|) \\
\cdot |(\bar{F}' + D_\alpha \psi_n | \lambda D_3 \psi_n) - (\bar{F} + D_\alpha \varphi_n | \lambda D_3 \varphi_n)|) dx
\]

(45)

\[
\leq 2C \|1 + h(1 + |(\bar{F}' + D_\alpha \psi_n | \lambda D_3 \psi_n)| + |(\bar{F} + D_\alpha \varphi_n | \lambda D_3 \varphi_n)|)\|_{L^M(Q)} \\
\cdot \|\bar{F}' - \bar{F}\|_{L^M(I)} + \|b'(\cdot) - b(\cdot)\|_{L^M(I)}
\]

and so by using the Luxemburg norm, we obtain that

(46)

\[
\sup_n \int_Q M(|(\bar{F}' + D_\alpha \varphi_n | \lambda D_3 \varphi_n)|) dx \leq C_2 \left( \int_I W(\bar{F}, b) dx_3 + 1 \right) < +\infty,
\]

and so by using the Luxemburg norm, we obtain that

\[
\sup_n \|M(|(\bar{F}' + D_\alpha \varphi_n | \lambda D_3 \varphi_n)|)\|_{L^M(Q)} \leq C_2 \left( \int_I W(\bar{F}, b) dx_3 + 1 \right) + 1,
\]

\[
\|1 + |(\bar{F}' + D_\alpha \varphi_n | \lambda D_3 \varphi_n + b'(\cdot) - b(\cdot))| + |(\bar{F} + D_\alpha \varphi_n | \lambda D_3 \varphi_n)|\|_{L^M(I)}
\]

(47)

\[
\leq C_3 \left( 1 + |\bar{F}'| + |\bar{F}| + \|b'(\cdot)\|_{L^M(I)} + \|b(\cdot)\|_{L^M(I)} \right) \\
+ 2C_2 \left( \int_I W(\bar{F}, b) dx_3 + 1 \right) + 2 =: \tilde{C}_4(\bar{F}, b, \bar{F}', b'),
\]

where \( C_3 := \|1\|_{L^M(Q)} + 1 \). By the Pluciennik-Tian-Wang lemma (see [35, Lemma 1]) for \( M \in \Delta_2 \), there exists a function \( r: [0, +\infty) \rightarrow [0, +\infty) \) such that \( \|\tilde{M}\|_{L^M(Q)} \leq a \Rightarrow \|h(|\tilde{M}|)\|_{L^M(Q)} \leq r(a) \). Define

(48)

\[
r_M[a] = \sup \left\{ \|h(|\tilde{M}|)\|_{L^M(Q)} : \|\tilde{M}\|_{L^M(Q)} \leq a \right\}.
\]

Then \( 0 \leq r_M[a] \leq r(a) < +\infty \) and \( r_M \) is nondecreasing. Hence (47) and (45) imply that
where $C_5 := \|1\|_{L^\infty(M^*(Q))}$.

By the upper bound condition (4) and $M \in \Delta_2$,

\begin{equation}
\left| \int_Q W(\bar{F}' + D_\alpha \psi_n | \lambda D_3 \psi_n)dx - \int_Q W(\bar{F} + D_\alpha \varphi_n | \lambda D_3 \varphi_n)dx \right|
\leq 2C(C_5 + r_M[\tilde{C}_4(\bar{F}, b, \bar{F}', b']) \cdot (|\bar{F}' - \bar{F}| + \|b' - b\|_{L^M(I)}) < +\infty,
\end{equation}

and for all $\bar{F}, \bar{F}', b, b'$. Here

\begin{equation}
q_M(|b|) := \int_I M(\|b(x_3)\|)dx_3, \quad q_M(|b'|) := \int_I M(\|b'(x_3)\|)dx_3.
\end{equation}

Hence (50) and (49) and the definition of $\tilde{C}_4(\bar{F}, b, \bar{F}', b')$ in (47) imply that

\begin{equation}
\left| \int_Q W(\bar{F}' + D_\alpha \psi_n | \lambda D_3 \psi_n)dx - \int_Q W(\bar{F} + D_\alpha \varphi_n | \lambda D_3 \varphi_n)dx \right|
\leq \tilde{C}_7(\bar{F}, b, \bar{F}', b') \cdot (|\bar{F}' - \bar{F}| + \|b' - b\|_{L^M(I)}) < +\infty,
\end{equation}

where

\begin{equation}
\tilde{C}_7(\bar{F}, b, \bar{F}', b') := 2C \left( C_5 + r_M[C_3(1 + |\bar{F}'| + \|\bar{F}'\|_{L^M(I)} + \|b'\|_{L^M(I)}) + 2C_2(1 + M(|\bar{F}|) + M(|\bar{F}'|) + q_M(|b|) + q_M(|b'|)) + 1 + 2 \right) < \infty.
\end{equation}

