Discussiones Mathematicae
Differential Inclusions, Control and Optimization 36 (2016) 7–31
doi:10.7151/dmdico.1179

EFFECTIVE ENERGY INTEGRAL FUNCTIONALS FOR THIN FILMS WITH THREE DIMENSIONAL BENDING MOMENT IN THE ORLICZ-SOBOLEV SPACE SETTING

Włodzimierz Laskowski

School of Mathematics
West Pomeranian University of Technology
Al. Piastów 48, 70–311 Szczecin, Poland

e-mail: wlaskowski@zut.edu.pl

and

Hong Thai Nguyen

Institute of Mathematics
Szczecin University
ul. Wielkopolska 15, 70–451 Szczecin, Poland

e-mail: nguyenhthaimathuspl@yahoo.com

Abstract

In this paper we consider an elastic thin film $\omega \subset \mathbb{R}^2$ with the bending moment depending also on the third thickness variable. The effective energy functional defined on the Orlicz-Sobolev space over ω is described by Γ-convergence and $3D$-$2D$ dimension reduction techniques. Then we prove the existence of minimizers of the film energy functional. These results are proved in the case when the energy density function has the growth prescribed by an Orlicz convex function M. Here M is assumed to be non-power-growth-type and to satisfy the conditions Δ_2 and ∇_2.

Keywords: Γ-convergence, 3D-2D dimension reduction, quasiconvex relaxation, minimizers of variational integral functionals, thin films, elastic membranes, effective energy integral functional, bulk and surface energy, equilibrium states of the film, non-power-growth-type bulk energy density, reflexive Orlicz and Orlicz-Sobolev spaces.

2010 Mathematics Subject Classification: 49J45, 74B20, 74K35, 74K15, 46E30, 46E35, 47H30.

1. Introduction

We consider an elastic thin film as a bounded open subset $\omega \subset \mathbb{R}^2$ with Lipschitz boundary. The set $\Omega_\varepsilon := \omega \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \subset \mathbb{R}^3$ for a small thickness ε is considered as an elastic cylinder approximate to the film ω. A three-dimensional deformation $U : \Omega_\varepsilon \to \mathbb{R}^3$ defined on the thin cylinder Ω_ε has the re-scaled elastic total energy represented by the difference of the re-scaled bulk and re-scaled surface energies

$$E_\varepsilon(U) = \frac{1}{\varepsilon^\alpha} \int_{\Omega_\varepsilon} W(DU)dx - \frac{1}{\varepsilon^\beta} Q_\varepsilon(U), \tag{1}$$

where $W : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is so-called the energy density function satisfying the growth and coercivity conditions

$$\frac{1}{C}(M(\|F\|) - 1) \leq W(F) \leq C(1 + M(\|F\|)) \quad (\forall F \in \mathbb{R}^{3\times 3}) \tag{2}$$

for some $C > 0$. Here $M \colon \mathbb{R} \to [0, \infty)$ is some Orlicz convex N-function.

The purpose of this type of research is to investigate, as the thickness ε goes to zero, the Γ-convergence limit of the sequence of these re-scaled energies and to understand the behavior of minimizers subject to appropriate boundary conditions.

The values of exponents α and β in the definition (1) are important. It turns out that when $\alpha = 1$, $\beta = 0$ and $M(t) = |t|^p$ $(1 < p < \infty)$ the form of the functional (1) leads through the use of Γ-convergence to the nonlinear membrane theory in the reflexive Sobolev spaces $W^{1,p}$ [26, 27]. It is important to note that the papers [26, 27] by H. Le Dret and A. Raoult published in 1993-1995 contain the first precise convergence results for the re-scaled energy functionals in the nonlinear theory of thin membranes through the use of Γ-convergence. For the case $\alpha = 1$, $\beta = 1$ and $M(t) = |t|^p$ $(1 < p < \infty)$, one has to deal with the additional two and three dimensional bending moment in the nonlinear membrane theory in the reflexive Sobolev spaces $W^{1,p}$, cf. [4] and [5] published in 2004 and 2009, respectively.

Many results through the use of Γ-convergence were established for other values of the exponents α and β in the reflexive Sobolev spaces $W^{1,p}$ (see, e.g., [16, 31]).

We assume that M is of the non-power-growth-type and satisfies the conditions Δ_2 and ∇_2 (that is equivalent to the reflexivity of Orlicz and Orlicz-Sobolev spaces generated by M).

In our previous papers (see [24, 25]) we consider the case $\alpha = 1$, $\beta = 0$ and the case $\alpha = 1$, $\beta = 1$ with the 2-dimensional bending moment in the Orlicz-Sobolev space setting $W^{1,M}$.

In Theorem 1, the effective energy functional for the thin film ω with the 3-dimensional bending moment is obtained, by Γ-convergence and $3D$-$2D$ dimension reduction techniques applied to the sequence of the re-scaled total energy integral functionals of the elastic cylinders Ω_ε as $\varepsilon \to 0$. In Corollary 2, the existence of minimizers of the energy functional for the thin film is established by showing that some sequence of re-scaled minimizers weakly converges in an appropriate Orlicz-Sobolev space to a minimizer of the film energy functional.

In Section 5, we give the proofs of Theorem 1 and Corollary 2. For these proofs we apply also results: for Orlicz convex functions [20, Proposition 4], for the Orlicz-Sobolev spaces [22, Theorem 5, Theorem 7] (cf. [10]), [17, Proposition 2.1], for differentiability properties of the Orlicz-Sobolev functions [3, Lemma 3.1, Lemma 3.2], for the sub-differential operator in Orlicz spaces [35, Lemma 1], for quasiconvex integral functionals and quasiconvexification in the Orlicz-Sobolev space setting [13], for the L^M-version [34, Homogenization Theorem 7.1, Remark p. 121] for the Riemann-Lebesgue Lemma.

Examples of Orlicz N-functions M with $M \in \Delta_2 \cap \nabla_2$ are $M(t) = |t|^p(\log(1+|t|))^q$, where $p > 1$ i $q > 1$ or $M(t) = |t|^p(\log(1+|t|))^{q_1} \cdot (\log(\log(1+|t|)))^{q_2}$, where $p > 1$ i $q_1, q_2 > 1$. Many other examples of M with $M \in \Delta_2 \cap \nabla_2$ can be found in [23, Theorem 7.1, pp. 58–59] and [28, 29]. Furthermore, the assumption $M \in \Delta_2 \cap \nabla_2$ is indispensable in the regularity study of minimizers of multiple variational integrals with the M-growth on Orlicz-Sobolev spaces (see discussions and references for many other concrete examples in [12, 7]).

2. Some terminology and notation

>From now on, unless stated to the contrary, $M\colon \mathbb{R} \to [0, \infty)$ is assumed to be a non-power-growth-type Orlicz N-function (i.e., even convex function satisfying $\lim_{t\to 0} \frac{M(t)}{t} = 0$ and $\lim_{t\to +\infty} \frac{M(t)}{t} = +\infty$).

We assume $M \in \Delta_2 \cap \nabla_2$. Here the condition $M \in \Delta_2$ means that $M(2t) \leq c\, M(t) \quad (t \geq t_0)$ for some $t_0 \in [0, \infty)$ and $c \in (0, \infty)$. The condition Δ_2 implies that $M(2t) \leq a(2) + b(2)M(t)$ for all $t \geq 0$, where $b(2) \in (0, +\infty)$ and $a(2) \in [0, +\infty)$. The condition $M \in \nabla_2$ means that $\exists\, l > 1, \exists\, t_* \in [0, \infty)$ such that $M(t) \leq \frac{1}{2l} M(lt)$ for all $t \geq t_*$.

Let M^* be the complementary (conjugate) Orlicz N-function of M defined by $M^*(\tau) := \sup\{t\tau - M(t) : t \in \mathbb{R}\}$. It is known that the condition $M \in \nabla_2$ is equivalent to the condition $M^* \in \Delta_2$.

Denote by $|v|$ the Euclidean norm of v and by (u, v) the scalar product. Given an open bounded subset $G \subset \mathbb{R}^N$ with Lipschitz (e.g., C^2-smooth) boundary ∂G equipped with the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$. Denote by $L^M(G; \mathbb{R}^m)$ the Orlicz space of all (equivalent classes of) measurable functions

$u\colon G \to \mathbb{R}^m$ equipped with the Luxemburg norm

$$\|u\|_{L^M(G;\mathbb{R}^m)} := \inf\left\{\lambda > 0 : \int_\Omega M(|u(x)|/\lambda)dx \leq 1\right\}.$$

It is known that $M \in \Delta_2 \cap \nabla_2$ is equivalent to the reflexivity of $L^M(G;\mathbb{R}^m)$.

Recall that the Orlicz-Sobolev space $W^{1,M}(G;\mathbb{R}^3)$ is defined as the Banach space of $\mathbb{R}^3$-valued functions u of $L^M(G;\mathbb{R}^3)$ with the Sobolev-Schwartz distributional derivative $Du \in L^M(G;\mathbb{R}^{3\times N})$ equipped with the norm

$$\|u\|_{W^{1,M}(G;\mathbb{R}^3)} := \|u\|_{L^M(G;\mathbb{R}^3)} + \|Du\|_{L^M(G;\mathbb{R}^{3\times N})} < \infty.$$

The subspace $W_0^{1,M}(G;\mathbb{R}^3)$ is defined as the closure in $\|\cdot\|_{W^{1,M}(G;\mathbb{R}^3)}$-norm of the set $C_0^\infty(G;\mathbb{R}^3)$ of C^∞-smooth $\mathbb{R}^3$-valued functions with compact support in G. Since ∂G is Lipschitz and $M, M^* \in \Delta_2$, by [15, Theorems 2.1, 2.3] (cf. [21]) there exists the bounded linear trace operator

$$\mathrm{Tr} : W^{1,M}(G;\mathbb{R}^3) \to L^M(\partial G;\mathbb{R}^3)$$

such that: (i) $\mathrm{Tr}(u) = u_{|\partial G}$ ($\forall u \in C^\infty(\overline{G})$) and (ii) $u \in W_0^{1,M}(G;\mathbb{R}^3)$ if and only if $\mathrm{Tr}(u) = 0$. So, for the simplicity of notation we will write "$u(x) = \varphi(x)$ on A" for $u \in W^{1,M}(G;\mathbb{R}^3)$ and $\varphi \in L^M(\partial G;\mathbb{R}^3)$ and $A \subset \partial G$ if $\mathrm{Tr}(u)(x) = \varphi(x)$ for almost every $x \in A$. Due to this reason, we also denote by "u on A" for "$\mathrm{Tr}(u)$ on A", etc.

Let $(X, \|\cdot\|_{W^{1,M}(G;\mathbb{R}^3)})$ be normed subspace of $W^{1,M}(G;\mathbb{R}^3)$. By [2, Proof of Theorem 3.9] and [19, Proof of Lemma 2.2] for every $\Lambda \in X^*$ there exist elements $h_0, h_1, \ldots, h_N \in L^{M^*}(G;\mathbb{R}^3)$ such that

$$\Lambda(u) = \int_G (h_0, u)dx + \sum_{i=1}^N \int_G \left(h_i, \frac{\partial u}{\partial x_i}\right) dx \quad (u \in X). \tag{3}$$

Conversely, every functional Λ defined by (3) in the case $h_0, h_1, \ldots, h_N \in L^{M^*}(G;\mathbb{R}^3)$, is an element of X^*.

