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ENTROPY SOLUTION FOR DOUBLY NONLINEAR ELLIPTIC ANISOTROPIC PROBLEMS WITH FOURIER BOUNDARY CONDITIONS

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Abstract

The goal of this paper is to study nonlinear anisotropic problems with Fourier boundary conditions. We first prove, by using the technic of monotone operators in Banach spaces, the existence of weak solutions, and by approximation methods, we prove a result of existence and uniqueness of entropy solution.

Keywords: anisotropic Sobolev spaces, variable exponent, monotone operator, Fourier boundary conditions, entropy solutions.

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1. Introduction

We consider in this paper the following nonlinear anisotropic elliptic Fourier boundary value problem:

$$\tag{1.1}\begin{cases} -\displaystyle\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)+b(u)=f & \text{in} \quad \Omega \\ \displaystyle\sum_{i=1}^{N}a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\eta_i+\lambda u=g & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary and $meas(\Omega) > 0$, $f \in L^1(\Omega)$, $g \in L^1(\partial\Omega)$, $\eta = (\eta_1, \dots, \eta_N)$ is the unit outward normal on $\partial\Omega$ and $\lambda > 0$ is a constant.

The problem (1.1) is the anisotropic case of the nonlinear isotropic problem

$$\tag{1.2}\begin{cases} b(u) - div\ a(x,\nabla u) = f & \text{in} \quad \Omega \\ a(x,\nabla u)\eta + \lambda u = g & \text{on} \quad \partial\Omega, \end{cases}$$

studied in [14] by Nyanquini and Ouaro. The authors use the minimization technic used in [12] (see also [6, 10, 15]) to prove the existence of weak solution when f and g are bounded, namely $f \in L^\infty(\Omega)$, $g \in L^\infty(\partial\Omega)$ and by approximation methods they obtain the entropy solution when $f \in L^1(\Omega)$, $g \in L^1(\partial\Omega)$.

The study of problems involving variable exponents has received considerable attention (see [10]–[15]) due to the fact that they can model various phenomena (see [1, 7, 13]).

All papers concerned by problems like (1.1) considered particular cases of function b. Indeed, in [3], Bonzi *et al.* adopted the technic used in [14] to study the following anisotropic problem:

$$\tag{1.3}\begin{cases} -\displaystyle\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)+|u|^{p_M(x)-2}u=f & \text{in} \quad \Omega \\ \displaystyle\sum_{i=1}^{N}a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\eta_i+\lambda u=g & \text{on} \quad \partial\Omega, \end{cases}$$

where $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$. In the present paper, as the function b is more general, it is not possible to use minimization technic to get the existence of entropy solution. Therefore, we used the technic of monotone operators in Banach spaces (see [16]) to obtain the existence of entropy solutions of problem (1.1). Indeed, we define an approximation problem, and we prove that this problem has a solution u_n which converges to u, an entropy solution of (1.1).

For presenting our main result, we first have to describe the data involved in our problem. Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary domain $\partial\Omega$ and $\vec{p}(\cdot) = \big(p_1(\cdot), \dots, p_N(\cdot)\big)$ such that for any $i = 1, \dots, N$, $p_i(\cdot) : \overline{\Omega} \longrightarrow [2; N)$ is a continuous function with

$$1 < p_i^- := ess \inf_{x\in\Omega} p_i(x) \le ess \sup_{x\in\Omega} p_i(x) := p_i^+ < +\infty. \tag{1.4}$$

For any $i = 1, \dots, N$, let $a_i : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function satisfying:

• there exists a positive constant C_1 such that

$$|a_i(x, \xi)| \le C_1\Big(j_i(x) + |\xi|^{p_i(x)-1}\Big), \tag{1.5}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$, where j_i is a nonnegative function lying in $L^{p_i'(\cdot)}(\Omega)$, with $\frac{1}{p_i(x)} + \frac{1}{p_i'(x)} = 1$;

• for every $\xi, \eta \in \mathbb{R}$ with $\xi \ne \eta$ and for almost every $x \in \Omega$, there exists a positive constant C_2 such that

$$\big(a_i(x, \xi) - a_i(x, \eta)\big)(\xi - \eta) \ge \begin{cases} C_2|\xi - \eta|^{p_i(x)} & \text{if } |\xi - \eta| \ge 1 \\ C_2|\xi - \eta|^{p_i^-} & \text{if } |\xi - \eta| < 1; \end{cases} \tag{1.6}$$