By the definition of $Q_k^W(\bar{F}', b')$, (52) implies that

\begin{equation}
Q_k^W(\bar{F}', b') \leq \int_Q W(\bar{F}' + D_\alpha \psi_n | \lambda D_3 \psi_n)dx
\leq \int_Q W(\bar{F} + D_\alpha \varphi_n | \lambda D_3 \varphi_n)dx + \tilde{C}_7(\bar{F}, b, \bar{F}', b') \cdot (|\bar{F}' - \bar{F}| + \|b' - b\|_{L^M(I)}),
\end{equation}

and letting $n \to +\infty$, we infer that

\begin{equation}
Q_k^W(\bar{F}', b') \leq Q_k^W(\bar{F}|b) + \tilde{C}_7(\bar{F}, b, \bar{F}', b') \cdot (|\bar{F}' - \bar{F}| + \|b' - b\|_{L^M(I)}).
\end{equation}
Effective energy integral functionals for thin films with ... 25

Using the same arguments for the pair \((\bar{F}', b')\) in place of \((\bar{F}, b)\), we deduce that
\begin{equation}
Q_k^M W(\bar{F}|b) \leq Q_k^M W(\bar{F}'|b') + \tilde{C}_\gamma(\bar{F}, b, \bar{F}', b') \cdot (\|\bar{F}' - \bar{F}\| + \|b'(-) - b(-)\|_{\mathcal{L}^M(I)})
\end{equation}

Taking infimum over \(\lambda > 0\) and then letting \(k \to +\infty\) by (55) and (56) we deduce, that
\begin{equation}
|Q_k W(\bar{F}|b) - Q_k W(\bar{F}'|b')|, |Q_\infty W(\bar{F}|b) - Q_\infty W(\bar{F}'|b')|
\leq \tilde{C}_\gamma(\bar{F}, b, \bar{F}', b') \cdot (\|\bar{F}' - \bar{F}\| + \|b'(-) - b(-)\|_{\mathcal{L}^M(I)}) < +\infty.
\end{equation}

By \(M \in \Delta_2\) and the definition of \(\tilde{C}_\gamma(\bar{F}, b, \bar{F}', b')\) in (53), we deduce that (57) implies the continuity of \(Q_k W\) and \(Q_\infty W\) on \(\mathbb{R}^{3 \times 2} \times \mathcal{L}^M(I; \mathbb{R}^3)\).

**Lemma 13.** Let \(W\) be a quasiconvex continuous function satisfying the hypothesis (4) and \(M \in \Delta_2 \cap \nabla_2\). Consider the \(\Gamma\)-lower limit \(E_0\) defined in (40). Then
\begin{equation}
E_0(u, b, A) \geq \int_A Q_\infty W(D_\alpha u|b(x_\alpha, \cdot))dx_\alpha
\end{equation}
for all \((u, b, A) \in \mathcal{Z} \times \mathcal{L}^M(\Omega; \mathbb{R}^3) \times \mathcal{A}(\omega)\).

**Proof.** By Proposition 12, \(Q_\infty W(D_\alpha u|b)\) and \(Q_k W(\bar{F}|b)\) are measurable non-negative functions.

**Step 13.1.** Let \(k \in \mathbb{N}, b \in \mathcal{L}^M(I; \mathbb{R}^3), u(x_\alpha) := \bar{F} x_\alpha + u_0\) with \(\bar{F} \in \mathbb{R}^{3 \times 2}, u_0 \in \mathbb{R}^3\).