3. Setup

Define $I := (-\frac{1}{2}, \frac{1}{2})$, $\Omega := \omega \times I$, $S^\pm := \omega \times \{\pm\frac{1}{2}\}$, $\Gamma := \partial\omega \times I$, and for each $\varepsilon > 0$, $S^\pm_\varepsilon := \omega \times \{\pm\frac{\varepsilon}{2}\}$, $\Gamma_\varepsilon := \partial\omega \times \varepsilon I$. Greek indexes will be used to distinguish the first two components of a vector, for instance (x_α) and (x_α, x_3), designates (x_1, x_2) and (x_1, x_2, x_3), respectively. We denote by $\mathbb{R}^{3\times 3}$ and $\mathbb{R}^{3\times 2}$ the vector spaces of respectively 3×3 and 3×2 real-valued matrices. Given $\bar{F} \in \mathbb{R}^{3\times 2}$ and $b \in \mathbb{R}^3$, denote by $(\bar{F}|b)$ the 3×3 matrix whose first two columns are those of $\bar{F}$ and the last column is b. By the analogous way, set $e_\alpha := (e_1|e_2) \in \mathbb{R}^{3\times 2}$ where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. Set $D_\alpha U := (\frac{\partial U}{\partial x_1}|\frac{\partial U}{\partial x_2})$, $D_3 U := \frac{\partial U}{\partial x_3}$, $DU := (D_\alpha U|D_3 U)$ for an $\mathbb{R}^3$-valued function U. Denote by C, $\widetilde{C}$ generic positive constants that may vary from line to line.

Let $W\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ be a continuous function satisfying the M-growth-type and coercivity conditions:

$$\frac{1}{C}(M(|F|)-1) \leq W(F) \leq C(1+M(|F|)) \quad \left(\forall F \in \mathbb{R}^{3\times 3}\right) \tag{4}$$

for some $C \in (0,\infty)$.

Set

$$\widetilde{\Psi}_\varepsilon := \{U \in W^{1,M}(\Omega_\varepsilon;\mathbb{R}^3) : U(\tilde{x}) = \tilde{x} \text{ on } \Gamma_\varepsilon\}.$$

We consider the variational integral functional $\widetilde{J}_\varepsilon : \widetilde{\Psi}_\varepsilon \to \mathbb{R}$, where $\widetilde{J}_\varepsilon(U)$ (the re-scaled total energy of the elastic cylinder Ω_ε under a deformation $U : \Omega_\varepsilon \to \mathbb{R}^3$) is represented by the difference of the re-scaled bulk and re-scaled surface energies:

$$\begin{aligned} \widetilde{J}_\varepsilon(U) := {} & \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} W(DU)d\tilde{x} - \frac{1}{\varepsilon}\int_{\Omega_\varepsilon}(f_\varepsilon, U)d\tilde{x} - \frac{1}{\varepsilon}\int_{S_\varepsilon^+}(\varepsilon g_0^+ + g, U)d\mathcal{H}^2 \\ & + \frac{1}{\varepsilon}\int_{S_\varepsilon^-}(\varepsilon g_0^- + g, U)d\mathcal{H}^2. \end{aligned} \tag{5}$$

Here, $f_\varepsilon := f\left(\tilde{x}_\alpha, \frac{\tilde{x}_3}{\varepsilon}\right)$, $f \in L^{M^*}(\Omega;\mathbb{R}^3)$, $g_0^\pm, g \in L^{M^*}(\omega;\mathbb{R}^3)$ and $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure in $\mathbb{R}^3$. Set

$$\overline{\Psi}_0 := \{\bar{u} \in W^{1,M}(\omega;\mathbb{R}^3) : \bar{u}(x_\alpha) = (x_\alpha, 0) \text{ on } \partial\omega\}.$$

Let $J_0 : \overline{\Psi}_0 \times L^M(\Omega;\mathbb{R}^3) \to \mathbb{R}$ be defined by

$$J_0(\bar{u}, b) := \int_\omega \mathcal{Q}_\infty W(D_\alpha u(x_\alpha)|b(x_\alpha,\cdot))dx_\alpha - P_0(\bar{u}, b), \tag{6}$$

where

$$\mathcal{Q}_\infty W(\bar{F}, \tilde{b}) := \sup_{k\in\mathbb{N}} \mathcal{Q}_k W(\bar{F}, \tilde{b}) \quad (\bar{F} \in \mathbb{R}^{3\times 2},\ \tilde{b} \in L^M(I;\mathbb{R}^3)) \tag{7}$$

and

$$\begin{aligned} \mathcal{Q}_k W(\bar{F}, \tilde{b}) := \inf\Bigg\{ & \int_Q W(\bar{F} + D_\alpha\varphi|\lambda D_3\varphi)dx : \lambda > 0, \varphi \in W^{1,M}(Q;\mathbb{R}^3), \\ & \varphi(\cdot, x_3) \text{ is } Q' \text{ periodic } \mathcal{L}^1 \text{ a.e. } x_3 \in I, \left|\int_Q \lambda D_3\varphi\theta_i dx - \int_I \tilde{b}\theta_i dx_3\right| < \frac{1}{k}(i=1,\ldots,k)\Bigg\}, \end{aligned} \tag{8}$$

for a fixed countable dense family $\{\theta_i\}_{i\in\mathbb{N}} \subset L^{M^*}(I;\mathbb{R}^3)$ (here, we assume that $M \in \Delta_2 \cap \nabla_2$) and

$$P_0(\bar{u}, b) := \int_\omega (\bar{f}, \bar{u}) dx_\alpha + \int_\omega (g_0^+ - g_0^-, \bar{u})\ dx_\alpha + \int_\Omega (g, b) dx,$$

with $\bar{f}(x_\alpha) := \int_I f(x_\alpha, x_3)\ dx_3$. By (see [23, p. 81]), we may choose $\{\theta_i\}_{i\in\mathbb{N}}$ from $L^\infty(I;\mathbb{R}^3)$. Later on, Proposition 12 shows that $\mathcal{Q}_\infty W$ and $\mathcal{Q}_k W$ are continuous.

4. The formulation of main results

Let $\mathcal{Z}$ be the space of membrane deformations defined by

$$\mathcal{Z} = \{z \in W^{1,M}(\Omega;\mathbb{R}^3) :\ D_3 z = 0,\ z(x) = (x_\alpha, 0) \text{ on } \Gamma\}. \tag{9}$$

Observe that $\mathcal{Z}$ is canonically isomorphic to $\overline{\Psi}_0$ [30, Theorem 1.1.3/1]. Let $\bar{z}$ denote the element of $\overline{\Psi}_0$ that is associated with $z \in \mathcal{Z}$ through this isomorphism:

$$z(x_\alpha, x_3) = \bar{z}(x_\alpha) \text{ a.e.} \tag{10}$$

Since we want to identify the sequence convergence with the thickness of our domain tending to zero, for simplicity we assume this thickness parameter ε takes its values in a sequence $\varepsilon_n \to 0$.

Theorem 1. *Let $\widetilde{J}_\varepsilon$ be defined in* (14) *and J_0 be defined in* (6)*. Assume $M \in \Delta_2 \cap \nabla_2$. Assume that the continuous function $W \colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfies the hypothesis* (4)*. Let $\{U_\varepsilon\} \in \widetilde{\Psi}_\varepsilon$. For each $\varepsilon > 0$ and $\tilde{x} = (\tilde{x}_\alpha, \tilde{x}_3) \in \Omega_\varepsilon$ we associate $x = (x_\alpha, x_3) := \big(\tilde{x}_\alpha, \frac{1}{\varepsilon}\tilde{x}_3\big) \in \Omega$ and we set $z_\varepsilon(x_\alpha, x_3) := U_\varepsilon(\tilde{x}_\alpha, \tilde{x}_3)$.*
Then the sequence $\widetilde{J}_\varepsilon$ converges to J_0 in the following sense:

(i) (*lower bound*) *if $z_\varepsilon \rightharpoonup z$ weakly in $W^{1,M}(\Omega;\mathbb{R}^3)$, $\|z_\varepsilon\|_{W^{1,M}(\Omega;\mathbb{R}^3)} < +\infty$ and $z \in \mathcal{Z}$ with $z(x_\alpha, x_3) = \bar{z}(x_\alpha)$ through the isomorphism* (10) *and $\frac{1}{\varepsilon} D_3 z_\varepsilon \rightharpoonup b$ weakly in $L^M(\Omega;\mathbb{R}^3)$ and $\|\frac{1}{\varepsilon} D_3 z_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} < +\infty$, then*

$$\liminf_{\varepsilon\to 0} \widetilde{J}_\varepsilon(U_\varepsilon) \geq J_0(\bar{z}, b);$$

(ii) (*upper bound*) *for every pair $(\bar{z}, b) \in \overline{\Psi}_0 \times L^M(\Omega;\mathbb{R}^3)$, there exists a sequence $U_\varepsilon \in W^{1,M}(\Omega;\mathbb{R}^3)$ such that $z_\varepsilon \rightharpoonup z$ weakly in $W^{1,M}(\Omega;\mathbb{R}^3)$, $\|z_\varepsilon\|_{W^{1,M}(\Omega;\mathbb{R}^3)} < +\infty$ and $z \in \mathcal{Z}$ with $z(x_\alpha, x_3) = \bar{z}(x_\alpha)$ through the isomorphism* (10) *and $\frac{1}{\varepsilon} D_3 z_\varepsilon \rightharpoonup b$ weakly in $L^M(\Omega;\mathbb{R}^3)$ and $\|\frac{1}{\varepsilon} D_3 z_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} < +\infty$ and*

$$\lim_{\varepsilon\to 0} \widetilde{J}_\varepsilon(U_\varepsilon) = J_0(\bar{z}, b).$$

Consider the asymptotic behavior of $U_\varepsilon \in \widetilde{\Psi}_\varepsilon$ such that

$$\widetilde{J}_\varepsilon(U_\varepsilon) \leq \inf_{U\in\widetilde{\Psi}_\varepsilon} \widetilde{J}_\varepsilon(U) + \gamma(\varepsilon), \tag{11}$$

where γ is a positive function such that $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Corollary 2 (The minimization problem)**.** *Assume that $U_\varepsilon \in \widetilde{\Psi}_\varepsilon$ satisfies* (11)*. Let the functions M, W and z_ε, $\bar{z}$ be such as in Theorem* 1*. Then:*

