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POINTWISE STRONG APPROXIMATION OF ALMOST PERIODIC FUNCTIONS

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Abstract

We consider the class $GM\,({}_2\beta)$ in pointwise estimate of the deviations in strong mean of almost periodic functions from matrix means of partial sums of their Fourier series.

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1. Introduction

Let S^p $(1 < p < \infty)$ $[p = \infty]$ be the class of all almost periodic functions in the sense of Stepanov [the class of all almost periodic functions in the sense of Bohr] with the norm

$$\|f\|_{S^p} := \begin{cases} \sup\limits_u \left\{ \frac{1}{\pi} \int_u^{u+\pi} \mid f(t) \mid^p dt \right\}^{1/p} & \text{when} \quad 1 < p < \infty, \\ \sup\limits_u \mid f(u) \mid & \text{when} \quad p = \infty. \end{cases}$$

Suppose that the Fourier series of $f \in S^p$ has the form

$$Sf\,(x) = \sum_{\nu=-\infty}^{\infty} A_\nu\,(f)\,e^{i\lambda_\nu x}, \quad \text{where} \quad A_\nu\,(f) = \lim_{L\to\infty} \frac{1}{L} \int_0^L f(t) e^{-i\lambda_\nu t} dt,$$

with the partial sums

$$S_{\gamma_k} f(x) = \sum_{|\lambda_\nu| \le \gamma_k} A_\nu(f) e^{i\lambda_\nu x}$$

and that $0 = \lambda_0 < \lambda_\nu < \lambda_{\nu+1}$ if $\nu \in \mathbb{N} = \{1, 2, 3 \dots\}$, $\lim_{v \to \infty} \lambda_\nu = \infty$, $\lambda_{-\nu} = -\lambda_\nu$, $|A_\nu| + |A_{-\nu}| > 0$. Let $\Omega_{\alpha,p}$, with some fixed positive α , be the set of functions of class S^p bounded on $U = (-\infty, \infty)$ whose Fourier exponents satisfy the condition

$$\lambda_{\nu+1} - \lambda_\nu \ge \alpha \quad (\nu \in \mathbb{N}).$$

In the case $f \in \Omega_{\alpha,p}$

$$S_{\lambda_k} f(x) = \int_0^\infty \{f(x+t) + f(x-t)\} \Psi_{\lambda_k, \lambda_k+\alpha}(t)\, dt,$$

where

$$\Psi_{\lambda,\eta}(t) = \frac{2 \sin \frac{(\eta-\lambda)t}{2} \sin \frac{(\eta+\lambda)t}{2}}{\pi(\eta-\lambda) t^2} \quad (0 < \lambda < \eta, \ |t| > 0).$$

Let $A := (a_{n,k})$ be an infinite matrix of real nonnegative numbers such that

$$\sum_{k=0}^{\infty} a_{n,k} = 1, \text{ where } n = 0, 1, 2, \dots \,. \tag{1}$$

Let us consider the strong mean

$$H^q_{n,A,\gamma} f(x) = \left\{ \sum_{k=0}^{\infty} a_{n,k} |S_{\gamma_k} f(x) - f(x)|^q \right\}^{1/q} \quad (q > 0). \tag{2}$$

As measures of approximation by the quantity (2), we use the best approximation of f by entire functions g_σ of exponential type σ bounded on the real axis, shortly $g_\sigma \in B_\sigma$ and the moduli of continuity of f defined by the formulas

$$E_\sigma(f)_{S^p} = \inf_{g_\sigma} \|f - g_\sigma\|_{S^p},$$

$$\omega f(\delta)_{S^p} = \sup_{|t| \le \delta} \|f(\cdot + t) - f(\cdot)\|_{S^p},$$

and

$$w_x f(\delta)_{S^p} := \left\{ \frac{1}{\delta} \int_0^\delta |\varphi_x(t)|^p \, dt \right\}^{1/p},$$

where $\varphi_x(t) := f(x+t) + f(x-t) - 2f(x)$, respectively.

Recently, Leindler [4] defined the new class of sequences named as sequences of rest bounded variation, briefly denoted by $RBVS$, i.e.,

$$RBVS = \left\{ a := (a_n) \in \mathbb{C} : \sum_{k=m}^{\infty} |a_k - a_{k+1}| \leq K(a) |a_m| \text{ for all } m \in \mathbb{N} \right\}, \tag{3}$$

where here and throughout the paper $K(a)$ always indicates a constant depending only on a.

Denote by MS the class of nonnegative and nonincreasing sequences and by GM the class of general monotone coefficients defined as follows (see [9]):

$$GM = \left\{ a := (a_n) \in \mathbb{C} : \sum_{k=m}^{2m-1} |a_k - a_{k+1}| \leq K(a) |a_m| \text{ for all } m \in \mathbb{N} \right\}. \tag{4}$$

Then it is obvious that

$$MS \subset RBVS \subset GM.$$

In [5, 9, 10, 11] it was defined the class of β-general monotone sequences as follows:

Definition 1. Let $\beta := (\beta_n)$ be a nonnegative sequence. The sequence of complex numbers $a := (a_n)$ is said to be β-general monotone, or $a \in GM(\beta)$, if the relation

$$\sum_{k=m}^{2m-1} |a_k - a_{k+1}| \leq K(a) \beta_m \tag{5}$$

holds for all m.

In the paper [11] Tikhonov considered, among others, the following examples of the sequences β_n :

(1) ${}_1\beta_n = |a_n|$,

(2) ${}_2\beta_n = \sum_{k=[n/c]}^{[cn]} \frac{|a_k|}{k}$ for some $c > 1$.

It is clear that $GM({}_1\beta) = GM$ and (see [11, Remark 2.1])

$$GM({}_1\beta +_2 \beta) \equiv GM({}_2\beta).$$

Moreover, we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, that there exists a constant K such that

$$0 \leq K(\alpha_n) \leq K$$

holds for all n, where $K(\alpha_n)$ denote the sequence of constants appearing in the inequalities (3)–(5) for the sequences $\alpha_n := (a_{n,k})_{k=0}^{\infty}$.

Now we give the conditions to be used later on. We assume that for all n

$$\sum_{k=m}^{2m-1}|a_{n,k}-a_{n,k+1}|\leq K\sum_{k=[m/c]}^{[cm]}\frac{a_{n,k}}{k} \tag{6}$$

holds if $\alpha_n=(a_{n,k})_{k=0}^{\infty}$ belongs to $GM\left({}_2\beta\right)$, for $n=1,2,\ldots$

In this paper we consider the class $GM\left({}_2\beta\right)$ in pointwise estimate of the quantity $H^q_{n,A,\gamma}f$. Thus we present some analog of the following result of Pych-Taberska (see [8, Theorem 5]):

Theorem 2. *If $f\in\Omega_{\alpha,\infty}$, $\alpha>0$ and $q\geq 2$, then*

$$\left\|H^q_{n,A,\gamma}f\right\|_{S^\infty}\ll\left\{\frac{1}{n+1}\sum_{k=0}^{n}\left[\omega f\left(\frac{\pi}{k+1}\right)_{S^\infty}\right]^q\right\}^{1/q}+\frac{\|f\|_{S^\infty}}{(n+1)^{1/q}},$$

for $n=0,1,2,\ldots$, where $\gamma=(\gamma_k)$ is a sequence with $\gamma_k=\frac{\alpha k}{2}$, $a_{n,k}=\frac{1}{n+1}$ when $k\leq n$ and $a_{n,k}=0$ otherwise.