By Lemma 10 we may restrict ourselves, in the definition (40) to sequences having the same trace as it’s limit. Consider the sequence
\[w_n(x) := \varphi_n(x) + (\bar{F} x_\alpha + u_0),\]
where \(\varphi_n \in W^{1,M}(Q; \mathbb{R}^3)\) is such that \(\varphi_n = 0\) on \(\partial Q' \times I\), \(\varphi_n \rightharpoonup 0\) weakly in \(W^{1,M}(Q; \mathbb{R}^3)\) and \(\lambda_n D_3 \varphi_n \rightharpoonup b\) weakly in \(\mathcal{L}^M(Q; \mathbb{R}^3)\). Then \(\varphi_n(x_\alpha)\) is \(Q'\)-periodic. For any \(\psi \in \mathcal{L}^{M'}(Q'; \mathbb{R}^3)\) by the Jensen inequality [23], together with
\(M \in \Delta_2\), we infer that
\begin{equation}
\int_A M^* \left(\int_I \theta_i(x_3) \psi(x_\alpha) dx_\alpha\right) dx_\alpha \leq \int_A \int_I M^* (\|\theta_i(x_3) \psi(x_\alpha)\|_{L^\infty(I; \mathbb{R}^3)}) dx_3 dx_\alpha
\end{equation}
\begin{align*}
&\leq \int_A \int_I M^* (\|\theta_i(x_3)\|_{L^\infty(I; \mathbb{R}^3)}) \psi(x_\alpha) dx_3 dx_\alpha \\
&\leq a (\|\theta_i(x_3)\|_{L^\infty(I; \mathbb{R}^3)}) \int_A \int_I M^* (\|\psi(x_\alpha)\|) dx_3 dx_\alpha \\
&+ b (\|\theta_i(x_3)\|_{L^\infty(I; \mathbb{R}^3)}) \cdot \mathcal{L}^2(A) < \infty,
\end{align*}
for all $i \in \mathbb{N}$. Therefore using $\theta_i(x_3) \cdot \psi(x_\alpha)$ as a test function from $L^{M^*}(Q' \times I; \mathbb{R}^3)$, we deduce that
\[
\int_{Q'} (\lambda_n D_3 \varphi_n \theta_i dx_3 - \bar{b}_i) \psi(x_\alpha) dx_\alpha = \int_{Q'} \int_I (\lambda_n D_3 \varphi_n - b(s)) \theta_i \psi(x_\alpha) dx_\alpha ds \to 0,
\]
and so $\int_I \lambda_n D_3 \varphi_n \theta_i dx_3 \to \bar{b}_i$ weakly in $L^M(Q'; \mathbb{R}^3)$ for $i = 1, \ldots, k$, where $\bar{b}_i := \int_I b(s) \theta_i ds$. Therefore there exists $n_k \in \mathbb{N}$ such that, for $n \geq n_k$ we have
\[
\left| \int_I \lambda_n D_3 \varphi_n \theta_i dx - \bar{b}_i \right| \leq \frac{1}{k} \quad (i = 1, \ldots, k).
\]
Thus $\varphi_n$ are admissible functions for the definition of $Q_k W$ and by the definition of $Q_k W$, we have
\[
\liminf_{n \to +\infty} \int_Q W(\bar{F} + D_\alpha \varphi_n \mid \lambda_n D_3 \varphi_n) dx \geq Q_k W(\bar{F} \mid b).
\]
By taking supremum over all $k \in \mathbb{N}$
\[
(60) \quad \liminf_{n \to +\infty} \int_Q W(\bar{F} + D_\alpha \varphi_n \mid \lambda_n D_3 \varphi_n) dx \geq Q_\infty W(\bar{F} \mid b).
\]
Since $\{w_n\}_{n \in \mathbb{N}}$ is admissible for the definition of (40), we complete the proof of (58) for the case in Step 13.1 by taking the infimum over all admissible sequences in (60), and then we get the inequality
\[
E_0(\bar{F} x_\alpha + u_0, b, Q') \geq \int_{Q'} Q_\infty W(\bar{F} \mid b) dx_\alpha
= \mathcal{L}^2(Q') \cdot Q_\infty W(\bar{F} \mid b) = Q_\infty W(\bar{F} \mid b).
\]

We omit the general case for the proof of the inequality (58), since it is analogous to Step 2 in Proposition 3.4 [5]. It is enough to apply $W^{1,M}$-generalization in [13, Theorem 3.1] for the Acerbi-Fusco weak l.s.c. $W^{1,p}$-theorem [1, Theorem II.5], the fact that embedding $W^{1,M}(A \times I; \mathbb{R}^3) \hookrightarrow L^M_{\text{loc}}(A \times I; \mathbb{R}^3)$ is compact (see Donaldson-Trudinger [10, Theorem 3.9] together with Gossez [18, Proposition 4.3]) and differentiability properties of the Orlicz-Sobolev functions [3, Lemma 3.1, Lemma 3.2].

**Lemma 14.** Under the hypothesis of Lemma 13, we have
\[
E_0(u, b, A) \leq \int_{A} Q_\infty W(D_\alpha u \mid b(x_\alpha, \cdot)) dx_\alpha
\]
for all $(u, b, A) \in Z \times L^M(\Omega; \mathbb{R}^3) \times \mathcal{A}(\omega)$. 

Proof. By Proposition 12, $Q_\infty W(D_\alpha u|b)$ and $Q_k W(\bar{F}|b)$ are measurable non-negative functions.