- (i) *the sequence $\left(z_\varepsilon, \frac{1}{\varepsilon}D_3 z_\varepsilon\right)$ is relatively weakly compact in $W^{1,M}(\Omega;\mathbb{R}^3) \times L^M(\Omega;\mathbb{R}^3)$;*
- (ii) *the set $\mathcal{C}_{\text{film}}$ of cluster points of the sequence $\left(z_\varepsilon, \frac{1}{\varepsilon}D_3 z_\varepsilon\right)$ in the weak topology is a non-empty subset of $\mathcal{Z} \times L^M(\Omega;\mathbb{R}^3)$;*
- (iii) *any point (z_∞, b) of $\mathcal{C}_{\text{film}}$ can be identified with $(\bar{z}_\infty, b) \in \overline{\Psi}_0 \times L^M(\Omega;\mathbb{R}^3)$ by the 3D-2D dimension reduction isomorphism* (10) *and $(\bar{z}_\infty, \bar{b})$ is a solution of the minimization problem*

$$\inf_{\bar{u}\in\overline{\Psi}_0} \left\{ J_0(\bar{u}, b) : b \in L^M(\Omega;\mathbb{R}^3) \right\}.$$

5. THE PROOFS OF THEOREM 1 AND COROLLARY 2

We will reformulate Theorem 1 and Corollary 2 by the use of the following equivalent functionals $\bar{J}^*_\varepsilon$ and $\bar{J}^*_0$ (see the re-formulation in Theorem 3 and Corollary 4). Define

$$u_{0,\varepsilon}(x) := (x_\alpha, \varepsilon x_3), \quad u_{0,0}(x) := (x_\alpha, 0). \tag{12}$$

Notice that after the change of variables as in Theorem 1 with the association

$$x = (x_\alpha, x_3) := \left(\tilde{x}_\alpha, \frac{1}{\varepsilon}\tilde{x}_3\right),\ u(x_\alpha, x_3) := U(\tilde{x}_\alpha, \tilde{x}_3), \tag{13}$$

and by the Fubini Theorem the re-scaled energy $\widetilde{J}_\varepsilon(U)$ in (14) can be rewritten in the equivalent form

$$\begin{aligned}
J_\varepsilon(u) &= \int_\Omega W\left(D_\alpha u \Big| \frac{1}{\varepsilon}D_3 u\right) dx - \int_\Omega (f, u) dx \\
&- \int_{S^+} (g_0^+, u) d\mathcal{H}^2 + \int_{S^-} (g_0^-, u) d\mathcal{H}^2 - \int_\omega \left(g, \frac{u^+ - u^-}{\varepsilon}\right) dx_\alpha \\
&= \int_\Omega W\left(D_\alpha u \Big| \frac{1}{\varepsilon}D_3 u\right) dx - \int_\Omega (f, u) dx
\end{aligned} \tag{14}$$

$$-\int_{S^+}(g_0^+,u)d\mathcal{H}^2+\int_{S^-}(g_0^-,u)d\mathcal{H}^2-\int_\omega\left(g,\frac{1}{\varepsilon}\int_I D_3u dx_3\right)dx_\alpha$$
$$=\int_\Omega W\left(D_\alpha u\Big|\frac{1}{\varepsilon}D_3u\right)dx-\int_\Omega(f,u)dx$$
$$-\int_{S^+}(g_0^+,u)d\mathcal{H}^2+\int_{S^-}(g_0^-,u)d\mathcal{H}^2-\int_\Omega\left(g,\frac{1}{\varepsilon}D_3u\right)dx,$$

where $u^\pm(x_\alpha):=\mathrm{Tr}_{S^\pm}(u)(x_\alpha)$ and u is an element of

$$\Psi_\varepsilon:=\{u\in W^{1,M}(\Omega;\mathbb{R}^3):u(x)=u_{0,\varepsilon}(x)\text{ on }\Gamma\}.$$

In order to individualize this new sequence $\frac{1}{\varepsilon}D_3u$ and since the direct consideration of J_ε would imply the study involving the weak topology of the Orlicz-Sobolev space $W^{1,M}(\Omega;\mathbb{R}^3)$ which is non-metrizable on unbounded sets, then it is needed to consider the new functional $\bar{J}_\varepsilon:W^{1,M}(\Omega;\mathbb{R}^3)\times L^M(\Omega;\mathbb{R}^3)\to\mathbb{R}\cup\{+\infty\}$ defined by

$$\bar{J}_\varepsilon(u,b):=\begin{cases}\int_\Omega W(D_\alpha u|\frac{1}{\varepsilon}D_3u)dx-P_\varepsilon(u) & \text{if } \frac{1}{\varepsilon}D_3u=b \text{ and } u\in\Psi_\varepsilon\\ +\infty & \text{otherwise,}\end{cases}\tag{15}$$

where

$$P_\varepsilon(u):=\int_\Omega(f,u)dx-\int_{S^+}(g_0^+,u)d\mathcal{H}^2+\int_{S^-}(g_0^-,u)d\mathcal{H}^2+\int_\Omega\left(g,\frac{1}{\varepsilon}D_3u\right)dx.$$

Observe that the re-scaled displacement $v=u-u_{0,\varepsilon}$ belongs to the set

$$V=W^{1,M}_\Gamma(\Omega;\mathbb{R}^3):=\{v\in W^{1,M}(\Omega;\mathbb{R}^3):v(x)=0\text{ on }\Gamma\}$$

and

$$J_\varepsilon(v+u_{0,\varepsilon})=\int_\Omega W\left(e_\alpha+D_\alpha v\Big|e_3+\frac{1}{\varepsilon}D_3v\right)dx-\int_\Omega(f,v+u_{0,\varepsilon})dx$$
$$-\int_{S^+}(g_0^+,v+u_{0,\varepsilon})d\mathcal{H}^2+\int_{S^-}(g_0^-,v+u_{0,\varepsilon})d\mathcal{H}^2-\int_\Omega\left(g,\frac{1}{\varepsilon}D_3v+e_3\right)dx.$$

Define $\bar{J}^*_\varepsilon:W^{1,M}(\Omega;\mathbb{R}^3)\times L^M(\Omega;\mathbb{R}^3)\to\mathbb{R}\cup\{+\infty\}$ by
(16)

$$\bar{J}^*_\varepsilon(v,b):=\begin{cases}\int_\Omega W(e_\alpha+D_\alpha v|\frac{1}{\varepsilon}D_3v+e_3)dx-P_\varepsilon(v+u_{0,\varepsilon}) & \text{if } \frac{1}{\varepsilon}D_3v+e_3=b \text{ and } v\in V\\ +\infty & \text{otherwise .}\end{cases}$$

Let $\mathcal{V}$ be the space of membrane displacements defined by

$$\mathcal{V} = \{v \in W^{1,M}(\Omega;\mathbb{R}^3) : D_3 v = 0,\, v(x) = 0 \text{ on } \Gamma\} \subset V. \tag{17}$$

Similarly as in (9)–(10), $\mathcal{V}$ is canonically isomorphic to $W_0^{1,M}(\omega;\mathbb{R}^3)$ [30, Theorem 1.1.3/1]. Let $\bar{v}$ denote the element of $W_0^{1,M}(\omega;\mathbb{R}^3)$ that is associated with $v \in \mathcal{V}$ through this isomorphism:

$$v(x_\alpha, x_3) = \bar{v}(x_\alpha) \text{ a.e.} \tag{18}$$

Analogously for $v \in \mathcal{V}$ and $b \in L^M(\Omega;\mathbb{R}^3)$ define the functional

$$\bar{J}_0^*(v,b) := \int_\omega \mathcal{Q}_\infty W(e_\alpha + D_\alpha \bar{v}(x_\alpha)|b(x_\alpha,\cdot) - e_3)dx_\alpha - P_0(\bar{v} + u_{0,0}, b + e_3). \tag{19}$$

In this notion we have for $U_\varepsilon \in \widetilde{\Psi}_\varepsilon$

$$\widetilde{J}_\varepsilon(U_\varepsilon) = J_\varepsilon(u_\varepsilon) = J_\varepsilon(v_\varepsilon + u_{0,\varepsilon}),$$

where $u_\varepsilon \in \Psi_\varepsilon$, $v_\varepsilon \in V$ with $u_\varepsilon = v_\varepsilon + u_{0,\varepsilon}$ and

$$J_0(\bar{z}, b) = \bar{J}_0^*(v,b) \quad (v \in \mathcal{V},\, \bar{z} = \bar{v} + u_{0,0} \in \overline{\Psi}_0).$$

Recall [9] that a sequence of functions I_ε from a topological space X to $\overline{R}$ is said to Γ-converge to I_0 for the topology of X if the following conditions are satisfied for all $x \in X$:

$$\begin{cases} \forall x_\varepsilon \to x, & I_0(x) \le \liminf I_\varepsilon(x_\varepsilon), \\ \exists y_\varepsilon \to y, & I_\varepsilon(y_\varepsilon) \to I_0(y). \end{cases} \tag{20}$$

Theorem 3. *Let $\bar{J}_\varepsilon^*$ be defined in* (16) *and $\bar{J}_0^*$ be defined in* (19). *Assume $M \in \Delta_2 \cap \nabla_2$. Suppose that the continuous function $W\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfies the hypothesis* (4). *Then the sequence $\bar{J}_\varepsilon^*$ $\Gamma-$converges to $\bar{J}_0^*$ in the weak topology of $W^{1,M}(\Omega;\mathbb{R}^3) \times L^M(\Omega;\mathbb{R}^3)$, as $\varepsilon \to 0$.*

Consider the asymptotic behavior of $u_\varepsilon \in \Psi_\varepsilon$ such that

$$J_\varepsilon(u_\varepsilon) \le \inf_{u \in \Psi_\varepsilon} J_\varepsilon(u) + \gamma(\varepsilon), \tag{21}$$

where γ is a positive function such that $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Corollary 4 (The minimization problem)**.** *Assume that $u_\varepsilon \in \Psi_\varepsilon$ satisfies* (21). *Let the functions M and W be such as in Theorem* 3. *Then:*

(i) *the sequence $\left(u_\varepsilon, \frac{1}{\varepsilon} D_3 u_\varepsilon\right)$ is relatively weakly compact in $W^{1,M}(\Omega;\mathbb{R}^3) \times L^M(\Omega;\mathbb{R}^3)$;*

(ii) *the set $\mathcal{C}_{\text{film}}$ of cluster points of the sequence $\left(u_\varepsilon, \frac{1}{\varepsilon} D_3 u_\varepsilon\right)$ in the weak topology is a non-empty subset of $\mathcal{Z} \times L^M(\Omega; \mathbb{R}^3)$;*

(iii) *any point (u_∞, b) of $\mathcal{C}_{\text{film}}$ can be identified with $(\bar{u}_\infty, b) \in \overline{\Psi}_0 \times L^M(\Omega; \mathbb{R}^3)$ by the 3D-2D dimension reduction isomorphism* (10) *and $(\bar{u}_\infty, b)$ is a solution of the minimization problem*

$$\inf_{\bar{u} \in \overline{\Psi}_0} \left\{ J_0(\bar{u}, b) : b \in L^M(\Omega; \mathbb{R}^3) \right\}.$$

We start the proofs of Theorem 3 and Corollary 4, with Lemmas 5–6.