• there exists a positive constant C_3 such that

$$a_i(x, \xi) \cdot \xi \ge C_3|\xi|^{p_i(x)}, \quad \text{for } \xi \in \mathbb{R}, \quad \text{for almost every } x \in \Omega. \tag{1.7}$$

The function b is such that

$$b : \mathbb{R} \longrightarrow \mathbb{R} \text{ is continuous, surjective, nondecreasing with } b(0) = 0. \tag{1.8}$$

Throughout the paper, for any $i = 1, \dots, N$, we assume that

$$\frac{\bar{p}(N-1)}{N(\bar{p}-1)} < p_i^- < \frac{\bar{p}(N-1)}{N-\bar{p}}, \qquad \frac{p_i^+ - p_i^- - 1}{p_i^-} < \frac{\bar{p} - N}{\bar{p}(N-1)} \tag{1.9}$$

and

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1, \tag{1.10}$$

where $\frac{N}{\bar{p}} = \sum_{i=1}^{N} \frac{1}{p_i^-}$.

We put for all $x \in \Omega$,

$$p_M(x) := \max\big\{p_1(x), \dots, p_N(x)\big\} \quad \text{and} \quad p_m(x) := \min\big\{p_1(x), \dots, p_N(x)\big\}$$

and for all $x \in \partial\Omega$,

$$p^{\partial}(x) = \begin{cases} \dfrac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

The hypotheses on a_i are classical in the study of nonlinear problems (see [4, 6]). A prototype example that is covered by our assumptions is the following anisotropic $\vec{p}(\cdot)$-harmonic system

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \right) = f, \tag{1.11}$$

which, in the particular case when $p_i = p$ for any $i = 1, \dots, N$, is a generalization of the classical p-Laplace equation

$$-div\big(|\nabla u|^{p-2}\nabla u\big) = f.$$

The rest of the paper is organized as follows. We first present some basic preliminary results including the variable exponent in Section 2. In Section 3, we study the existence and uniqueness of entropy solution.

2. Preliminaries

We recall in this section some definitions and basic properties of anisotropic Lebesgue and Sobolev spaces with variable exponent, which will be used in the next Section. Set

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) \text{ such that } \min_{x \in \overline{\Omega}} p(x) > 1 \right\}.$$

For any $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \longrightarrow \mathbb{R}, \text{ measurable such that } \int_\Omega |u|^{p(x)} dx < \infty \right\},$$

endowed with the so-called Luxemburg norm

$$|u|_{p(\cdot)} := \inf \left\{ \beta > 0 \text{ such that } \int_\Omega \left|\frac{u}{\beta}\right|^{p(x)} dx \leq 1 \right\}.$$

The $p(\cdot)$-*modular* of the $L^{p(\cdot)}(\Omega)$ space is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)} dx.$$

For any $u \in L^{p(\cdot)}(\Omega)$, the following inequality (see [8, 9]) will be used later.

$$\min\left\{|u|_{p(\cdot)}^{p^-};\ |u|_{p(\cdot)}^{p^+}\right\} \leq \rho_{p(\cdot)}(u) \leq \max\left\{|u|_{p(\cdot)}^{p^-};\ |u|_{p(\cdot)}^{p^+}\right\}. \tag{2.1}$$

For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ in Ω, we have the Hölder type inequality:

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)|u|_{p(\cdot)}|v|_{p'(\cdot)}. \tag{2.2}$$

If Ω is bounded and $p, q \in C_+(\overline{\Omega})$ such that $p(x) \leq q(x)$ for any $x \in \Omega$, then the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous (see [11], Theorem 2.8).

Herein, we need the following anisotropic Sobolev space with variable exponent.

$$W^{1,\vec{p}(\cdot)}(\Omega) := \left\{u \in L^{p_M(\cdot)}(\Omega) \ \text{ such that } \ \frac{\partial u}{\partial x_i} \in L^{p_i(\cdot)}(\Omega), \quad i = 1, \dots, N\right\}.$$

$W^{1,\vec{p}(\cdot)}(\Omega)$ is a separable and reflexive Banach space (see [12]) under the norm

$$||u||_{\vec{p}(\cdot)} = |u|_{p_M(\cdot)} + \sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}.$$

We need the following embedding and trace results.