We shall write $I_1\ll I_2$ if there exists a positive constant K, sometimes depended on some parameters, such that $I_1\leq KI_2$.

2. Statement of the results

Let us consider a function w_x of modulus of continuity type on the interval $[0,+\infty)$, i.e. a nondecreasing continuous function having the following properties: $w_x(0)=0$, $w_x(\delta_1+\delta_2)\leq w_x(\delta_1)+w_x(\delta_2)$ for any $\delta_1,\delta_2\geq 0$ with x such that the set

$$\Omega_{\alpha,p}(w_x)=\left\{f\in\Omega_{\alpha,p}:\left[\frac{1}{\delta}\int_0^{\delta}|\varphi_x(t)-\varphi_x(t\pm\gamma)|^p\,dt\right]^{1/p}\ll w_x(\gamma)\right.$$
$$\left.\text{and}\quad w_xf(\delta)_{S^p}\ll w_x(\delta)\,,\ \text{ where }\ \gamma,\delta>0\right\}$$

is nonempty. It is clear that $\Omega_{\alpha,p}(w_x)\subseteq\Omega_{\alpha,p'}(w_x)$, for $p'\leq p$.

We start with proposition

Proposition 3. *If $f\in\Omega_{\alpha,p}(w_x)$, $\alpha>0$ and $q>0$, then*

$$\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|S_{\frac{\alpha k}{2}}f(x)-f(x)\right|^q\right\}^{1/q}\ll w_x\left(\frac{\pi}{n+1}\right)+E_{\alpha n/2}(f)_{S^p}\,,$$

for $n=0,1,2,\ldots$

Our main results are following

Theorem 4. *If* $f \in \Omega_{\alpha,p}(w_x)$, $\alpha > 0$, $q > 0$, $(a_{n,k})_{k=0}^{\infty} \in GM({}_2\beta)$ *for all* n, (1) *and* $\lim_{n\to\infty} a_{n,0} = 0$ *hold, then*

$$H_{n,A,\gamma}^{q} f(x) \ll \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[w_x\left(\frac{\pi}{k+1}\right) + E_{\frac{\alpha k}{2^{1+[c]}}}(f)_{S^p} \right]^q \right\}^{1/q}$$

for some $c > 1$ *and* $n = 0, 1, 2, \ldots$, *where* $\gamma = (\gamma_k)$ *is a sequence with* $\gamma_k = \frac{\alpha k}{2}$.

Theorem 5. *If* $f \in \Omega_{\alpha,p}(w_x)$, $\alpha > 0$, $q > 0$, $(a_{n,k})_{k=0}^{\infty} \in MS$ *for all* n, (1) *and* $\lim_{n\to\infty} a_{n,0} = 0$ *hold, then*

$$H_{n,A,\gamma}^{q} f(x) \ll \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[w_x\left(\frac{\pi}{k+1}\right) + E_{\frac{\alpha k}{2}}(f)_{S^p} \right]^q \right\}^{1/q}$$

for $n = 0, 1, 2, \ldots$, *where* $\gamma = (\gamma_k)$ *is a sequence with* $\gamma_k = \frac{\alpha k}{2}$.

Remark 1. Since

$$\left\| \left[\frac{1}{\delta} \int_0^{\delta} |\varphi_{\cdot}(t) - \varphi_{\cdot}(t \pm \gamma)|^p \, dt \right]^{1/p} \right\|_{S^p} \leq \omega f(\gamma)_{S^p}$$

and

$$\| w_{\cdot} f(\delta)_{S^p} \|_{S^p} \leq \omega f(\delta)_{S^p},$$

the analysis of the proof of Proposition 3 shows that, the estimate from Theorem 5 implies the estimate from Theorem 2 with $p \geq q$ (without change q instead of q' in the estimate of the quantity $\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| I_3(k) \right|^q \right\}^{1/q}$). Thus, taking $a_{n,k} = \frac{1}{n+1}$ when $k \leq n$ and $a_{n,k} = 0$ otherwise, in the case $p = \infty$ we obtain the better estimate than this one from Theorem 2 [8, Theorem 5].

3. Proofs of the results

3.1. Proof of Proposition 3

In the proof we will use the following function $\Phi_x f(\delta, \nu) = \frac{1}{\delta} \int_{\nu}^{\nu+\delta} \varphi_x(u) \, du$, with $\delta = \delta_n = \frac{\pi}{n+1}$ and its estimate from [6, Lemma 1, p. 218]

$$|\Phi_x f(\delta_1, \delta_2)| \leq w_x(\delta_1) + w_x(\delta_2) \tag{7}$$

for $f \in \Omega_{\alpha,p}(w_x)$ and any $\delta_1, \delta_2 > 0$.

Since, for $n = 0$ our estimate is evident we consider $n > 0$, only.
Denote by $S_k^* f$ the sums of the form

$$S_{\frac{\alpha k}{2}} f(x) = \sum_{|\lambda_\nu| \leq \frac{\alpha k}{2}} A_\nu(f) e^{i\lambda_\nu x}$$

such that the interval $\left(\frac{\alpha k}{2}, \frac{\alpha(k+1)}{2}\right)$ does not contain any λ_ν. Applying Lemma 1.10.2 of [7] we easily verify that

$$S_k^* f(x) - f(x) = \int_0^\infty \varphi_x(t) \Psi_k(t)\, dt,$$

where $\varphi_x(t) := f(x+t) + f(x-t) - 2f(x)$ and $\Psi_k(t) = \Psi_{\frac{\alpha k}{2}, \frac{\alpha(k+1)}{2}}(t)$, i.e.,

$$\Psi_k(t) = \frac{4 \sin \frac{\alpha t}{4} \sin \frac{\alpha(2k+1)t}{4}}{\alpha \pi t^2}$$

(see also [3], p. 41). Evidently, if the interval $\left(\frac{\alpha k}{2}, \frac{\alpha(k+1)}{2}\right)$ contains a Fourier exponent λ_ν, then

$$S_{\frac{\alpha k}{2}} f(x) = S_{k+1}^* f(x) - \left(A_\nu(f) e^{i\lambda_\nu x} + A_{-\nu}(f) e^{-i\lambda_\nu x}\right).$$

We can also note that if $f \in \Omega_{\alpha,p}(w_x)$, with $p > 1$ and $q > 0$, then there exists $q' \geq q$ such that $q' \geq 2$ and $p' = \frac{q'}{q'-1} \leq p$. Thus,

$$\left\{\frac{1}{n+1} \sum_{k=n}^{2n} \left|S_{\frac{\alpha k}{2}} f(x) - f(x)\right|^q\right\}^{1/q} \leq \left\{\frac{1}{n+1} \sum_{k=n}^{2n} \left|S_{\frac{\alpha k}{2}} f(x) - f(x)\right|^{q'}\right\}^{1/q'}.$$