We consider the following condition (22):

$$\begin{aligned} &\exists\, i(M) \in [1, \infty), \exists\, c \in (0, \infty) \quad \text{such that} \\ &M(at) \le c\, a^{i(M)} M(t) \quad (\forall t \ge 0, \forall a \le 1). \end{aligned} \tag{22}$$

The condition (22) is equivalent to the condition

$$\begin{aligned} &\exists\, i(M) \in [1, \infty), \exists\, c \in (0, \infty) \quad \text{such that} \\ &\frac{1}{c}\, b^{i(M)} M(s) \le M(bs) \quad (\forall s \ge 0, \forall b \ge 1). \end{aligned} \tag{23}$$

Lemma 5 is a re-formulation of a part of [20, Proposition 4] (see the explanation in our previous paper [24, Lemma 4.3, pp. 592–593]).

Lemma 5. *Assume the dual Orlicz N-function M^* satisfies the condition Δ_2^{glob}, i.e., $M^*(2\tau) \le K\, M^*(\tau)$ for all $\tau \in [0, \infty)$ and for some $K \in (0, \infty)$. Then M satisfies the condition* (22) *for some $i(M) \in (1, \infty)$.*

Lemma 6 (compactness lemma)**.** *Let M and W be such as in Theorem* 3*. Let $v_\varepsilon \in W^{1,M}(\Omega; \mathbb{R}^3)$ and $b_\varepsilon \in L^M(\Omega; \mathbb{R}^3)$ be a sequence such that*

$$\sup_{\varepsilon \in (0,1)} \bar{J}_\varepsilon^*(v_\varepsilon, b_\varepsilon) \le d < +\infty. \tag{24}$$

Then there exists $\bar{d}_1 > 0$ and $\bar{d}_2 > 0$ such that:

(i)

$$\sup_{\varepsilon \in (0,1)} \|v_\varepsilon\|_{W^{1,M}(\Omega; \mathbb{R}^3)} \le \bar{d}_1 < +\infty \tag{25}$$

and

$$\sup_{\varepsilon \in (0,1)} \left\| \frac{1}{\varepsilon} D_3 v_\varepsilon \right\|_{L^M(\Omega; \mathbb{R}^3)} \le \bar{d}_2 < +\infty \tag{26}$$

and the sequence $(v_\varepsilon, \frac{1}{\varepsilon} D_3 v_\varepsilon)$ is relatively weakly compact in $W^{1,M}(\Omega; \mathbb{R}^3) \times L^M(\Omega; \mathbb{R}^3)$;

(ii) *the set of cluster points of the sequence* $(v_\varepsilon, \frac{1}{\varepsilon}D_3 v_\varepsilon)$ *in the weak topology of* $W^{1,M}(\Omega;\mathbb{R}^3) \times L^M(\Omega;\mathbb{R}^3)$ *is a non-empty subset of* $\mathcal{V} \times L^M(\Omega;\mathbb{R}^3)$.

Proof. We divide the proof into Steps 6.1–6.2, where in Step 6.2 we assume additionally $M^* \in \Delta_2^{glob}$.

Step 6.1. By (24) and (16) for $\bar{J}^*_\varepsilon$, $v_\varepsilon \in V$ for all $\varepsilon > 0$. Denote $u_\varepsilon = v_\varepsilon + u_{0,\varepsilon}$. We claim that

$$\begin{aligned}\int_\Omega M\left(\left|\left(D_\alpha u_\varepsilon \middle| \frac{D_3 u_\varepsilon}{\varepsilon}\right)\right|\right) dx \le C_1 + C_1\Bigg(\Big(\|f\|_{L^{M^*}(\Omega;\mathbb{R}^3)} \\ + \left(\|g_0^+\|_{L^{M^*}(S^+;\mathbb{R}^3)} + \|g_0^-\|_{L^{M^*}(S^-;\mathbb{R}^3)}\right)\|\operatorname{Tr}\|_{\mathcal{L}}\Big)\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})}\Bigg) \\ + C_1\|g\|_{L^{M^*}(\omega;\mathbb{R}^3)}\left\|\frac{1}{\varepsilon}D_3 u_\varepsilon\right\|_{L^M(\Omega;\mathbb{R}^3)}\end{aligned} \tag{27}$$

for some $C_1 \in (0,+\infty)$ and for all $\varepsilon \in (0,1)$. Here $\|\operatorname{Tr}\|_{\mathcal{L}} := N^+ + N^-$, where N^+ (resp., N^-) denotes the operator norm of the linear trace operator $\operatorname{Tr} : W^{1,M}(\Omega;\mathbb{R}^3) \to L^M(S^+;\mathbb{R}^3)$ (resp., $\operatorname{Tr} : W^{1,M}(\Omega;\mathbb{R}^3) \to L^M(S^-;\mathbb{R}^3)$).

For this, by the coercivity condition (4) together with (24) and Fubini Theorem, we infer that

$$\begin{aligned}&\frac{1}{C}\left(\int_\Omega M\left(\left|\left(D_\alpha u_\varepsilon \middle| \frac{D_3 u_\varepsilon}{\varepsilon}\right)\right|\right) dx - |\Omega|\right) \\ &\le d + \left|\int_\Omega (f, u_\varepsilon)dx\right| + \left|\int_{S^+}(g_0^+, u_\varepsilon)d\mathcal{H}^2\right| + \left|\int_{S^-}(g_0^-, u_\varepsilon)d\mathcal{H}^2\right| + \left|\int_\Omega\left(g, \frac{1}{\varepsilon}D_3 u_\varepsilon\right)dx\right|.\end{aligned}$$

By the generalized Hölder inequality (see, e.g., [33, Theorems 13.13, 13.11], [23]), we deduce that

$$\begin{aligned}\frac{1}{C}\left(\int_\Omega M\left(\left|\left(D_\alpha u_\varepsilon \middle| \frac{D_3 u_\varepsilon}{\varepsilon}\right)\right|\right) dx - |\Omega|\right) \le d + 2\|f\|_{L^{M^*}(\Omega;\mathbb{R}^3)}\|u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} \\ + 2\left(\|g_0^+\|_{L^{M^*}(S^+;\mathbb{R}^3)}\|u_\varepsilon^+\|_{L^M(S^+;\mathbb{R}^3)} + \|g_0^-\|_{L^{M^*}(S^-;\mathbb{R}^3)}\|u_\varepsilon^-\|_{L^M(S^-;\mathbb{R}^3)}\right) \\ + 2\|g\|_{L^{M^*}(\omega;\mathbb{R}^3)}\left\|\frac{1}{\varepsilon}D_3 u_\varepsilon\right\|_{L^M(\Omega;\mathbb{R}^3)} \le d + 2\|f\|_{L^{M^*}(\Omega;\mathbb{R}^3)}\|u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} \\ + 2\left(\|g_0^+\|_{L^{M^*}(S^+;\mathbb{R}^3)} + \|g_0^-\|_{L^{M^*}(S^-;\mathbb{R}^3)}\right)\|\operatorname{Tr}\|_{\mathcal{L}}\Big(\|u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} \\ + \|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})}\Big) + 2\|g\|_{L^{M^*}(\omega;\mathbb{R}^3)}\left\|\frac{1}{\varepsilon}D_3 u_\varepsilon\right\|_{L^M(\Omega;\mathbb{R}^3)}.\end{aligned} \tag{28}$$

By the $W^{1,M}$-generalization (see [22, Theorem 5 and 7] together with [10, Theorem 3.9], [18, Lemma 4.14], [17, Proposition 2.1]) for the Poincaré-Sobolev-type inequality (see [32, Theorem 3.6.4]), there exists $\widetilde{C} \in (0,\infty)$ such that

$$
\begin{aligned}
\|u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} &\le \widetilde{C}\left(\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})} + \int_\Gamma |u_\varepsilon| d\mathcal{H}^2\right) \\
&= \widetilde{C}\left(\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})} + \int_\Gamma |u_{0,\varepsilon}| d\mathcal{H}^2\right) \\
&\le \widetilde{C}\left(\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})} + \mathcal{H}^2(\Gamma)\sup_{x\in\Omega}|x|\right) < \infty \quad (\forall \varepsilon \in (0,1)).
\end{aligned} \tag{29}
$$

Then (28)–(29) imply (27).

Step 6.2. By the additional assumption $M^* \in \Delta_2^{glob}$, we may apply Lemma 5, and so M satisfies the condition (22) for some $i(M) \in (1,\infty)$.

We claim that

$$\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})} \le C_2 < \infty \quad (\forall \varepsilon \in (0,1)), \tag{30}$$

$$\|u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} \le C_3 < \infty \quad (\forall \varepsilon \in (0,1)), \tag{31}$$

$$\left\|\frac{1}{\varepsilon} D_3 u_\varepsilon\right\|_{L^M(\Omega;\mathbb{R}^3)} \le C_4 < \infty \quad (\forall \varepsilon \in (0,1)), \tag{32}$$

$$\int_\Omega M\left(\left|\left(D_\alpha u_\varepsilon \middle| \frac{D_3 u_\varepsilon}{\varepsilon}\right)\right|\right) dx \le C_5 < \infty \quad (\forall \varepsilon \in (0,1)) \tag{33}$$

for some C_2, C_3, C_4, C_5.

For this, by (27) we infer that

$$
\begin{aligned}
&\frac{1}{1 + \|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})} + \|\frac{1}{\varepsilon} D_3 u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)}} \int_\Omega M\left(\left|D_\alpha u_\varepsilon \middle| \frac{1}{\varepsilon} D_3 u_\varepsilon\right|\right) dx \\
&\le C_6 < \infty
\end{aligned} \tag{34}
$$

for all $\varepsilon \in (0,1)$ and for some C_6.