Theorem 2.1 ([8], Corollary 2.1). *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded open set and for all $i = 1, \dots, N$, $p_i \in L^\infty(\Omega)$, $p_i(x) \geq 1$ a.e. in Ω. Then, for any $q \in L^\infty(\Omega)$ with $q(x) \geq 1$ a.e. in Ω such that*

$$ess \inf_{x\in\Omega}\big(p_M(x) - q(x)\big) > 0,$$

we have the compact embedding

$$W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega). \tag{2.3}$$

Theorem 2.2 ([6], Theorem 6). *Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded open set with smooth boundary and let $\vec{p}(\cdot) \in \big(C_+(\overline{\Omega})\big)^N$, $r \in C(\overline{\Omega})$ satisfy the condition*

$$1 \leq r(x) < \min\left\{p_1^\partial(x), \dots, p_N^\partial(x)\right\}, \quad \forall\ x \in \partial\Omega. \tag{2.4}$$

Then, there is a compact boundary trace embedding

$$W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega).$$

In particular case

$$W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^1(\partial\Omega).$$

We introduce the numbers

$$q = \frac{N(\bar{p}-1)}{N-1} \quad \text{and} \quad q^* = \frac{N(\bar{p}-1)}{N-\bar{p}} = \frac{Nq}{N-q}.$$

The following result is due to Troisi (see [17]).

Theorem 2.3. *Let $p_1, \dots, p_N \in [1, +\infty)$; $g \in W^{1,(p_1,\dots,p_N)}(\Omega)$ and*

$$\begin{cases} q = (\bar{p})^* & if \quad (\bar{p})^* < N \\ q \in [1, +\infty) & if \quad (\bar{p})^* \geq N. \end{cases}$$

Then, there exists a constant $C_4 > 0$ depending on $N, p_1, \dots, p_N$ if $\bar{p} < N$ and also on q and $meas(\Omega)$ if $\bar{p} \geq N$ such that

$$||g||_{L^q(\Omega)} \leq C_4 \prod_{i=1}^{N} \left[||g||_{L^{p_M}(\Omega)} + \left|\left| \frac{\partial g}{\partial x_i} \right|\right|_{L^{p_i}(\Omega)} \right]^{1/N}. \tag{2.5}$$

In this paper, we will use the weak Lebesgue (Marcinkiewicz) space $\mathcal{M}^q(\Omega)$ $(1 < q < +\infty)$ as the set of measurable functions $h : \Omega \longrightarrow \mathbb{R}$ for which the distribution function

$$\lambda_h(k) = meas\big(\{x \in \Omega : \ |h(x)| > k\}\big), \quad k \geq 0 \tag{2.6}$$

satisfies an estimate of the form

$$\lambda_h(k) \leq C k^{-q}, \text{ for some finite constant } \ C > 0. \tag{2.7}$$

We will use the following pseudo norm in $\mathcal{M}^q(\Omega)$:

$$||h||_{\mathcal{M}^q(\Omega)} := \inf\{C > 0 : \lambda_h(k) \leq C k^{-q}, \ \ \forall \ k > 0\}. \tag{2.8}$$

For any $k > 0$, the truncation function T_k is defined on $\mathbb{R}$ by

$$T_k(s) = \max\{-k; \min\{k; s\}\}. \tag{2.9}$$

It is clear that $\lim_{k \to +\infty} T_k(s) = s$ and $|T_k(s)| = \min\{|s|; k\}$. In order to simplify the notation, for any $v \in W^{1,\vec{p}(\cdot)}(\Omega)$, we use v instead of $v_{|\partial\Omega}$ for the trace of v on $\partial\Omega$.

Set $\mathcal{T}^{1,\vec{p}(\cdot)}(\Omega)$ as the set of the measurable functions $u : \Omega \longrightarrow \mathbb{R}$ such that for any $k > 0$, $T_k(u) \in W^{1,\vec{p}(\cdot)}(\Omega)$. We define the space $\mathcal{T}^{1,\vec{p}(\cdot)}_{tr}(\Omega)$ as the set of functions

$u \in \mathcal{T}^{1,\vec{p}(\cdot)}(\Omega)$ such that there exists a sequence $(u_n)_n \subset W^{1,\vec{p}(\cdot)}(\Omega)$ satisfying

$$u_n \longrightarrow u \quad \text{a.e.} \quad \text{in} \quad \Omega, \tag{2.10}$$

$$\frac{\partial}{\partial x_i} T_k(u_n) \longrightarrow \frac{\partial}{\partial x_i} T_k(u) \quad \text{in} \quad L^1(\Omega), \quad \forall \ k > 0 \tag{2.11}$$

and there exists a measurable function v on $\partial\Omega$ such that

$$u_n \longrightarrow v \quad \text{a.e.} \quad \text{on} \quad \partial\Omega. \tag{2.12}$$

We need the following lemma proved in [5].