Since (see [1, p. 78] and [2, p. 7])

$$\left\{\sum_{\nu=-\infty}^{\infty} |A_\nu(f)|^{q'}\right\}^{1/q'} \leq \|f\|_{B^{p'}} \quad \text{and} \quad \|f\|_{B^{p'}} \leq \|f\|_{S^{p'}},$$

where $\|\cdot\|_{B^{p'}}$, with $p' \geq 1$, is the Besicovitch norm, so we have

$$|A_{\pm\nu}(f)| = \left|A_{\pm\nu}\left(f - g_{\alpha\mu/2}\right)\right| \leq \left\|f - g_{\alpha\mu/2}\right\|_{S^{p'}} \leq \left\|f - g_{\alpha\mu/2}\right\|_{S^p} = E_{\alpha\mu/2}(f)_{S^p},$$

for some $g_{\alpha\mu/2} \in B_{\alpha\mu/2}$, with $\alpha k/2 < \alpha\mu/2 < \lambda_\nu$. Therefore, the deviation

$$\left\{\frac{1}{n+1} \sum_{k=n}^{2n} \left|S_{\frac{\alpha k}{2}} f(x) - f(x)\right|^{q'}\right\}^{\frac{1}{q'}}$$

can be estimated from above by

$$\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|\int_0^{\infty}\varphi_x(t)\,\Psi_{k+\kappa}(t)\,dt\right|^{q'}\right\}^{1/q'}+\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left(E_{\alpha k/2}(f)_{S^p}\right)^{q'}\right\}^{1/q'}$$
$$\leq\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|\int_0^{\infty}\varphi_x(t)\,\Psi_{k+\kappa}(t)\,dt\right|^{q'}\right\}^{1/q'}+E_{\alpha n/2}(f)_{S^p},$$

where κ equals 0 or 1. Putting $h_n=2\pi/\alpha(n+1)$ we obtain

$$\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|\int_0^{\infty}\varphi_x(t)\,\Psi_{k+\kappa}(t)\,dt\right|^{q'}\right\}^{1/q'}$$
$$=\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|\left(\int_0^{h_n}+\int_{h_n}^{(n+1)h_n}+\int_{(n+1)h_n}^{\infty}\right)\varphi_x(t)\,\Psi_{k+\kappa}(t)\,dt\right|^{q'}\right\}^{1/q'}$$
$$\leq\left\{\frac{1}{n+1}\sum_{k=n}^{2n}|I_1(k)|^{q'}\right\}^{1/q'}+\left\{\frac{1}{n+1}\sum_{k=n}^{2n}|I_2(k)|^{q'}\right\}^{1/q'}+\left\{\frac{1}{n+1}\sum_{k=n}^{2n}|I_3(k)|^{q'}\right\}^{1/q'}.$$

So, for the first term we have

$$\left\{\frac{1}{n+1}\sum_{k=n}^{2n}|I_1(k)|^{q'}\right\}^{1/q'}$$
$$\leq\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|\frac{4}{\alpha\pi}\int_0^{h_n}\varphi_x(t)\,\frac{\sin\frac{\alpha t}{4}}{t^2}\sin\frac{\alpha t}{4}(2k+2\kappa+1)dt\right|^{q'}\right\}^{1/q'}$$
$$\leq\frac{\alpha\,(4n+2\kappa+1)}{4\pi}\int_0^{h_n}|\varphi_x(t)|\,dt\ll\left(2+\frac{\kappa}{n+1}\right)w_xf\left(\frac{2\pi}{\alpha(n+1)}\right)_{S^p}$$
$$\ll w_x\left(\frac{\pi}{n+1}\right).$$

Next, we estimate the second term. We have

$$\left\{\frac{1}{n+1}\sum_{k=n}^{2n}|I_2(k)|^{q'}\right\}^{1/q'}$$
$$\leq\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|\frac{4}{\alpha\pi}\int_{h_n}^{(n+1)h_n}\frac{\left(\Phi_xf\,(\delta_n,t)-\varphi_x(t)\right)\sin\frac{\alpha t}{4}}{t^2}\sin\frac{\alpha t}{4}(2k+2\kappa+1)dt\right|^{q'}\right\}^{1/q'}$$

$$+ \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| \frac{4}{\alpha\pi} \int_{h_n}^{(n+1)h_n} \frac{\Phi_x f(\delta_n, t) \sin\frac{\alpha t}{4}}{t^2} \sin\frac{\alpha t}{4} (2k+2\kappa+1) dt \right|^{q'} \right\}^{1/q'}$$

$$= \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{21}(k)|^{q'} \right\}^{1/q'} + \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{22}(k)|^{q'} \right\}^{1/q'}.$$

From the Hausdorff-Young inequality [12, Chap. XII, Theorem 3.3 II] we obtain

$$\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{21}(k)|^{q'} \right\}^{1/q'}$$

$$\leq \frac{8}{\alpha^2} \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| \frac{\alpha}{2\pi} \int_{h_n}^{(n+1)h_n} \frac{(\Phi_x f(\delta_n, t) - \varphi_x(t)) \sin\frac{\alpha t}{4}}{t^2} \sin\frac{\alpha t}{4} (2\kappa+1) \cos\frac{\alpha k t}{2} dt \right|^{q'} \right\}^{1/q'}$$

$$+ \frac{8}{\alpha^2} \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| \frac{\alpha}{2\pi} \int_{h_n}^{(n+1)h_n} \frac{(\Phi_x f(\delta_n, t) - \varphi_x(t)) \sin\frac{\alpha t}{4}}{t^2} \cos\frac{\alpha t}{4} (2\kappa+1) \sin\frac{\alpha k t}{2} dt \right|^{q'} \right\}^{1/q'}$$

$$\ll \frac{1}{(n+1)^{1/q'}} \left(\left\{ \int_{h_n}^{(n+1)h_n} \left| \frac{(\Phi_x f(\delta_n, t) - \varphi_x(t)) \sin\frac{\alpha t}{4}}{t^2} \sin\frac{\alpha t}{4} (2\kappa+1) \right|^{p'} dt \right\}^{1/p'} \right.$$

$$\left. + \left\{ \int_{h_n}^{(n+1)h_n} \left| \frac{(\Phi_x f(\delta_n, t) - \varphi_x(t)) \sin\frac{\alpha t}{4}}{t^2} \cos\frac{\alpha t}{4} (2\kappa+1) \right|^{p'} dt \right\}^{1/p'} \right)$$

$$\ll \frac{1}{(n+1)^{1/q'}} \left\{ \int_{h_n}^{(n+1)h_n} \left| \frac{\Phi_x f(\delta_n, t) - \varphi_x(t)}{t} \right|^{p'} dt \right\}^{1/p'}$$

$$= \frac{1}{(n+1)^{1/q'}} \left\{ \int_{h_n}^{(n+1)h_n} \left| \frac{1}{t\delta_n} \int_t^{t+\delta_n} (\varphi_x(u) - \varphi_x(t)) du \right|^{p'} dt \right\}^{1/p'}$$

$$\leq \frac{1}{(n+1)^{1/q'}} \left\{ \int_{h_n}^{(n+1)h_n} \left[\frac{1}{t\delta_n} \int_0^{\delta_n} |\varphi_x(u+t) - \varphi_x(t)| du \right]^{p'} dt \right\}^{1/p'}$$

and by the Minkowski inequality, for $p' > 1$, we have

$$\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{21}(k)|^{q'} \right\}^{1/q'}$$

$$\leq \frac{1}{(n+1)^{1/q'}} \frac{1}{\delta_n} \int_0^{\delta_n} \left[\int_{h_n}^{(n+1)h_n} \frac{|\varphi_x(u+t) - \varphi_x(t)|^{p'}}{t^{p'}} dt \right]^{1/p'} du$$