Consider the case when $\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})}/2 \ge 1 > 0$ and $\|\frac{1}{\varepsilon} D_3 u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)}/2 \ge 1 > 0$. Since $0 < \frac{\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})}}{2} < \|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})}$ and $0 < \frac{\|\frac{1}{\varepsilon} D_3 u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)}}{2} <$

$\|\frac{1}{\varepsilon}D_3u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)}$ by the definition of the Luxemburg norm and by (22), we deduce that

$$
\begin{aligned}
1 &< \int_\Omega M\left(\frac{|Du_\varepsilon|}{\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})}/2}\right)dx \\
&\leq \left(\frac{2}{\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})}}\right)^{i(M)} \int_\Omega M(|Du_\varepsilon|)dx \quad (\forall\varepsilon\in(0,1))
\end{aligned}
\tag{35}
$$

and

$$
\begin{aligned}
1 &< \int_\Omega M\left(\frac{\frac{1}{\varepsilon}|D_3u_\varepsilon|}{\|\frac{1}{\varepsilon}D_3u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)}/2}\right)dx \\
&\leq \left(\frac{2}{\|\frac{1}{\varepsilon}D_3u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)}}\right)^{i(M)} \int_\Omega M\left(\left|\frac{1}{\varepsilon}D_3u_\varepsilon\right|\right)dx \qquad (\forall\varepsilon\in(0,1)).
\end{aligned}
\tag{36}
$$

Obviously

$$
\begin{aligned}
&\int_\Omega M\left(\left|\left(Du_\varepsilon\right|\right)dx + \int_\Omega M\left(\left|\frac{1}{\varepsilon}D_3u_\varepsilon\right|\right)dx \\
&\leq 2\int_\Omega M\left(\left|\left(D_\alpha u_\varepsilon\middle|\frac{1}{\varepsilon}D_3u_\varepsilon\right)\right|\right)dx \qquad (\forall\varepsilon\in(0,1)).
\end{aligned}
\tag{37}
$$

Therefore, (34), (35)–(36) and (37) implies

$$
A\left(\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})}, \left\|\frac{1}{\varepsilon}D_3u_\varepsilon\right\|_{L^M(\Omega;\mathbb{R}^3)}\right) \leq C_6 < \infty \tag{38}
$$

whenever $\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})} \geq 2$ and $\|\frac{1}{\varepsilon}D_3u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} \geq 2$. Here

$$
A(s,t) := \frac{1}{2}\cdot\frac{s^{i(M)}+t^{i(M)}}{2^{i(M)}(1+s+t)}.
$$

Since $i(M) > 1$, $A(s,2) \to +\infty$ as $s \to +\infty$ and $A(2,t) \to +\infty$ as $t \to +\infty$ and so there exists $C_7, C_8 \in (0,\infty)$ such that $A(s,2) > C_6$ $(\forall s > C_7)$ and $A(2,t) > C_6$ $(\forall t > C_8)$. Hence, (38) implies the claims (30) and (32), where $C_2 = C_4 := \max\{C_7, C_8, 2\}$. By (29) and (27) we deduce the claims (31) and (33).

The remaining steps of the proof are analogous to Steps 4.3–4.6 in our previous paper [24]. The arguments of these Steps allow to consider also the general case of M^*. ■

Remind that the quasiconvex envelope $\mathcal{Q}g : \mathbb{R}^{m\times n} \to \mathbb{R}$ of a continuous function $g : \mathbb{R}^{m\times n} \to \mathbb{R}$ is defined (see, e.g., [8, Theorem 6.9]) by

$$\mathcal{Q}g(E) := \inf\left\{\frac{1}{\operatorname{meas}(B)}\int_B g(E+D\varphi)dx : \varphi \in C_0^\infty(B;\mathbb{R}^m)\right\}$$

for all $E \in \mathbb{R}^{m\times n}$ where B is the open unit ball of $\mathbb{R}^n$.

We recall that a sequence $w_n \in L^1(G;\mathbb{R}^d)$ is said L^1-equi-integrable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_E |w_n|dx < \varepsilon$ whenever $E \subset G$ with $\mathcal{L}^N(E) < \delta$, where $G \subset \mathbb{R}^N$.

The next lemma is the direct $W^{1,M}$-generalization in the case of $M \in \Delta_2 \cap \nabla_2$ of the Fonseca-Müller-Pedregal Decomposition Lemma in the Sobolev $W^{1,p}$-space [14].

Lemma 7 (decomposition lemma)**.** *Assume $M \in \Delta_2 \cap \nabla_2$. Let G be an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary. Let $w_n \in W^{1,M}(G;\mathbb{R}^d)$ be such that $w_n \rightharpoonup w_0$ weakly in $W^{1,M}(G;\mathbb{R}^d)$. Then there exists a subsequence of w_n (not relabelled) and a sequence $z_n \in W^{1,M}(G;\mathbb{R}^d)$ such that $z_n \rightharpoonup w_0$ weakly in $W^{1,M}(G;\mathbb{R}^d)$, $z_n = w_0$ in a neighborhood od ∂G,*

$$\mathcal{L}^N(\{x \in G : w_n(x) \neq z_n(x), Dw_n(x) \neq Dz_n(x)\}) \to 0,$$

as $n \to +\infty$, and the sequence $M(|Dz_n|)$ is L^1-equi-integrable on G.

Proposition 8. *Let $\mathcal{Q}_\infty W$ be defined by* (7) *and let $W\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ be a continuous function satisfying the hypothesis* (4)*. Then*

$$\mathcal{Q}_\infty \mathcal{Q}W(\bar{F}|b) = \mathcal{Q}_\infty W(\bar{F}|b), \tag{39}$$

where $\mathcal{Q}W$ denotes the quasiconvex envelope of W.

Proof of Proposition 8 is analogous to the proof of Proposition 2.6 in [5]. It is enough to apply $W^{1,M}$-generalization in [13, Theorem 3.1] for the Acerbi-Fusco weak l.s.c. $W^{1,p}$-theorem [1, Theorem II.5] and Decomposition Lemma 7.

Let $\mathcal{A}(\omega)$ be a family of all open subsets of ω. According to (15) define the functional $E_\varepsilon : W^{1,M}(\Omega;\mathbb{R}^3) \times L^M(\Omega;\mathbb{R}^3) \times \mathcal{A}(\omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$E_\varepsilon(u,b,A) = \begin{cases} \int_{A\times I} W(D_\alpha u|\frac{1}{\varepsilon}D_3 u)dx & \text{if } \frac{1}{\varepsilon}D_3 u = b \\ & \text{and } u \in \Psi_\varepsilon \\ +\infty & \text{otherwise.} \end{cases}$$

Denote by $E_0 : \mathcal{Z} \times L^M(\Omega;\mathbb{R}^3) \times \mathcal{A}(\omega) \to \mathbb{R} \cup \{+\infty\}$ the Γ- lower limit (see [9]) of E_ε, i.e.,

$$(40)\quad E_0(u,b,A) := \inf\left\{ \liminf_{n\to+\infty} \int_{A\times I} W(D_\alpha u_n|\lambda_n D_3 u_n)dx : u_n \rightharpoonup u \text{ weakly in } W^{1,M}(A\times I;\mathbb{R}^3), \lambda_n D_3 u_n \rightharpoonup b \text{ weakly in } L^M(A\times I;\mathbb{R}^3)\right\},$$

where $\lambda_n := (\varepsilon_n)^{-1}$. Later on, we say that $u_n \to u$ in $L^M_{\text{loc}}(A\times I;\mathbb{R}^3)$ if for any $D \subset\subset A$ $u_n \to u$ in $L^M(D\times I;\mathbb{R}^3)$-norm.

Lemma 9. *Let the functions M and W be such as in Theorem 3 and E_0 be defined by (40). Then for any sequence $\lambda_n \to +\infty$, there exists subsequence λ_{n_k} such that for each $(u,b) \in \mathcal{Z}\times L^M(\Omega;\mathbb{R}^3)$, the set function $E_0(u,b,\cdot)$ is a trace of a Radon measure, absolutely continuous with respect to the 2-dimensional Lebesgue measure.*

Lemma 10. *Let the functions M and W be such as in Theorem 3. Let $A \in \mathcal{A}(\omega)$, $L \in \mathbb{R}$, $u \in \mathcal{Z}$ and consider a sequences $u_n \in W^{1,M}(A\times I;\mathbb{R}^3)$ and $\lambda_n \in \mathbb{R}$ such that $\lambda_n \to +\infty$, $u_n \to u$ in $L^M_{loc}(A\times I;\mathbb{R}^3)$-norm, $\lambda_n D_3 u_n \rightharpoonup b$ weakly in $L^M(A\times I;\mathbb{R}^3)$ and*

$$\lim_{n\to+\infty} \int_{A\times I} W(D_\alpha u_n|\lambda_n D_3 u_n)dx = L.$$

Then there exists a subsequence λ_{n_k} of λ_n and a sequence $\tilde{u}_k \in W^{1,M}(A\times I;\mathbb{R}^3)$ such that $\tilde{u}_k = u$ on $\Theta_k(\partial A)\times I$ for some neighborhood $\Theta_k(\partial A)$, $\tilde{u}_k \to u$ in $L^M_{loc}(A\times I;\mathbb{R}^3)$-norm, $\lambda_n D_3\tilde{u}_k \rightharpoonup b$ weakly in $L^M(A\times I;\mathbb{R}^3)$ and

$$\limsup_{k\to+\infty} \int_{A\times I} W(D_\alpha \tilde{u}_k|\lambda_n D_3 \tilde{u}_k)dx \le L.$$

The proofs of Lemma 9 and Lemma 10 are analogous to the proofs of Lemma 2.1 and Lemma 2.2 in [4].

Lemma 11. *The infimum in (40) for E_0 remains unchanged if we replace W by its quasiconvex envelope $\mathcal{Q}W$.*

Proof of Lemma 11 is analogous to the proof of Proposition 2.7 in [5]. It is enough to apply $W^{1,M}$-generalization in [13, Theorem 3.1] for the Acerbi-Fusco weak l.s.c. $W^{1,p}$-theorem [1, Theorem II.5] and the fact that embedding $W^{1,M}(A\times I;\mathbb{R}^3) \hookrightarrow\hookrightarrow L^M_{\text{loc}}(A\times I;\mathbb{R}^3)$ is compact (see Donaldson-Trudinger [10, Theorem 3.9] together with Gossez [18, Proposition 4.3]).

Notice that by Proposition 8 and Lemma 11 we may assume without loss of generality that W is quasiconvex. Therefore by the hypothesis (4), $M \in \Delta_2$ together with Focardi [13, Proposition 3.2] W satisfies

$$(41)\qquad |W(\xi_1) - W(\xi_2)| \le C(1 + h(1 + |\xi_1| + |\xi_2|))|\xi_1 - \xi_2|$$

for some $C \in (0, +\infty)$ and for all $\xi_1, \xi_2 \in \mathbb{R}^{3\times 3}$, where h denotes the right derivative of M.

Proposition 12. *Assume that a quasiconvex function $W\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfies the hypothesis* (4) *and $M \in \Delta_2 \cap \nabla_2$. Then the functions*

$$(\bar{F}, b) \mapsto \mathcal{Q}_k W(\bar{F}|b) \in \mathbb{R}, \qquad (\bar{F}, b) \mapsto \mathcal{Q}_\infty W(\bar{F}|b) \in \mathbb{R} \tag{42}$$

are continuous on $\mathbb{R}^{3\times 2} \times L^M(I; \mathbb{R}^3)$.