Step 14.1. Let $u(x_\alpha) := \bar{F}x_\alpha + u_0$ with $\bar{F} \in \mathbb{R}^{3\times 2}$, $u_0 \in \mathbb{R}^3$ and $b \in L^M(I; \mathbb{R}^3)$. Since $Q_k W(\bar{F}|b)$ is nondecreasing in $k$, by Proposition 12

\begin{equation}
Q_k W(\bar{F}|b) \in \mathbb{R}, \quad Q_\infty W(\bar{F}|b) = \lim_{k \to \infty} Q_k W(\bar{F}|b) \in \mathbb{R}
\end{equation}

By the definition of $Q_k W(\bar{F}|b)$ there exists $t_k \in \mathbb{R}$, $\varphi^k \in W^{1,M}(Q; \mathbb{R}^3)$, $\varphi^k(\cdot, x_3)$ is $Q'$-periodic $L^1$ a.e. $x_3 \in I$,

\begin{equation}
\left| \int_Q t_k D_3 \varphi^k \theta_i dx - \int_I b \theta_i dx_3 \right| < \frac{1}{k} \quad (i = 1, \ldots, k)
\end{equation}

and

\begin{equation}
Q_k W(\bar{F}|b) \leq \int_Q W(\bar{F} + D_\alpha \varphi^k | t_k D_3 \varphi^k) dx < Q_k W(\bar{F}|b) + \frac{1}{k}.
\end{equation}

Extending $Q'$-periodically of the $Q'$-periodic function $\varphi$, we define $\varphi^k_n : \mathbb{R}^2 \times I \to \mathbb{R}^3$ by

$$
\varphi^k_n(x) := \frac{t_k}{\lambda_n} \varphi\left(\frac{\lambda_n}{t_k} x_\alpha, x_3 \right).
$$

Observe that the function

$$
y_\alpha \mapsto t_k \int_I D_3 \varphi^k(y_\alpha, x_3) \theta_i(x_3) dx_3
$$

belongs to $L^M(A; \mathbb{R}^3)$, since by the Jensen inequality [23] and $M \in \Delta_2$, we infer (cf. (59)) that

\begin{align*}
\int_A M \left( \left| \int_I D_3 \varphi^k(y_\alpha, x_3) \theta_i(x_3) dx_3 \right| \right) dx_\alpha &\leq \int_A \int_I M \left( \left| D_3 \varphi^k(y_\alpha, x_3) \theta_i(x_3) \right| \right) dx_3 dx_\alpha \\
&\leq a_2 \left( \| \theta_i \|_{L^\infty(I; \mathbb{R}^3)} \right) \int_A \int_I M \left( |D_3 \varphi^k(y_\alpha, x_3)| \right) dx_3 dx_\alpha \\
&\quad + b_2 \| \theta_i(x_3) \|_{L^\infty(I; \mathbb{R}^3)} \cdot \mathcal{L}^3(A \times I) < +\infty
\end{align*}

Applying the $L^M(Q')$-version (see Pedregal [34, Homogenization Theorem 7.1, Remark p. 121]) for the Riemann-Lebesgue lemma in $L^p(Q')$-spaces (see, e.g., [8]) we infer that

\begin{align}
&\lambda_n \int_I D_3 \varphi^k_n \theta_i dx_3 = t_k \int_I D_3 \varphi^k \left( \frac{\lambda_n}{t_k} x_\alpha, x_3 \right) \theta_i dx_3 \\
\rightarrow t_k \int_{Q'} D_3 \varphi^k(y_\alpha, x_3) dy_\alpha dx_3 =: \int_I b(s) \theta_i ds + r^k_i, \quad \text{as } n \to +\infty
\end{align}
weakly in $L^M(A; \mathbb{R}^3)$. By (64) $|r^k_i| < \frac{1}{k^2}$ ($i = 1, \ldots, k$). Define
\[ H^k(x_\alpha, x_3) := \left( \bar{F} + D_\alpha \varphi^k(x_\alpha, x_3) | t_k D_3 \varphi^k(x_\alpha, x_3) \right), \]
\[ \bar{W}^k(x_\alpha) := \int_I W \left( H^k(x_\alpha, x_3) \right) dx_3. \]