$$= \frac{1}{(n+1)^{1/q'}} \frac{1}{\delta_n} \int_0^{\delta_n} \left[\int_{h_n}^{(n+1)h_n} \frac{1}{t^{p'}} \frac{d}{dt} \left(\int_0^t |\varphi_x(u+s) - \varphi_x(s)|^{p'} ds \right) dt \right]^{1/p'} du$$

$$= \frac{1}{(n+1)^{1/q'}} \frac{1}{\delta_n} \int_0^{\delta_n} \left\{ \left[\frac{1}{t^{p'}} \int_0^t |\varphi_x(u+s) - \varphi_x(s)|^{p'} ds \right]_{h_n}^{(n+1)h_n} \right.$$

$$\left. + \ p' \int_{h_n}^{(n+1)h_n} \frac{1}{t^{p'}} \left(\frac{1}{t} \int_0^t |\varphi_x(u+s) - \varphi_x(s)|^{p'} ds \right) dt \right\}^{1/p'} du$$

$$\ll \frac{1}{(n+1)^{1/q'}} \frac{1}{\delta_n} \int_0^{\delta_n} \left\{ \left(\frac{\alpha}{2\pi} \right)^{p'} \int_0^{(n+1)h_n} |\varphi_x(u+s) - \varphi_x(s)|^{p'} ds \right.$$

$$\left. + \left(\frac{\alpha(n+1)}{2\pi} \right)^{p'} \int_0^{h_n} |\varphi_x(u+s) - \varphi_x(s)|^{p'} ds + \ p' \int_{h_n}^{(n+1)h_n} \frac{1}{t^{p'}} (w_x(u))^{p'} dt \right\}^{1/p'} du$$

$$\ll \frac{1}{(n+1)^{1/q'}} \frac{1}{\delta_n} \int_0^{\delta_n} \left\{ w_x^{p'}(u) \left(\left(\frac{\alpha}{2\pi} \right)^{p'-1} + \left(\frac{\alpha(n+1)}{2\pi} \right)^{p'-1} \right. \right.$$

$$\left. \left. + \ p' \int_{h_n}^{(n+1)h_n} \frac{1}{t^{p'}} dt \right) \right\}^{1/p'} du$$

$$\ll \frac{1}{(n+1)^{1/q'}} w_x(\delta_n) \left\{ 1 + (n+1)^{p'-1} + p' \left[\frac{t^{1-p'}}{1-p'} \right]_{h_n}^{(n+1)h_n} \right\}^{1/p'}$$

$$\ll w_x(\delta_n)(n+1)^{-1/q'} (n+1)^{1-1/p'} = w_x\left(\frac{\pi}{n+1} \right).$$

Moreover,

$$\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{22}(k)|^{q'} \right\}^{1/q'}$$

$$= \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| \frac{4}{\alpha\pi} \int_{h_n}^{(n+1)h_n} \frac{\Phi_x f(\delta_n, t) \sin \frac{\alpha t}{4}}{t^2} \sin \frac{\alpha t}{4}(2k+2\kappa+1) dt \right|^{q'} \right\}^{1/q'}$$

$$= \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| \frac{4}{\alpha\pi} \int_{h_n}^{(n+1)h_n} \Phi_x f(\delta_n, t) \frac{\sin \frac{\alpha t}{4}}{t^2} \frac{d}{dt} \left(\frac{-\cos \frac{\alpha t}{4}(2k+2\kappa+1)}{\frac{\alpha}{4}(2k+2\kappa+1)} \right) dt \right|^{q'} \right\}^{1/q'}$$

$$= \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| \frac{4}{\alpha\pi} \left[\Phi_x f(\delta_n, t) \frac{\sin \frac{\alpha t}{4}}{t^2} \left(\frac{-\cos \frac{\alpha t}{4}(2k+2\kappa+1)}{\frac{\alpha}{4}(2k+2\kappa+1)} \right) \right]_{h_n}^{(n+1)h_n} \right. \right.$$

$$+\ \frac{4}{\alpha\pi}\int_{h_n}^{(n+1)h_n}\frac{d}{dt}\left(\Phi_x f\left(\delta_n,t\right)\frac{\sin\frac{\alpha t}{4}}{t^2}\right)\frac{\cos\frac{\alpha t}{4}(2k+2\kappa+1)}{\frac{\alpha}{4}(2k+2\kappa+1)}dt\Bigg|^{q'}\Bigg\}^{1/q'}$$

$$\leq\ \Bigg\{\frac{1}{n+1}\sum_{k=n}^{2n}\Bigg|\frac{-4}{\alpha\pi}\Phi_x f\left(\delta_n,\frac{2\pi}{\alpha}\right)\left(\frac{\alpha}{2\pi}\right)^2\frac{\cos\frac{\pi}{2}(2k+2\kappa+1)}{\frac{\alpha}{4}(2k+2\kappa+1)}$$

$$+\ \frac{4}{\alpha\pi}\ \Phi_x f\left(\delta_n,\frac{2\pi}{\alpha(n+1)}\right)\frac{\sin\frac{\pi}{2(n+1)}}{\left(\frac{2\pi}{\alpha(n+1)}\right)^2}\frac{\cos\frac{\pi}{2(n+1)}(2k+2\kappa+1)}{\frac{\alpha}{4}(2k+2\kappa+1)}\Bigg|^{q'}\Bigg\}^{1/q'}$$

$$+\ \Bigg\{\frac{1}{n+1}\sum_{k=n}^{2n}\Bigg|\frac{4}{\alpha\pi}\int_{h_n}^{(n+1)h_n}\frac{d}{dt}\left(\Phi_x f\left(\delta_n,t\right)\frac{\sin\frac{\alpha t}{4}}{t^2}\right)\frac{\cos\frac{\alpha t}{4}(2k+2\kappa+1)}{\frac{\alpha}{4}(2k+2\kappa+1)}dt\Bigg|^{q'}\Bigg\}^{1/q'}$$

$$=\ \left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|I_{221}(k)\right|^{q'}\right\}^{1/q'}+\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|I_{222}(k)\right|^{q'}\right\}^{1/q'}.$$

For the first term, using inequality (7), we obtain

$$\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|I_{221}(k)\right|^{q'}\right\}^{1/q'}$$

$$\leq\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left(\frac{4}{\pi^3}\left|\Phi_x f\left(\delta_n,\frac{2\pi}{\alpha}\right)\right|\frac{1}{(2k+2\kappa+1)}\right)^{q'}\right\}^{1/q'}$$

$$+\ \left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left(\frac{2(n+1)}{\pi^2}\left|\Phi_x f\left(\delta_n,\frac{2\pi}{\alpha(n+1)}\right)\right|\frac{1}{(2k+2\kappa+1)}\right)^{q'}\right\}^{1/q'}$$

$$\leq\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left(\frac{4}{(k+1)\,\pi^3}\left(w_x\left(\delta_n\right)+w_x\left(\frac{2\pi}{\alpha}\right)\right)\right)^{q'}\right\}^{1/q'}$$

$$\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left(\frac{2(n+1)}{(k+1)\,\pi^2}\left(w_x\left(\delta_n\right)+w_x\left(\frac{2\pi}{\alpha(n+1)}\right)\right)\right)^{q'}\right\}^{1/q'}$$

$$\ll\ w_x\left(\delta_n\right)+w_x\left(\frac{2\pi}{\alpha(n+1)}\right)\ \left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left(\frac{1}{k+1}w_x\left(\frac{2\pi}{\alpha}\right)\right)^{q'}\right\}^{1/q'}$$

$$\ll\ w_x\left(\frac{\pi}{n+1}\right)+\left\{\frac{1}{n+1}\sum_{k=n}^{2n}w_x^{q'}\left(\frac{\pi}{k+1}\right)\right\}^{1/q'}\leq 2w_x\left(\frac{\pi}{n+1}\right).$$