Proof. Let $\lambda > 0$ and $k \in \mathbb{N}$ be fixed and define

$$\begin{aligned} \mathcal{Q}_k^\lambda W(\bar{F}, b) := \inf \Bigg\{ \int_Q W(\bar{F} + D_\alpha\varphi|\lambda D_3\varphi)dx : \varphi \in W^{1,M}(Q; \mathbb{R}^3), \varphi(\cdot, x_3) \\ \text{is } Q' \text{ periodic } \mathcal{L}^1 \text{ a.e. } x_3 \in I, \left| \int_Q \lambda D_3\varphi\theta_i dx - \int_I b\theta_i dx_3 \right| < \frac{1}{k}\ (i = 1, \ldots, k) \Bigg\}. \end{aligned} \tag{43}$$

Let $(\bar{F}, b), (\bar{F}', b') \in \mathbb{R}^{3\times 2} \times L^M(I; \mathbb{R}^3)$. For any infimizing sequence $\{\varphi_n\}$ in the definition of $\mathcal{Q}_k^\lambda W(\bar{F}, b)$ consider the sequence $\psi_n := \varphi_n + \frac{\int_0^{x_3}(b'(s)-b(s))ds}{\lambda}$. By (4) and by considering the function $\varphi := \frac{\int_0^{x_3} b(s)ds}{\lambda}$ in the definition (43), we obtain that

$$-\frac{1}{C}\mathcal{L}^3(Q) \leq \mathcal{Q}_k^\lambda W(\bar{F}, b) \leq \int_I W(\bar{F}, b)dx_3.$$

Hence we may assume that

$$\begin{aligned} \int_Q W(\bar{F} + D_\alpha\varphi_n|\lambda D_3\varphi_n)dx &\leq \mathcal{Q}_k^\lambda W(\bar{F}, b) + 1 \\ &\leq \int_I W(\bar{F}, b)dx_3 + 1. \end{aligned} \tag{44}$$

Since

$$D_\alpha\psi_n = D_\alpha\varphi_n, \quad D_3\psi_n = D_3\varphi_n + \frac{b' - b}{\lambda},$$

then

$$\left| \int_Q \lambda D_3\psi_n\theta_i dx - \int_I b'\theta_i dx_3 \right| = \left| \int_Q \lambda D_3\varphi_n\theta_i dx - \int_I b\theta_i dx_3 \right| < \frac{1}{k}\ (i = 1, \ldots, k),$$

which implies, that $\{\psi_n\}$ is admissible for the definition of $\mathcal{Q}_k^\lambda W(\bar{F}', b')$. Observe that $\int_Q M\left(\frac{|a(x_3)|}{\alpha}\right)dx = \int_I M\left(\frac{|a(x_3)|}{\alpha}\right)dx_3$ for $a \in L^M(I)$, and so $\|a\|_{L^M(Q)} = \|a\|_{L^M(I)}$ follows. By [13, Proposition 3.2] the quasiconvex function W satisfies (41). Thus by the Hölder inequality in L^M-norm [23], we deduce that

$$
\begin{aligned}
&\left|\int_Q W(\bar F' + D_\alpha\psi_n|\lambda D_3\psi_n)dx - \int_Q W(\bar F + D_\alpha\varphi_n|\lambda D_3\varphi_n)dx\right| \\
&\le C\int_Q (1 + h(1 + |(\bar F' + D_\alpha\psi_n|\lambda D_3\psi_n)| + |(\bar F + D_\alpha\varphi_n|\lambda D_3\varphi_n)|) \\
&\cdot |(\bar F' + D_\alpha\psi_n|\lambda D_3\psi_n) - (\bar F + D_\alpha\varphi_n|\lambda D_3\varphi_n)|)dx \\
&\le 2C\|1 + h(1 + |(\bar F' + D_\alpha\psi_n|\lambda D_3\psi_n)| + |(\bar F + D_\alpha\varphi_n|\lambda D_3\varphi_n)|)\|_{L^{M^*}(Q)} \\
&\cdot \||\bar F' - \bar F| + |b'(\cdot) - b(\cdot)|\|_{L^M(I)} \\
&= 2C\|1 + h(1 + |(\bar F' + D_\alpha\varphi_n|\lambda D_3\varphi_n + b'(\cdot) - b(\cdot))| \\
&+ |(\bar F + D_\alpha\varphi_n|\lambda D_3\varphi_n)|)\|_{L^{M^*}(Q)} \cdot \||\bar F' - \bar F| + |b'(\cdot) - b(\cdot)|\|_{L^M(I)}.
\end{aligned} \tag{45}
$$

By the coercivity condition (4), (44) implies that

$$
\sup_n \int_Q M(|(\bar F + D_\alpha\varphi_n|\lambda D_3\varphi_n)|)dx \le C_2\left(\int_I W(\bar F, b)dx_3 + 1\right) < +\infty, \tag{46}
$$

and so by using the Luxemburg norm, we obtain that

$$
\sup_n \|M(|(\bar F + D_\alpha\varphi_n|\lambda D_3\varphi_n)|)\|_{L^M(Q)} \le C_2\left(\int_I W(\bar F, b)dx_3 + 1\right) + 1,
$$

$$
\begin{aligned}
&\|1 + |(\bar F' + D_\alpha\varphi_n|\lambda D_3\varphi_n + b'(\cdot) - b(\cdot))| + |(\bar F + D_\alpha\varphi_n|\lambda D_3\varphi_n)|\|_{L^M(Q)} \\
&\le C_3\left(1 + |\bar F'| + |\bar F| + \|b'(\cdot)\|_{L^M(I)} + \|b(\cdot)\|_{L^M(I)}\right) \\
&+ 2C_2\left(\int_I W(\bar F, b)dx_3 + 1\right) + 2 =: \widetilde C_4(\bar F, b, \bar F', b'),
\end{aligned} \tag{47}
$$

where $C_3 := \|1\|_{L^M(Q)} + 1$. By the Płuciennik-Tian-Wang lemma (see [35, Lemma 1]) for $M \in \Delta_2$, there exists a function $r\colon [0, +\infty) \to [0, +\infty)$ such that $\|z\|_{L^M(Q)} \le a \Rightarrow \|h(|z|)\|_{L^{M^*}(Q)} \le r(a)$. Define

$$
r_M[a] = \sup\left\{\|h(|z|)\|_{L^{M^*}(Q)} : \|z\|_{L^M(Q)} \le a\right\}. \tag{48}
$$

Then $0 \le r_M[a] \le r(a) < +\infty$ and r_M is nondecreasing. Hence (47) and (45) imply that

$$
\begin{aligned}
(49)\quad &\left|\int_Q W(\bar F' + D_\alpha\psi_n|\lambda D_3\psi_n)dx - \int_Q W(\bar F + D_\alpha\varphi_n|\lambda D_3\varphi_n)dx\right| \\
&\le 2C(C_5 + r_M[\widetilde C_4(\bar F, b, \bar F', b')]) \cdot (|\bar F' - \bar F| + \|b'(\cdot) - b(\cdot)\|_{L^M(I)}) < +\infty,
\end{aligned}
$$

where $C_5 := \|1\|_{L^{M^*}(Q)}$.

By the upper bound condition (4) and $M \in \Delta_2$,

$$
\begin{aligned}
(50)\qquad \int_I W(\bar F|b(x_3))dx_3 \le \widetilde C_6(1 &+ M(|\bar F|) \\
&+ M(|\bar F'|) + \varrho_M(|b|) + \varrho_M(|b'|)) < +\infty
\end{aligned}
$$

for some $\widetilde C_6 \in (0, +\infty)$ and for all $\bar F, \bar F', b, b'$. Here

$$
(51)\qquad \varrho_M(|b|) := \int_I M(|b(x_3)|)dx_3, \quad \varrho_M(|b'|) := \int_I M(|b'(x_3)|)dx_3.
$$

Hence (50) and (49) and the definition of $\widetilde C_4(\bar F, b, \bar F', b')$ in (47) imply that

$$
\begin{aligned}
(52)\quad &\left|\int_Q W(\bar F' + D_\alpha\psi_n|\lambda D_3\psi_n)dx - \int_Q W(\bar F + D_\alpha\varphi_n|\lambda D_3\varphi_n)dx\right| \\
&\le \widetilde C_7(\bar F, b, \bar F', b') \cdot (|\bar F' - \bar F| + \|b'(\cdot) - b(\cdot)\|_{L^M(I)}) < +\infty,
\end{aligned}
$$

where

$$
\begin{aligned}
&\widetilde C_7(\bar F, b, \bar F', b') \\
(53)\quad &:= 2C\Big(C_5 + r_M[C_3(1 + |\bar F'| + |\bar F| + \|b'(\cdot)\|_{L^M(I)} + \|b(\cdot)\|_{L^M(I)}\Big) \\
&\quad + 2C_2\Big(\widetilde C_6\left(1 + M(|\bar F|) + M(|\bar F'|) + \varrho_M(|b|) + \varrho_M(|b'|)\right) + 1) + 2\Big) < \infty.
\end{aligned}
$$

By the definition of $\mathcal{Q}_k^\lambda W(\bar F', b')$, (52) implies that

$$
\begin{aligned}
(54)\quad &\mathcal{Q}_k^\lambda W(\bar F', b') \le \int_Q W(\bar F' + D_\alpha\psi_n|\lambda D_3\psi_n)dx \\
&\le \int_Q W(\bar F + D_\alpha\varphi_n|\lambda D_3\varphi_n)dx + \widetilde C_7(\bar F, b, \bar F', b') \cdot (|\bar F' - \bar F| + \|b'(\cdot) - b(\cdot)\|_{L^M(I)}),
\end{aligned}
$$

and letting $n \to +\infty$, we infer that

$$
(55)\quad \mathcal{Q}_k^\lambda W(\bar F', b') \le \mathcal{Q}_k^\lambda W(\bar F|b) + \widetilde C_7(\bar F, b, \bar F', b') \cdot (|\bar F' - \bar F| + \|b'(\cdot) - b(\cdot)\|_{L^M(I)}).
$$

Using the same arguments for the pair $(\bar{F}'|b')$ in place of $(\bar{F}|b)$, we deduce that

$$\mathcal{Q}_k^\lambda W(\bar{F}|b) \leq \mathcal{Q}_k^\lambda W(\bar{F}'|b') + \widetilde{C}_7(\bar{F}, b, \bar{F}', b') \cdot (|\bar{F}' - \bar{F}| + \|b'(\cdot) - b(\cdot)\|_{L^M(I)}). \tag{56}$$

Taking infimum over $\lambda > 0$ and then letting $k \to +\infty$ by (55) and (56) we deduce, that

$$\begin{aligned} &|\mathcal{Q}_k W(\bar{F}|b) - \mathcal{Q}_k W(\bar{F}'|b')|, |\mathcal{Q}_\infty W(\bar{F}|b) - \mathcal{Q}_\infty W(\bar{F}'|b')| \\ &\leq \widetilde{C}_7(\bar{F}, b, \bar{F}', b') \cdot (|\bar{F}' - \bar{F}| + \|b'(\cdot) - b(\cdot)\|_{L^M(I)}) < +\infty. \end{aligned} \tag{57}$$

By $M \in \Delta_2$ and the definition of $\widetilde{C}_7(\bar{F}, b, \bar{F}', b')$ in (53), we deduce that (57) implies the continuity of $\mathcal{Q}_k W$ and $\mathcal{Q}_\infty W$ on $\mathbb{R}^{3\times 2} \times L^M(I;\mathbb{R}^3)$. ∎

Lemma 13. *Let W be a quasiconvex continuous function satisfying the hypothesis* (4) *and $M \in \Delta_2 \cap \nabla_2$. Consider the Γ-lower limit E_0 defined in* (40)*. Then*

$$E_0(u, b, A) \geq \int_A \mathcal{Q}_\infty W(D_\alpha u | b(x_\alpha, \cdot)) dx_\alpha \tag{58}$$

for all $(u, b, A) \in \mathcal{Z} \times L^M(\Omega;\mathbb{R}^3) \times \mathcal{A}(\omega)$.