Lemma 2.1. *Let h be a nonnegative function in $W^{1,\vec{p}(\cdot)}(\Omega)$. Assume $\bar{p} < N$ and there exists a constant $C > 0$ such that*

$$\int_\Omega |T_k(h)|^{p_M^-} dx + \sum_{i=1}^N \int_{\{|h| \le k\}} \left| \frac{\partial h}{\partial x_i} \right|^{p_i^-} dx \le C(1+k), \quad \forall \ k > 0. \tag{2.13}$$

Then, there exists a constant D, depending on C, such that

$$||h||_{\mathcal{M}^{q^*}(\Omega)} \le D, \tag{2.14}$$

where $q^* = N(\bar{p}-1)/(N-\bar{p})$.

3. Entropy solution

The notion of entropy solution to problem (1.1) is the following.

Definition 3.1. A measurable function $u \in \mathcal{T}_{tr}^{1,\vec{p}(\cdot)}(\Omega)$ is an entropy solution of problem (1.1) if $b(u) \in L^1(\Omega)$, $u \in L^1(\partial\Omega)$ and

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u-\varphi) dx + \int_\Omega b(u) T_k(u-\varphi) dx \\ \displaystyle + \lambda \int_{\partial\Omega} u T_k(u-\varphi) d\sigma \le \int_\Omega f T_k(u-\varphi) dx + \int_{\partial\Omega} g T_k(u-\varphi) d\sigma, \end{cases} \tag{3.1}$$

for every $\varphi \in W^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and for every $k > 0$.

Remark that as we have in the definition $\varphi \in W^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ then $(u-\varphi) \in \mathcal{T}_{tr}^{1,\vec{p}(\cdot)}(\Omega)$, hence $T_k(u-\varphi) \in W^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$. Consequently the first, the second, the third and the fifth integrals in (3.1) are well defined.

The existence result is the following theorem.

Theorem 3.1. *Assume* (1.4)–(1.10). *There exists at least one entropy solution of the problem* (1.1).

Proof. The proof is done in three steps.

Step 1. The approximate problem.

We define the reflexive space

$$E = W^{1,\vec{p}(\cdot)}(\Omega) \times L^{p_M(\cdot)}(\partial\Omega).$$

Let X_0 be the subspace of E defined by

$$X_0 = \{(u,v) \in E : v = \tau(u)\},$$

where $\tau(u)$ is the trace of $u \in \mathcal{T}^{1,\vec{p}(\cdot)}_{tr}(\Omega)$ in the usual sense, since $u \in W^{1,\vec{p}(\cdot)}(\Omega)$. In the sequel, we will identify an element $(u,v) \in X_0$ with its representative $u \in W^{1,\vec{p}(\cdot)}(\Omega)$.

For any $n \in \mathbb{N}$ and $\varepsilon > 0$, we consider the sequence of approximate problems

$$\left\{\begin{array}{l} \displaystyle\sum_{i=1}^{N} \int_\Omega a_i\Big(x, \frac{\partial u_n}{\partial x_i}\Big) \frac{\partial v}{\partial x_i} dx + \varepsilon \int_\Omega |u_n|^{p_M(x)-2} u_n v dx \\ \displaystyle + \int_\Omega T_n\big(b(u_n)\big) v dx + \lambda \int_{\partial\Omega} T_n(u_n) v d\sigma = \int_\Omega f_n v dx + \int_{\partial\Omega} g_n v d\sigma, \end{array}\right. \tag{3.2}$$

where $f_n = T_n(f)$ and $g_n = T_n(g)$.

Remark that $(f_n)_n$ and $(g_n)_n$ are sequences of bounded functions which converge strongly to $f \in L^1(\Omega)$ and to $g \in L^1(\partial\Omega)$, respectively. Moreover,

$$||f_n||_{L^1(\Omega)} \le ||f||_{L^1(\Omega)}, \quad ||g_n||_{L^1(\partial\Omega)} \le ||g||_{L^1(\partial\Omega)} \quad \text{for all} \quad n \in \mathbb{N}$$

and

$$\|f_n\|_\infty \le \frac{\|f\|_{L^1(\Omega)}}{meas(\Omega)}, \quad \|g_n\|_\infty \le \frac{\|g\|_{L^1(\partial\Omega)}}{meas(\Omega)} \quad \text{for all} \quad n \in \mathbb{N}.$$