For the second term, using (7) and the Hausdorff-Young inequality [12, Chap. XII, Theorem 3.3 II] we have

$$\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|I_{222}(k)\right|^{q'}\right\}^{1/q'}$$

$$\leq \frac{1}{(n+1)^{1/q'+1}}\left\{\sum_{k=n}^{2n}\left|\frac{16}{\alpha^2\pi}\int_{h_n}^{(n+1)h_n}\frac{d}{dt}\left(\Phi_x f\left(\delta_n,t\right)\frac{\sin\frac{\alpha t}{4}}{t^2}\right)\cos\frac{\alpha t}{4}(2k+2\kappa+1)dt\right|^{q'}\right\}^{1/q'}$$

$$\ll \frac{1}{(n+1)^{1/q'+1}}\left\{\int_{h_n}^{(n+1)h_n}\left|\frac{d}{dt}\left(\Phi_x f\left(\delta_n,t\right)\frac{\sin\frac{\alpha t}{4}}{t^2}\right)\right|^{p'}dt\right\}^{1/p'}$$

$$\leq \frac{1}{(n+1)^{1/q'+1}}\left\{\int_{h_n}^{(n+1)h_n}\left(\frac{1}{\delta_n}\left|\varphi_x\left(\delta_n+t\right)-\varphi_x\left(t\right)\right|\frac{\sin\frac{\alpha t}{4}}{t^2}\right.\right.$$

$$\left.\left.+\ \left|\Phi_x f\left(\delta_n,t\right)\right|\frac{\left|\frac{\alpha}{4}t^2\cos\frac{\alpha t}{4}-2t\sin\frac{\alpha t}{4}\right|}{t^4}\right)^{p'}dt\right\}^{1/p'}$$

$$\ll \frac{1}{(n+1)^{1/q'+1}}\left[\left\{\int_{h_n}^{(n+1)h_n}\left(\frac{1}{\delta_n t}\left|\varphi_x\left(\delta_n+t\right)-\varphi_x\left(t\right)\right|\right)^{p'}dt\right\}^{1/p'}\right.$$

$$\left.+\left\{\int_{h_n}^{(n+1)h_n}\left(\frac{1}{t^2}\left(w_x\left(\delta_n\right)+w_x\left(t\right)\right)\right)^{p'}dt\right\}^{1/p'}\right].$$

Further, by partial integration the considered term does not exceed

$$\frac{1}{(n+1)^{1/q'+1}}\left[\left\{\frac{1}{\delta_n^{p'}}\left[t^{-p'}\int_0^t\left|\varphi_x\left(\delta_n+u\right)-\varphi_x\left(u\right)\right|^{p'}du\right]_{h_n}^{(n+1)h_n}\right.\right.$$

$$\left.+\ \frac{p'}{\delta_n^{p'}}\int_{h_n}^{(n+1)h_n}t^{-p'}\left(\frac{1}{t}\int_0^t\left|\varphi_x\left(\delta_n+u\right)-\varphi_x\left(u\right)\right|^{p'}du\right)dt\right\}^{1/p'}$$

$$\left.+\left\{\int_{h_n}^{(n+1)h_n}\left(\frac{w_x\left(\delta_n\right)}{t^2}\right)^{p'}dt\right\}^{1/p'}+\left\{\int_{h_n}^{(n+1)h_n}\left(\frac{w_x\left(t\right)/t}{t}\right)^{p'}dt\right\}^{1/p'}\right]$$

$$\ll \frac{1}{(n+1)^{1/q'+1}}\left[\frac{1}{\delta_n}\left\{\frac{\alpha}{2\pi}\int_0^{(n+1)h_n}\left|\varphi_x\left(\delta_n+u\right)-\varphi_x\left(u\right)\right|^{p'}du\right\}^{1/p'}\right.$$

$$+\ \frac{1}{\delta_n}(n+1)^{1-1/p'}\left\{\frac{\alpha(n+1)}{2\pi}\int_0^{h_n}\left|\varphi_x\left(\delta_n+u\right)-\varphi_x\left(u\right)\right|^{p'}du\right\}^{1/p'}$$

$$+\frac{1}{\delta_n}\left\{\int_{h_n}^{(n+1)h_n} t^{-p'}\left(w_x\left(\delta_n\right)\right)^{p'}dt\right\}^{1/p'}$$

$$+\ w_x\left(\delta_n\right)\left\{\left[\frac{t^{-2p'+1}}{-2p'+1}\right]_{h_n}^{(n+1)h_n}\right\}^{1/p'}+(n+1)w_x\left(\frac{\pi}{n+1}\right)\left\{\left[\frac{t^{-p'+1}}{-p'+1}\right]_{h_n}^{(n+1)h_n}\right\}^{1/p'}\Bigg]$$

$$\ll\frac{1}{(n+1)^{1/q'+1}}\left[\frac{1}{\delta_n}w_x\left(\delta_n\right)+\frac{1}{\delta_n}w_x\left(\delta_n\right)(n+1)^{1-1/p'}\right.$$

$$\left.+\ w_x\left(\delta_n\right)(n+1)^{2-1/p'}+(n+1)w_x\left(\frac{\pi}{n+1}\right)(n+1)^{1-1/p'}\right]$$

$$\ll w_x\left(\frac{\pi}{n+1}\right).$$

For the third term we obtain

$$\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|I_3(k)\right|^{q'}\right\}^{1/q'}$$

$$\leq\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|\sum_{\mu=1}^{\infty}\int_{(n+1)h_n\mu}^{(n+1)h_n(\mu+1)}\left[\varphi_x\left(t\right)-\Phi_x f\left(\delta_k,t\right)\right]\Psi_{k+\kappa}\left(t\right)dt\right|^{q'}\right\}^{1/q'}$$

$$+\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|\sum_{\mu=1}^{\infty}\int_{(n+1)h_n\mu}^{(n+1)h_n(\mu+1)}\Phi_x f\left(\delta_k,t\right)\Psi_{k+\kappa}\left(t\right)dt\right|^{q'}\right\}^{1/q'}$$

$$=\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|I_{31}(k)\right|^{q'}\right\}^{1/q'}+\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|I_{32}(k)\right|^{q'}\right\}^{1/q'}$$

and

$$\begin{aligned}\left|I_{31}(k)\right| &\leq\frac{4}{\alpha\pi}\sum_{\mu=1}^{\infty}\int_{(n+1)h_n\mu}^{(n+1)h_n(\mu+1)}\left|\varphi_x\left(t\right)-\Phi_x f\left(\delta_k,t\right)\right|t^{-2}dt\\ &\leq\frac{4}{\alpha\pi}\sum_{\mu=1}^{\infty}\int_{(n+1)h_n\mu}^{(n+1)h_n(\mu+1)}\left[\frac{1}{\delta_k t^2}\int_0^{\delta_k}\left|\varphi_x\left(t\right)-\varphi_x\left(t+u\right)\right|du\right]dt\\ &=\frac{4}{\alpha\pi}\frac{1}{\delta_k}\int_0^{\delta_k}\sum_{\mu=1}^{\infty}\left\{\int_{(n+1)h_n\mu}^{(n+1)h_n(\mu+1)}\frac{1}{t^2}\left|\varphi_x\left(t\right)-\varphi_x\left(t+u\right)\right|dt\right\}du\end{aligned}$$