Proof. By Proposition 12, $\mathcal{Q}_\infty W(D_\alpha u|b)$ and $\mathcal{Q}_k W(\bar{F}|b)$ are measurable non-negative functions.

Step 13.1. Let $k \in \mathbb{N}$, $b \in L^M(I;\mathbb{R}^3)$, $u(x_\alpha) := \bar{F}x_\alpha + u_0$ with $\bar{F} \in \mathbb{R}^{3\times 2}$, $u_0 \in \mathbb{R}^3$. By Lemma 10 we may restrict ourselves, in the definition (40) to sequences having the same trace as it's limit. Consider the sequence

$$w_n(x) := \varphi_n(x) + (\bar{F}x_\alpha + u_0),$$

where $\varphi_n \in W^{1,M}(Q;\mathbb{R}^3)$ is such that $\varphi_n = 0$ on $\partial Q' \times I$, $\varphi_n \rightharpoonup 0$ weakly in $W^{1,M}(Q;\mathbb{R}^3)$ and $\lambda_n D_3\varphi_n \rightharpoonup b$ weakly in $L^M(Q;\mathbb{R}^3)$. Then $\varphi_n(\cdot, x_3)$ is Q'-periodic. For any $\psi \in L^{M^*}(Q';\mathbb{R}^3)$ by the Jensen inequality [23], together with $M \in \Delta_2$, we infer that

$$\begin{aligned} &\int_A M^*\left(\left|\int_I \theta_i(x_3)\psi(x_\alpha)dx_3\right|\right) dx_\alpha \leq \int_A \int_I M^*(|\theta_i(x_3)\psi(x_\alpha)|)dx_3 dx_\alpha) \\ &\leq \int_A \int_I M^*(\|\theta_i(x_3)\|_{L^\infty(I;\mathbb{R}^3)}|\psi(x_\alpha)|)dx_3 dx_\alpha \\ &\leq a\left(\|\theta_i(x_3)\|_{L^\infty(I;\mathbb{R}^3)}\right) \int_A \int_I M^*(|\psi(x_\alpha)|)dx_\alpha \\ &+ b\left(\|\theta_i(x_3)\|_{L^\infty(I;\mathbb{R}^3)}\right) \cdot \mathcal{L}^2(A) < \infty, \end{aligned} \tag{59}$$

for all $i \in \mathbb{N}$. Therefore using $\theta_i(x_3)\cdot\psi(x_\alpha)$ as a test function from $L^{M^*}(Q'\times I;\mathbb{R}^3)$, we deduce that

$$\begin{aligned}&\int_{Q'}\left(\lambda_n D_3\varphi_n\theta_i dx_3 - \bar{b}_i\right)\psi(x_\alpha)dx_\alpha\\ &= \int_{Q'}\int_I \left(\lambda_n D_3\varphi_n - b(s)\right)\theta_i\psi(x_\alpha)dx_\alpha ds \to 0,\end{aligned}$$

and so $\int_I \lambda_n D_3\varphi_n\theta_i dx_3 \rightharpoonup \bar{b}_i$ weakly in $L^M(Q';\mathbb{R}^3)$ for $i = 1,\dots,k$, where $\bar{b}_i := \int_I b(s)\theta_i ds$. Therefore there exists $n_k \in \mathbb{N}$ such that, for $n \geq n_k$ we have

$$\left|\int_Q \lambda_n D_3\varphi_n\theta_i dx - \bar{b}_i\right| \leq \frac{1}{k} \quad (i = 1,\dots,k).$$

Thus φ_n are admissible functions for the definition of $\mathcal{Q}_kW$ and by the definition of $\mathcal{Q}_kW$, we have

$$\liminf_{n\to+\infty}\int_Q W(\bar{F} + D_\alpha\varphi_n|\lambda_n D_3\varphi_n)dx \geq \mathcal{Q}_kW(\bar{F}|b).$$

By taking supremum over all $k \in \mathbb{N}$

$$\liminf_{n\to+\infty}\int_Q W(\bar{F} + D_\alpha\varphi_n|\lambda_n D_3\varphi_n)dx \geq \mathcal{Q}_\infty W(\bar{F}|b). \tag{60}$$

Since $\{w_n\}_{n\in\mathbb{N}}$ is admissible for the definition of (40), we complete the proof of (58) for the case in Step 13.1 by taking the infimum over all admissible sequences in (60), and then we get the inequality

$$\begin{aligned}E_0(\bar{F}x_\alpha + u_0, b, Q') &\geq \int_{Q'}\mathcal{Q}_\infty W(\bar{F}|b)dx_\alpha\\ &= \mathcal{L}^2(Q')\cdot\mathcal{Q}_\infty W(\bar{F}|b) = \mathcal{Q}_\infty W(\bar{F}|b).\end{aligned} \tag{61}$$

We omit the general case for the proof of the inequality (58), since it is analogous to Step 2 in Proposition 3.4 [5]. It is enough to apply $W^{1,M}$-generalization in [13, Theorem 3.1] for the Acerbi-Fusco weak l.s.c. $W^{1,p}$-theorem [1, Theorem II.5], the fact that embedding $W^{1,M}(A\times I;\mathbb{R}^3) \hookrightarrow\hookrightarrow L^M_{\mathrm{loc}}(A\times I;\mathbb{R}^3)$ is compact (see Donaldson-Trudinger [10, Theorem 3.9] together with Gossez [18, Proposition 4.3]) and differentiability properties of the Orlicz-Sobolev functions [3, Lemma 3.1, Lemma 3.2]. ■

Lemma 14. *Under the hypothesis of Lemma* 13, *we have*

$$E_0(u,b,A) \leq \int_A \mathcal{Q}_\infty W(D_\alpha u|b(x_\alpha,\cdot))dx_\alpha \tag{62}$$

for all $(u,b,A) \in \mathcal{Z}\times L^M(\Omega;\mathbb{R}^3)\times\mathcal{A}(\omega)$.

Proof. By Proposition 12, $\mathcal{Q}_\infty W(D_\alpha u|b)$ and $\mathcal{Q}_k W(\bar{F}|b)$ are measurable non-negative functions.

Step 14.1. Let $u(x_\alpha) := \bar{F}x_\alpha + u_0$ with $\bar{F} \in \mathbb{R}^{3\times 2}$, $u_0 \in \mathbb{R}^3$ and $b \in L^M(I;\mathbb{R}^3)$. Since $\mathcal{Q}_k W(\bar{F}|b)$ is nondecreasing in k, by Proposition 12

$$\mathcal{Q}_k W(\bar{F}|b) \in \mathbb{R},\ \mathcal{Q}_\infty W(\bar{F}|b) = \lim_{k\to\infty} \mathcal{Q}_k W(\bar{F}|b) \in \mathbb{R} \tag{63}$$

By the definition of $\mathcal{Q}_k W(\bar{F}|b)$ there exists $t_k \in \mathbb{R}$, $\varphi^k \in W^{1,M}(Q;\mathbb{R}^3)$, $\varphi^k(\cdot, x_3)$ is Q'-periodic $\mathcal{L}^1$ a.e. $x_3 \in I$,

$$\left| \int_Q t_k D_3\varphi^k \theta_i dx - \int_I b\theta_i dx_3 \right| < \frac{1}{k} \quad (i = 1, \dots, k) \tag{64}$$

and

$$\mathcal{Q}_k W(\bar{F}|b) \leq \int_Q W(\bar{F} + D_\alpha\varphi^k | t_k D_3\varphi^k) dx < \mathcal{Q}_k W(\bar{F}|b) + \frac{1}{k}. \tag{65}$$

Extending Q'-periodically of the Q'-periodic function φ, we define $\varphi_n^k : \mathbb{R}^2 \times I \to \mathbb{R}^3$ by

$$\varphi_n^k(x) := \frac{t_k}{\lambda_n}\varphi\left(\frac{\lambda_n}{t^k}x_\alpha, x_3\right).$$

Observe that the function

$$y_\alpha \mapsto t_k \int_I D_3\varphi^k(y_\alpha, x_3)\theta_i(x_3)dx_3$$

belongs to $L^M(A;\mathbb{R}^3)$, since by the Jensen inequality [23] and $M \in \Delta_2$, we infer (cf. (59)) that

$$\begin{aligned}
\int_A M\left(\left|\int_I D_3\varphi^k(y_\alpha, x_3)\theta_i(x_3)dx_3\right|\right)dx_\alpha &\leq \int_A\int_I M\left(\left|D_3\varphi^k(y_\alpha, x_3)\theta_i(x_3)\right|\right)dx_3 dx_\alpha \\
&\leq a_2\left(\|\theta_i\|_{L^\infty(I;\mathbb{R}^3)}\right)\int_A\int_I M(|D_3\varphi^k(y_\alpha, x_3)|)dx_3 dx_\alpha \\
&\quad + b_2\|\theta_i(x_3)\|_{L^\infty(I;\mathbb{R}^3)} \cdot \mathcal{L}^3(A \times I) < +\infty
\end{aligned}$$

Applying the $L^M(Q')$-version (see Pedregal [34, Homogenization Theorem 7.1, Remark p. 121]) for the Riemann-Lebesgue lemma in $L^p(Q')$-spaces (see, e.g., [8]) we infer that

$$\begin{aligned}
&\lambda_n \int_I D_3\varphi_n^k\theta_i dx_3 = t_k \int_I D_3\varphi^k\left(\frac{\lambda_n}{t_k}x_\alpha, x_3\right)\theta_i dx_3 \\
&\rightharpoonup\ t_k \int_I\int_{Q'} D_3\varphi^k(y_\alpha, x_3)dy_\alpha dx_3 =: \int_I b(s)\theta_i ds + r_i^k, \quad \text{as } n \to +\infty
\end{aligned} \tag{66}$$

weakly in $L^M(A;\mathbb{R}^3)$. By (64) $|r_i^k| < \frac{1}{k}$ $(i = 1, \dots, k)$. Define

$$H^k(x_\alpha, x_3) := \left(\bar F + D_\alpha \varphi^k(x_\alpha, x_3) | t_k D_3 \varphi^k(x_\alpha, x_3)\right),$$
$$\widetilde{W}^k(x_\alpha) := \int_I W\left(H^k(x_\alpha, x_3)\right) dx_3.$$

By the coercivity condition (4) and $H^k \in W^{1,M}(Q;\mathbb{R}^3)$,

$$\int_{Q'} \widetilde{W}^k(x_\alpha) dx_\alpha = \int_Q W\left(H^k(x_\alpha, x_3)\right) dx_3 dx_\alpha$$
$$\le C_2 \left(1 + \int_Q M(|\bar F + D_\alpha \varphi^k(x_\alpha, x_3)| t_k D_3 \varphi^k(x_\alpha, x_3)|) dx_\alpha dx_3\right) < \infty,$$

and so $\widetilde{W}^k \in L^1(Q';\mathbb{R}^3)$. Using the Riemann-Lebesgue lemma, we deduce that