By the coercivity condition (4) and $H^k \in W^{1,M}(Q; \mathbb{R}^3)$,
\[ \int_{Q'} \bar{W}^k(x_\alpha) dx_\alpha = \int_Q W \left( H^k(x_\alpha, x_3) \right) dx_3 dx_\alpha \]
\[ \leq C_2 \left( 1 + \int_Q M(|\bar{F} + D_\alpha \varphi^k(x_\alpha, x_3) | t_k D_3 \varphi^k(x_\alpha, x_3)) dx_\alpha dx_3 \right) < \infty, \]
and so $\bar{W}^k \in L^1(Q'; \mathbb{R}^3)$. Using the Riemann-Lebesgue lemma, we deduce that
\[ \lim_{n \to +\infty} \int_{A \times I} W(\bar{F} + D_\alpha \varphi_n^k | \lambda_n D_3 \varphi^k) dx = \lim_{n \to +\infty} \int_{Q'} 1_A(x_\alpha) \bar{W}^k \left( \frac{\lambda_n x_\alpha}{t_k} \right) dx_\alpha \]
\[ = \int_{Q'} 1_A(x_\alpha) \left( \int_{Q'} \bar{W}^k(y_\alpha) dy_\alpha \right) dx_\alpha = \mathcal{L}^2(A) \int_Q W(\bar{F} + D_\alpha \varphi^k | t_k D_3 \varphi^k) dx. \]

In view of the coercivity condition (4) and by the Alaoglu-Bourbaki theorem [11, Theorem V.4.2] using the Moore lemma [11, Lemma I.7.6], by (63)–(67) we may find a subsequence $\{\lambda_{n_k}\}$ and $\{\varphi_{n_k}^k\}$ such that $\varphi_{n_k}^k \to 0$ weakly in $W^{1,M}(A \times I; \mathbb{R}^3)$, $\lambda_{n_k} \int_I D_3 \varphi_{n_k}^k \theta_i dx_3 \to \int_I b \theta_i dx_3$ weakly in $L^M(A; \mathbb{R}^3)$ for all $i \in \mathbb{N}$ and
\[ \lim_{k \to +\infty} \int_{A \times I} W(\bar{F} + D_\alpha \varphi_{n_k}^k | \lambda_{n_k} D_3 \varphi_{n_k}^k) dx = \mathcal{L}^2(A) \mathcal{Q}_\infty W(\bar{F}|b). \]

Hence by the argument in [23, pp. 81–82] together with the Stone-Weierstrass approximation theorem and by our choice of the family $\{\theta_i\}_{i \in \mathbb{N}}$ for the definition (8), we deduce that $\lambda_{n_k} D_3 \varphi_{n_k}^k \to b$ weakly in $L^M(A \times I; \mathbb{R}^3)$ as $k \to \infty$. Consequently,
\[ E_0(u, b, A) \leq \mathcal{L}^2(A) \mathcal{Q}_\infty W(\bar{F}|b). \]

**Step 14.2.** By Lemma 9 and Step 14.1 the inequality (62) holds for piecewise affine functions.

**Step 14.3.** By Proposition 12 and Step 14.2 the inequality (62) holds for any $(u, b, A) \in W^{1,M}(\omega; \mathbb{R}^3) \times L^M(\Omega; \mathbb{R}^3) \times A(\omega).$
Proof of Theorem 3. Let $u_\varepsilon \in \Psi_\varepsilon$ be such that $u_\varepsilon \rightharpoonup \bar{u}$ weakly in $W^{1,M}(\Omega; \mathbb{R}^3)$, $\frac{1}{\varepsilon} D_3 u_\varepsilon \rightharpoonup b$ weakly in $L^M(\Omega; \mathbb{R}^3)$. It is easy to check by the representation (3), the isomorphism (18) and by the Fubini theorem that $P_\varepsilon(u_\varepsilon) \to P_0(\bar{u}, b)$ and $P_\varepsilon(v_\varepsilon + u_{0,\varepsilon}) \to P_0(\bar{v} + u_{0,0}, b + e_3)$ as $\varepsilon \to 0$, with $u_\varepsilon = v_\varepsilon + u_{0,\varepsilon}$ and $\bar{u} = \bar{v} + u_{0,0}$, where $v_\varepsilon \in V$. By the same argument analog to the one used in A. Braides, I. Fonseca and G. Francfort [6, Step 2, Theorem 2.5], in order to show that $J^*_\varepsilon \Gamma$-converges to $J^*_0$ it is enough to prove that the $\Gamma$-lower limit $E_0$ of any subsequence of $E_\varepsilon$ coincides with $J_0$. Therefore the assertions of Theorem 3 follows from Lemmas 13–14 applied to the sequence $u_\varepsilon = v_\varepsilon + u_{0,\varepsilon}$.

We omit the proof of Corollary 4 since it is analogous to the proof of Corollary 4.2 in our previous paper [24].

References


Received 16 September 2015
Revised 26 January 2016