$$= \frac{4}{\alpha\pi}\frac{1}{\delta_k}\int_0^{\delta_k}\sum_{\mu=1}^{\infty}\left\{\left[\frac{1}{t^2}\int_0^t |\varphi_x(s)-\varphi_x(s+u)|\,ds\right]_{(n+1)h_n\mu}^{(n+1)h_n(\mu+1)}\right.$$

$$\left.+\ 2\int_{(n+1)h_n\mu}^{(n+1)h_n(\mu+1)}\left[\frac{1}{t^3}\int_0^t |\varphi_x(s)-\varphi_x(s+u)|\,ds\right]dt\right\}du$$

$$\ll \left|\frac{1}{\delta_k}\int_0^{\delta_k}\sum_{\mu=1}^{\infty}\left\{\frac{1}{[(n+1)h_n(\mu+1)]^2}\int_0^{(n+1)h_n(\mu+1)} |\varphi_x(s)-\varphi_x(s+u)|\,ds\right.\right.$$

$$\left.\left.-\ \frac{1}{[(n+1)h_n\mu]^2}\int_0^{(n+1)h_n\mu} |\varphi_x(s)-\varphi_x(s+u)|\,ds\right\}du\right|$$

$$+\ \frac{1}{\delta_k}\int_0^{\delta_k}\sum_{\mu=1}^{\infty}\left\{\int_{(n+1)h_n\mu}^{(n+1)h_n(\mu+1)}\left[\frac{1}{t^3}\int_0^t |\varphi_x(s)-\varphi_x(s+u)|\,ds\right]dt\right\}du.$$

Since $f\in\Omega_{\alpha,p}(w_x)$, thus for any x

$$\begin{aligned}\lim_{\zeta\to\infty}\frac{1}{\zeta^2}\int_0^{\zeta}|\varphi_x(s)-\varphi_x(s+u)|\,ds &\le \lim_{\zeta\to\infty}\frac{1}{\zeta}w_x(u)\le \lim_{\zeta\to\infty}\frac{1}{\zeta}w_x(\delta_k)\\ &\le \lim_{\zeta\to\infty}\frac{1}{\zeta}w_x(\pi)=0,\end{aligned}$$

and therefore,

$$\begin{aligned}|I_{31}(k)| &\le \frac{1}{\delta_k}\int_0^{\delta_k}\frac{\alpha}{2\pi}\left[\frac{\alpha}{2\pi}\int_0^{2\pi/\alpha}|\varphi_x(s)-\varphi_x(s+u)|\,ds\right]du\\ &\quad+\frac{1}{\delta_k}\int_0^{\delta_k}w_x(u)\,du\sum_{\mu=1}^{\infty}\left\{\int_{(n+1)h_n\mu}^{(n+1)h_n(\mu+1)}\frac{1}{t^2}dt\right\}\\ &\ll \frac{1}{\delta_k}\int_0^{\delta_k}w_x(u)\,du+w_x(\delta_k)\sum_{\mu=1}^{\infty}\frac{1}{(n+1)h_n\mu^2}\ll w_x(\delta_k)\,.\end{aligned}$$

Next, we will estimate the term $|I_{32}(k)|$. So,

$$\begin{aligned}I_{32}(k) &= \frac{2}{\alpha\pi}\sum_{\mu=1}^{\infty}\int_{(n+1)h_n\mu}^{(n+1)h_n(\mu+1)}\frac{\Phi_x f(\delta_k,t)}{t^2}\frac{d}{dt}\left(-\frac{\cos\frac{\alpha t(k+\kappa)}{2}}{\frac{\alpha(k+\kappa)}{2}}+\frac{\cos\frac{\alpha t(k+\kappa+1)}{2}}{\frac{\alpha(k+\kappa+1)}{2}}\right)dt\\ &= \frac{2}{\alpha\pi}\sum_{\mu=1}^{\infty}\left[\frac{\Phi_x f(\delta_k,t)}{t^2}\left(-\frac{\cos\frac{\alpha t(k+\kappa)}{2}}{\frac{\alpha(k+\kappa)}{2}}+\frac{\cos\frac{\alpha t(k+\kappa+1)}{2}}{\frac{\alpha(k+\kappa+1)}{2}}\right)\right]_{(n+1)h_n\mu}^{(n+1)h_n(\mu+1)}\end{aligned}$$

$$+\frac{2}{\alpha\pi}\sum_{\mu=1}^{\infty}\int_{(n+1)h_n\mu}^{(n+1)h_n(\mu+1)}\frac{d}{dt}\left(\frac{\Phi_x f\left(\delta_k,t\right)}{t^2}\right)\left(\frac{\cos\frac{\alpha t(k+\kappa)}{2}}{\frac{\alpha(k+\kappa)}{2}}-\frac{\cos\frac{\alpha t(k+\kappa+1)}{2}}{\frac{\alpha(k+\kappa+1)}{2}}\right)dt$$
$$= I_{321}\left(k\right)+I_{322}\left(k\right).$$

Since $f\in\Omega_{\alpha,p}$, thus for x (using (7))

$$\lim_{\zeta\to\infty}\left|\frac{\Phi_x f\left(\delta_k,\frac{2\pi}{\alpha}\zeta\right)}{\left[\frac{2\pi}{\alpha}\zeta\right]^2}\left(-\frac{\cos\left[\pi\zeta(k+\kappa)\right]}{\frac{\alpha(k+\kappa)}{2}}+\frac{\cos\left[\pi\zeta\left(k+\kappa+1\right)\right]}{\frac{\alpha(k+\kappa+1)}{2}}\right)\right|$$
$$\le\lim_{\zeta\to\infty}\frac{w_x\left(\delta_k\right)+w_x\left(\frac{2\pi}{\alpha}\zeta\right)}{2\pi^2\zeta^2k}\ll\lim_{\zeta\to\infty}\frac{w_x\left(\delta_k\right)+\zeta w_x\left(\frac{2\pi}{\alpha}\right)}{\zeta^2k}\ll w_x\left(\pi\right)\lim_{\zeta\to\infty}\frac{1+\zeta}{\zeta^2}=0,$$

and therefore,

$$\begin{aligned}I_{321}\left(k\right) &= \frac{2}{\alpha\pi}\sum_{\mu=1}^{\infty}\left[\frac{\Phi_x f\left(\delta_k,\frac{2\pi}{\alpha}\left(\mu+1\right)\right)}{\left[\frac{2\pi}{\alpha}\left(\mu+1\right)\right]^2}\left(-\frac{\cos\left[\pi\left(\mu+1\right)\left(k+\kappa\right)\right]}{\frac{\alpha(k+\kappa)}{2}}\right.\right.\\ &\quad\left.+\frac{\cos\left[\pi\left(\mu+1\right)\left(k+\kappa+1\right)\right]}{\frac{\alpha(k+\kappa+1)}{2}}\right)\\ &\quad\left.-\frac{\Phi_x f\left(\delta_k,\frac{2\pi}{\alpha}\mu\right)}{\left[\frac{2\pi}{\alpha}\mu\right]^2}\left(-\frac{\cos\left[\pi\mu(k+\kappa)\right]}{\frac{\alpha(k+\kappa)}{2}}+\frac{\cos\left[\pi\mu\left(k+\kappa+1\right)\right]}{\frac{\alpha(k+\kappa+1)}{2}}\right)\right]\\ &= -\frac{2}{\alpha\pi}\frac{\Phi_x f\left(\delta_k,2\pi/\alpha\right)}{\left[2\pi/\alpha\right]^2}\left(-\frac{\left(-1\right)^{(k+\kappa)}}{\frac{\alpha(k+\kappa)}{2}}+\frac{\left(-1\right)^{(k+\kappa+1)}}{\frac{\alpha(k+\kappa+1)}{2}}\right)\\ &= -\frac{1}{\pi^3}\Phi_x f\left(\delta_k,2\pi/\alpha\right)\left(-1\right)^{(k+\kappa+1)}\left(\frac{1}{k+\kappa+1}+\frac{1}{k+\kappa}\right).\end{aligned}$$