(67)
$$\lim_{n\to+\infty} \int_{A\times I} W(\bar F + D_\alpha \varphi_n^k | \lambda_n D_3 \varphi_n^k) dx = \lim_{n\to+\infty} \int_{Q'} 1_A(x_\alpha) \widetilde{W}^k\left(\frac{\lambda_n}{t_k} x_\alpha\right) dx_\alpha$$
$$= \int_{Q'} 1_A(x_\alpha) \left(\int_{Q'} \widetilde{W}^k(y_\alpha) dy_\alpha\right) dx_\alpha = \mathcal{L}^2(A) \int_Q W(\bar F + D_\alpha \varphi^k | t_k D_3 \varphi^k) dx.$$

In view of the coercivity condition (4) and by the Alaoglu-Bourbarki theorem [11, Theorem V.4.2] using the Moore lemma [11, Lemma I.7.6], by (63)–(67) we may find a subsequence $\{\lambda_{n_k}\}$ and $\{\varphi_{n_k}^k\}$ such that $\varphi_{n_k}^k \rightharpoonup 0$ weakly in $W^{1,M}(A\times I;\mathbb{R}^3)$, $\lambda_{n_k} \int_I D_3 \varphi_{n_k}^k \theta_i dx_3 \rightharpoonup \int_I b\theta_i dx_3$ weakly in $L^M(A;\mathbb{R}^3)$ for all $i \in \mathbb{N}$ and

$$\lim_{k\to+\infty} \int_{A\times I} W(\bar F + D_\alpha \varphi_{n_k}^k | \lambda_{n_k} D_3 \varphi_{n_k}^k) dx = \mathcal{L}^2(A) \mathcal{Q}_\infty W(\bar F | b).$$

Hence by the argument in [23, pp. 81–82] together with the Stone-Weierstrass approximation theorem and by our choice of the family $\{\theta_i\}_{i\in\mathbb{N}}$ for the definition (8), we deduce that $\lambda_{n_k} D_3 \varphi_{n_k}^k \rightharpoonup b$ weakly in $L^M(A\times I;\mathbb{R}^3)$ as $k \to \infty$. Consequently,

$$E_0(u, b, A) \le \mathcal{L}^2(A) \mathcal{Q}_\infty W(\bar F | b).$$

Step 14.2. By Lemma 9 and Step 14.1 the inequality (62) holds for piecewise affine functions.

Step 14.3. By Proposition 12 and Step 14.2 the inequality (62) holds for any $(u, b, A) \in W^{1,M}(\omega;\mathbb{R}^3) \times L^M(\Omega;\mathbb{R}^3) \times \mathcal{A}(\omega)$. ∎

***Proof of Theorem* 3.** Let $u_\varepsilon \in \Psi_\varepsilon$ be such that $u_\varepsilon \rightharpoonup \bar{u}$ weakly in $W^{1,M}(\Omega;\mathbb{R}^3)$, $\frac{1}{\varepsilon}D_3u_\varepsilon \rightharpoonup b$ weakly in $L^M(\Omega;\mathbb{R}^3)$. It is easy to check by the representation (3), the isomorphism (18) and by the Fubini theorem that $P_\varepsilon(u_\varepsilon) \to P_0(\bar{u}, b)$ and $P_\varepsilon(v_\varepsilon + u_{0,\varepsilon}) \to P_0(\bar{v} + u_{0,0}, b + e_3)$ as $\varepsilon \to 0$, with $u_\varepsilon = v_\varepsilon + u_{0,\varepsilon}$ and $\bar{u} = \bar{v} + u_{0,0}$, where $v_\varepsilon \in V$. By the same argument analog to the one used in A. Braides, I. Fonseca and G. Francfort [6, Step 2, Theorem 2.5], in order to show that $\bar{J}^*_\varepsilon$ Γ-converges to $\bar{J}^*_0$ it is enough to prove that the Γ-lower limit E_0 of any subsequence of E_ε coincides with J_0. Therefore the assertions of Theorem 3 follows from Lemmas 13–14 applied to the sequence $u_\varepsilon = v_\varepsilon + u_{0,\varepsilon}$. ∎

We omit the proof of Corollary 4 since it is analogous to the proof of Corollary 4.2 in our previous paper [24].

References

[1] E. Acerbi and N. Fusco, *Semicontinuity problems in the calculus of variations*, Arch. Ration. Mech. Anal. **86** (1984) 125–145. doi:10.1007/BF00275731

[2] R.A. Adams and J.J.F. Fournier, Sobolev Spaces, 2 ed. (Academic Press, 2003).

[3] A. Alberico and A. Cianchi, *Differentiability properties of Orlicz-Sobolev functions*, Ark. Mat. **43** (2005) 1–28. doi:10.1007/BF02383608

[4] G. Bouchitté, I. Fonseca and M.L. Mascarenhas, *Bending moment in membrane theory*, J. Elasticity **73** (2004) 75–99. doi:10.1023/B:ELAS.0000029996.20973.92

[5] G. Bouchitté, I. Fonseca and M.L. Mascarenhas, *The Cosserat Vector In Membrane Theory: A Variational Approach*, J. Convex Anal. **16** (2009) 351–365.

[6] A. Braides, I. Fonseca and G. Francfort, *3D-2D-asymptotic analysis for inhomogeneous thin films*, Indiana Univ. Math. J. **49** (2000) 1367–1404. doi:10.1512/iumj.2000.49.1822

[7] D. Breit, B. Stroffolini and A. Verde, *A general regularity theorem for functionals with φ-growth*, J. Math. Anal. Appl. **383** (2011) 226–233. doi:10.1016/j.jmaa.2011.05.012

[8] B. Dacorogna, Direct Methods in the Calculus of Variations, 2nd revised edition (Springer, Berlin, 2008).

[9] G. Dal Maso, An Introduction to Γ-Convergence (Birkhäuser, Boston, 1993). doi:10.1007/978-1-4612-0327-8

[10] T.K. Donaldson and N.S. Trudinger, *Orlicz-Sobolev spaces and imbedding theorems*, J. Funct. Anal. **8** (1971) 52–75. doi:10.1016/0022-1236(71)90018-8

[11] N. Dunford and J.T. Schwartz, Linear Operators, Part I: General Theory (Interscience, New York, 1957).

[12] A. Fiorenza and M. Krbec, *Indices of Orlicz spaces and some applications*, Comment. Math. Univ. Carolinae **38** (1997) 433–451.

[13] M. Focardi, *Semicontinuity of vectorial functionals in Orlicz-Sobolev spaces*, Rend. Istit. Mat. Univ. Trieste **29** (1997) 141–161.

[14] I. Fonseca, S. Müller and P.Pedregal, *Analysis of concentration and oscillation effects generated by gradients*, SIAM J. Math. Anal. **29** (1998) 736–756. doi:10.1137/S0036141096306534

[15] A. Fougères, *Théoremès de trace et de prolongement dans les espaces de Sobolev et Sobolev-Orlicz*, C.R. Acad. Sci. Paris Sér. A–B **274** (1972) A181–A184.

[16] G. Friesecke, R.D. James and S. Müller, *A hierarchy of plate models derived from nonlinear elasticity by Gamma-convergence*, Arch. Ration. Mech. Anal. **180** (2006) 183–236. doi:10.1007/s00205-005-0400-7

[17] M. García-Huidobro, V.K. Le, R. Manásevich and K. Schmitt, *On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz-Sobolev space setting*, NoDEA Nonlinear Differential Equations Appl. **6** (1999) 207–225. doi:10.1007/s000300050073

[18] J.-P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coeffcients*, Trans. Am. Math. Soc. **190** (1974) 163–205. doi:10.1090/S0002-9947-1974-0342854-2

[19] H. Hudzik, *The problems of separability, duality, reflexivity and of comparison for generalized Orlicz-Sobolev spaces* $W^k_M(\Omega)$, Comment. Math. Prace Mat. **21** (1980) 315–324.

[20] A. Kamińska and B. Turett, *Type and cotype in Musielak-Orlicz spaces*, in: Geometry of Banach Spaces, London Mathematical Society Lecture Note Series **158** (1991) 165–180, Cambridge University Press. doi:10.1017/CBO9780511662317.015

[21] A. Kałamajska and M. Krbec, *Traces of Orlicz-Sobolev functions under general growth restrictions*, Mathematische Nachrichten **286** (2013) 730–742. doi:10.1002/mana.201100185

[22] V.S. Klimov, *On imbedding theorems for anisotropic classes of functions*, Mathematics of the USSR-Sbornik **55** (1986) 195–205. doi:10.1070/SM1986v055n01ABEH002999

[23] M.A. Krasnosel'skii and Ya.B. Rutickii, Convex Functions and Orlicz Spaces (P. Noordhoof LTD., Groningen, 1961).

[24] W. Laskowski and H.T. Nguyen, *Effective energy integral functionals for thin films in the Orlicz-Sobolev space setting*, Demonstratio Math. **46** (2013) 589–608.

[25] W. Laskowski and H.T. Nguyen, *Effective energy integral functionals for thin films with bending moment in the Orlicz-Sobolev space setting*, Banach Center Publ., Polish Acad. Sci., Warsaw **102** (2014) 143–167.

[26] H. Le Dret and A. Raoult, *Le modèle de membrane non linéaire comme limite variationnelle de l'élasticité non linéaire tridimensionnelle*, C.R. Acad. Sci. Paris Sér. I Math. **317** (1993) 221–226.

[27] H. Le Dret and A. Raoult, *The nonlinear membrane model as variational limit of nonlinear threedimensional elasticity*, J. Math. Pures Appl. **74** (1995) 549–578.

[28] L. Maligranda, *Indices and interpolation*, Dissertationes Math. (Rozprawy Mat.) **234** (1985) 1–49.

[29] L. Maligranda, Orlicz Spaces and Interpolation, Seminars in Mathematics **5**, Campinas SP, Univ. of Campinas (Brazil, 1989).

[30] V. Mazja, Sobolev Spaces (Springer, New York, 1985). doi:10.1007/978-3-662-09922-3

[31] M.G. Mora and L. Scardia, *Convergence of equilibria of thin elastic plates under physical growth conditions for the energy density*, J. Diff. Equ. **252** (2012) 35–55. doi:10.1016/j.jde.2011.09.009

[32] C.B.Jr. Morrey, Multiple Integrals in the Calculus of Variations (Classics in Mathematics) (Springer, New York, 1966, Reprint in 2008).

[33] J. Musielak, Orlicz Spaces and Modular Spaces, Lecture Notes in Math. **1034** (Springer, Berlin, 1983).

[34] P. Pedregal, Parametrized Measures and Variational Principles (Birkhäuser, Basel, 1997). doi:10.1007/978-3-0348-8886-8

[35] R. Płuciennik, S. Tian and Y. Wang, *Non-convex integral functionals on Musielak-Orlicz spaces*, Comment. Math. Prace Mat. **30** (1990) 113–123.

Received 16 September 2015
Revised 26 January 2016