Using (7), we get

$$\left|I_{321}\left(k\right)\right|\ll\frac{1}{\pi^3}\frac{2}{k+1}\left|\Phi_x f\left(\delta_k,2\pi/\alpha\right)\right|\le\frac{2}{\pi^3\left(k+1\right)}\left(w_x\left(\delta_k\right)+w_x\left(2\pi/\alpha\right)\right).$$

Similarly

$$\begin{aligned}I_{322}\left(k\right) &= \frac{2}{\alpha\pi}\sum_{\mu=1}^{\infty}\int_{(n+1)h_n\mu}^{(n+1)h_n(\mu+1)}\left(\frac{\frac{d}{dt}\Phi_x f\left(\delta_k,t\right)}{t^2}-\frac{2\Phi_x f\left(\delta_k,t\right)}{t^3}\right)\\ &\quad\cdot\left(\frac{\cos\frac{\alpha t(k+\kappa)}{2}}{\frac{\alpha(k+\kappa)}{2}}-\frac{\cos\frac{\alpha t(k+\kappa+1)}{2}}{\frac{\alpha(k+\kappa+1)}{2}}\right)dt\end{aligned}$$

and

$$
\begin{aligned}
\left|I_{322}(k)\right| \ll\ & \frac{8}{\alpha^{2}(k+1)\pi}\sum_{\mu=1}^{\infty}\left[\int_{(n+1)h_{n}\mu}^{(n+1)h_{n}(\mu+1)}\frac{\left|\varphi_{x}(t+\delta_{k})-\varphi_{x}(t)\right|}{\delta_{k}t^{2}}dt\right. \\
& \left. +2\int_{(n+1)h_{n}\mu}^{(n+1)h_{n}(\mu+1)}\frac{\left|\Phi_{x}f(\delta_{k},t)\right|}{t^{3}}dt\right] \\
\leq\ & \frac{8}{\alpha^{2}(k+1)\pi\delta_{k}}\sum_{\mu=1}^{\infty}\int_{(n+1)h_{n}\mu}^{(n+1)h_{n}(\mu+1)}\frac{\left|\varphi_{x}(t+\delta_{k})-\varphi_{x}(t)\right|}{t^{2}}dt \\
& +\frac{16}{\alpha^{2}(k+1)\pi}\sum_{\mu=1}^{\infty}\int_{(n+1)h_{n}\mu}^{(n+1)h_{n}(\mu+1)}\frac{w_{x}(\delta_{k})+w_{x}(t)}{t^{3}}dt \\
\ll\ & \frac{1}{(k+1)\delta_{k}}w_{x}(\delta_{k})+\frac{1}{k+1}\sum_{\mu=1}^{\infty}\left[\left(w_{x}(\delta_{k})+w_{x}\left(\frac{2\pi(\mu+1)}{\alpha}\right)\right)\frac{\alpha^{2}}{4\pi^{2}\mu^{3}}\right] \\
\ll\ & w_{x}(\delta_{k})+\frac{1}{k+1}\left[w_{x}(\delta_{k})\sum_{\mu=1}^{\infty}\frac{1}{\mu^{3}}+\sum_{\mu=1}^{\infty}\frac{w_{x}\left(\frac{2\pi(\mu+1)}{\alpha}\right)}{\mu^{3}}\right] \\
\ll\ & w_{x}(\delta_{k})+\frac{1}{k+1}\left(w_{x}(\delta_{k})+w_{x}\left(\frac{4\pi}{\alpha}\right)\sum_{\mu=1}^{\infty}\frac{\mu+1}{\mu^{3}}\right) \\
\ll\ & w_{x}(\delta_{k})+\frac{1}{k+1}\left(w_{x}(\delta_{k})+w_{x}\left(\frac{4\pi}{\alpha}\right)\right).
\end{aligned}
$$

Therefore,

$$
\left|I_{3}(k)\right| \ll w_{x}(\delta_{k})+\frac{1}{k+1}\left(w_{x}(\delta_{k})+w_{x}\left(\frac{2\pi}{\alpha}\right)+w_{x}\left(\frac{4\pi}{\alpha}\right)\right)
$$

and thus

$$
\begin{aligned}
\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|I_{3}(k)\right|^{q'}\right\}^{1/q'} \ll\ & \left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left(w_{x}\left(\frac{\pi}{k+1}\right)+\frac{1}{k+1}w_{x}\left(\frac{\pi}{\alpha}\right)\right)^{q'}\right\}^{1/q'} \\
\ll\ & \left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left(w_{x}\left(\frac{\pi}{k+1}\right)\right)^{q'}\right\}^{1/q'} \leq w_{x}\left(\frac{\pi}{n+1}\right).
\end{aligned}
$$

and the desired result follows. ∎

3.2. Proof of Theorem 4

For some $c > 1$

$$H_{n,A,\gamma}^{q} f(x) = \left\{ \sum_{k=0}^{2^{[c]}-1} a_{n,k} \left| S_{\frac{\alpha k}{2}} f(x) - f(x) \right|^{q} + \sum_{k=2^{[c]}}^{\infty} a_{n,k} \left| S_{\frac{\alpha k}{2}} f(x) - f(x) \right|^{q} \right\}^{1/q}$$

$$\ll \left\{ \sum_{k=0}^{2^{[c]}-1} a_{n,k} \left| S_{\frac{\alpha k}{2}} f(x) - f(x) \right|^{q} \right\}^{1/q} + \left\{ \sum_{m=[c]}^{\infty} \sum_{k=2^{m}}^{2^{m+1}-1} a_{n,k} \left| S_{\frac{\alpha k}{2}} f(x) - f(x) \right|^{q} \right\}^{1/q}$$

$$= I_1(x) + I_2(x).$$

Using Proposition 3 and denoting the left hand side of the inequality from its by F_n, i.e., $F_n = w_x\left(\frac{\pi}{n+1}\right) + E_n(f)_{S^p}$, we get

$$\begin{aligned} I_1(x) &\le \left\{ \sum_{k=0}^{2^{[c]}-1} a_{n,k} \frac{k/2+1}{k/2+1} \sum_{l=k/2}^{k} \left| S_{\frac{\alpha l}{2}} f(x) - f(x) \right|^{q} \right\}^{1/q} \\ &\le \left\{ 2^{[c]} \sum_{k=0}^{2^{[c]}-1} a_{n,k} \frac{1}{k/2+1} \sum_{l=k/2}^{k} \left| S_{\frac{\alpha l}{2}} f(x) - f(x) \right|^{q} \right\}^{1/q} \\ &\ll \left\{ \sum_{k=0}^{2^{[c]}-1} a_{n,k} F_{\alpha k/4}^{q} \right\}^{1/q}. \end{aligned}$$

By partial summation, our Proposition 3 gives

$$\begin{aligned} I_2^{q}(x) &= \sum_{m=[c]}^{\infty} \left[\sum_{k=2^{m}}^{2^{m+1}-2} (a_{n,k} - a_{n,k+1}) \sum_{l=2^{m}}^{k} \left| S_{\frac{\alpha l}{2}} f(x) - f(x) \right|^{q} \right. \\ &\quad \left. + a_{n,2^{m+1}-1} \sum_{l=2^{m}}^{2^{m+1}-1} \left| S_{\frac{\alpha l}{2}} f(x) - f(x) \right|^{q} \right] \\ &\ll \sum_{m=[c]}^{\infty} \left[2^{m} \sum_{k=2^{m}}^{2^{m+1}-2} |a_{n,k} - a_{n,k+1}| F_{\alpha 2^{m}/2}^{q} + 2^{m} a_{n,2^{m+1}-1} F_{\alpha 2^{m}/2}^{q} \right] \\ &= \sum_{m=[c]}^{\infty} 2^{m} F_{\alpha 2^{m}/2}^{q} \left[\sum_{k=2^{m}}^{2^{m+1}-2} |a_{n,k} - a_{n,k+1}| + a_{n,2^{m+1}-1} \right]. \end{aligned}$$

Since (6) holds, we have

$$a_{n,s+1} - a_{n,r} \leq |a_{n,r} - a_{n,s+1}| \leq \sum_{k=r}^{s} |a_{n,k} - a_{n,k+1}|$$
$$\leq \sum_{k=2^m}^{2^{m+1}-2} |a_{n,k} - a_{n,k+1}| \ll \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k} \quad \left(2 \leq 2^m \leq r \leq s \leq 2^{m+1} - 2\right),$$

whence

$$a_{n,s+1} \ll a_{n,r} + \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k} \quad \left(2 \leq 2^m \leq r \leq s \leq 2^{m+1} - 2\right)$$

and

$$2^m a_{n,2^{m+1}-1} = \frac{2^m}{2^m - 1} \sum_{r=2^m}^{2^{m+1}-2} a_{n,2^{m+1}-1} \ll \sum_{r=2^m}^{2^{m+1}-2} \left(a_{n,r} + \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k} \right)$$
$$\ll \sum_{r=2^m}^{2^{m+1}-1} a_{n,r} + 2^m \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k}.$$

Thus

$$I_2^q(x) \ll \sum_{m=[c]}^{\infty} \left\{ 2^m F^q_{\alpha 2^m/2} \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k} + F^q_{\alpha 2^m/2} \sum_{k=2^m}^{2^{m+1}-1} a_{n,k} \right\}.$$

Finally, by elementary calculations we get

$$I_2^q(x) \ll \sum_{m=[c]}^{\infty} \left\{ 2^m F^q_{\alpha 2^m/2} \sum_{k=2^{m-[c]}}^{2^{m+[c]}} \frac{a_{n,k}}{k} + F^q_{\alpha 2^m/2} \sum_{k=2^m}^{2^{m+1}} a_{n,k} \right\}$$
$$\ll \sum_{m=[c]}^{\infty} F^q_{\alpha 2^m/2} \sum_{k=2^{m-[c]}}^{2^{m+[c]}} a_{n,k}$$
$$= \sum_{m=[c]}^{\infty} F^q_{\alpha 2^m/2} \sum_{k=2^{m-[c]}}^{2^m-1} a_{n,k} + \sum_{m=[c]}^{\infty} F^q_{\alpha 2^m/2} \sum_{k=2^m}^{2^{m+[c]}} a_{n,k}$$
$$\ll \sum_{m=[c]}^{\infty} \sum_{k=2^{m-[c]}}^{2^m-1} a_{n,k} F^q_{\alpha k/2} + \sum_{m=[c]}^{\infty} \sum_{k=2^m}^{2^{m+[c]}} a_{n,k} F^q_{\frac{\alpha k}{2^{1+[c]}}}$$
$$= \sum_{m=[c]}^{\infty} \sum_{k=2^{m-[c]}}^{2^m-1} a_{n,k} F^q_{\alpha k/2} + \sum_{m=[c]}^{\infty} \sum_{k=2^m}^{2^{m+[c]}-1} a_{n,k} F^q_{\frac{\alpha k}{2^{1+[c]}}} + \sum_{m=[c]}^{\infty} F^q_{\frac{\alpha 2^m}{2}} a_{n,2^{m+[c]}}$$

$$= \sum_{m=[c]}^{\infty} \sum_{r=1}^{[c]} \sum_{k=2^{m-r}}^{2^{m-r+1}-1} a_{n,k} F^q_{\alpha k/2} + \sum_{m=[c]}^{\infty} \sum_{r=0}^{[c]-1} \sum_{k=2^{m+r}}^{2^{m+r+1}-1} a_{n,k} F^q_{\frac{\alpha k}{2^{1+[c]}}}$$
$$+ \sum_{m=[c]}^{\infty} F^q_{\frac{\alpha 2^m}{2}} a_{n,2^{m+[c]}}$$
$$\leq \sum_{r=1}^{[c]} \sum_{k=2^{[c]-r}}^{\infty} a_{n,k} F^q_{\alpha k/2} + \sum_{r=0}^{[c]-1} \sum_{k=2^{[c]+r}}^{\infty} a_{n,k} F^q_{\frac{\alpha k}{2^{1+[c]}}} + \sum_{k=2^{2[c]}}^{\infty} a_{n,k} F^q_{\frac{\alpha k}{2^{1+[c]}}}$$
$$\ll \sum_{k=0}^{\infty} a_{n,k} F^q_{\frac{\alpha k}{2^{1+[c]}}}.$$

Thus we obtain the desired result. ■

3.3. Proof of Theorem 5

If $(a_{n,k})_{k=0}^{\infty} \in MS$ then $(a_{n,k})_{k=0}^{\infty} \in GM\left({}_2\beta\right)$ and using Theorem 4 we obtain

$$H^q_{n,A,\gamma} f(x)$$
$$\ll \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[w_x\left(\frac{\pi}{k+1}\right) \right]^q \right\}^{1/q} + \left\{ \sum_{k=0}^{\infty} \sum_{m=k2^{[c]}}^{(k+1)2^{[c]}-1} a_{n,m} \left[E_{\frac{\alpha m}{2^{1+[c]}}} (f)_{S^p} \right]^q \right\}^{1/q}$$
$$\leq \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[w_x\left(\frac{\pi}{k+1}\right) \right]^q \right\}^{1/q} + \left\{ \sum_{k=0}^{\infty} \sum_{m=k2^{[c]}}^{(k+1)2^{[c]}-1} a_{n,m} \left[E_{\frac{\alpha k}{2}} (f)_{S^p} \right]^q \right\}^{1/q}$$
$$\leq \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[w_x\left(\frac{\pi}{k+1}\right) \right]^q \right\}^{1/q} + \left\{ \sum_{k=0}^{\infty} 2^{[c]} a_{n,k2^{[c]}} \left[E_{\frac{\alpha k}{2}} (f)_{S^p} \right]^q \right\}^{1/q}$$
$$\ll \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[w_x\left(\frac{\pi}{k+1}\right) + E_{\frac{\alpha k}{2}} (f)_{S^p} \right]^q \right\}^{1/q}.$$

This ends our proof. ■

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