We define operators A_n by

$$\langle A_n(u), v\rangle = \langle A(u), v\rangle + \int_\Omega T_n\big(b(u)\big) v dx + \lambda \int_{\partial\Omega} T_n(u) v d\sigma \quad \forall\ u, v \in X_0,$$

where

$$\langle A(u), v\rangle = \sum_{i=1}^{N} \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial v}{\partial x_i} dx + \varepsilon \int_\Omega |u|^{p_M(x)-2} u v dx.$$

Assertion 1. *The operator A is of type M.*

• *The operator A is monotone.* Indeed, for $u, v \in W^{1,\vec{p}(\cdot)}(\Omega)$, we have

$$\begin{aligned}
&\langle A(u)-A(v), u-v\rangle = \langle A(u), u-v\rangle + \langle A(v), v-u\rangle \\
&= \int_\Omega \sum_{i=1}^N a_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\frac{\partial (u-v)}{\partial x_i}dx + \varepsilon\int_\Omega |u|^{p_M(x)-2}u(u-v)dx \\
&+ \int_\Omega \sum_{i=1}^N a_i\Big(x, \frac{\partial v}{\partial x_i}\Big)\frac{\partial (v-u)}{\partial x_i}dx + \varepsilon\int_\Omega |v|^{p_M(x)-2}v(v-u)dx \\
&= \int_\Omega \sum_{i=1}^N \Big[a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) - a_i\Big(x, \frac{\partial v}{\partial x_i}\Big)\Big]\Big(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\Big)dx \\
&+ \varepsilon\int_\Omega \Big(|u|^{p_M(x)-2}u - |v|^{p_M(x)-2}v\Big)(u-v)dx.
\end{aligned}$$

Therefore,

$$\langle A(u)-A(v), u-v\rangle \geq 0, \tag{3.3}$$

since for $i = 1, \dots, N$, for almost every $x \in \Omega$, $a_i(x,\cdot)$ and $t \longmapsto |t|^{p_M(x)-2}t$ are monotone.

• *The operator A is hemicontinuous.* Indeed, for every u, v in $W^{1,\vec{p}(\cdot)}(\Omega)$, let

$$\varphi : t \in \mathbb{R} \longmapsto \varphi(t) = \langle A(u+tv), v\rangle$$

and let $t,\ t_0 \in \mathbb{R}$ such that $t \longrightarrow t_0$. We have $w = u + tv \longrightarrow w_0 = u + t_0 v$ in $W^{1,\vec{p}(\cdot)}(\Omega)$. Using the Hölder type inequality, there exists $i_0 \in \{1, \dots, N\}$ such that

$$\begin{aligned}
|\varphi(t)-\varphi(t_0)| &= |\langle A(u+tv), v\rangle - \langle A(u+t_0v), v\rangle| \\
&\leq \sum_{i=1}^N \int_\Omega \Big|a_i\Big(x, \frac{\partial w}{\partial x_i}\Big) - a_i\Big(x, \frac{\partial w_0}{\partial x_i}\Big)\Big|\Big|\frac{\partial v}{\partial x_i}\Big|dx \\
&+ \varepsilon\int_\Omega \big||w|^{p_M(x)-2}w - |w_0|^{p_M(x)-2}w_0\big||v|dx \\
&\leq N\Big(\frac{1}{p_{i_0}^-} + \frac{1}{(p'_{i_0})^-}\Big)\Big|a_{i_0}\Big(x, \frac{\partial w}{\partial x_{i_0}}\Big) - a_{i_0}\Big(x, \frac{\partial w_0}{\partial x_{i_0}}\Big)\Big|_{p'_{i_0}(\cdot)}\Big|\frac{\partial v}{\partial x_{i_0}}\Big|_{p_{i_0}(\cdot)} \\
&+ \varepsilon\Big(\frac{1}{p_M^-} + \frac{1}{(p'_M)^-}\Big)\big||w|^{p_M(x)-2}w - |w_0|^{p_M(x)-2}w_0\big|_{p'_M(\cdot)}|v|_{p_M(\cdot)}.
\end{aligned}$$

Let's denote $\psi_{i_0}(x,w) = a_{i_0}\big(x, \frac{\partial w}{\partial x_{i_0}}\big)$. Using assumption (1.5) and ([11], Theorems 4.1 and 4.2) we have $\psi_{i_0}(x,w) \longrightarrow \psi_{i_0}(x,w_0)$ in $L^{p'_{i_0}(\cdot)}(\Omega)$. Then, we deduce that φ is continuous, namely the operator A is hemicontinuous.

Since the operator A is *monotone* and *hemicontinuous*, then according to the Lemma 2.1 in [16], A is of type M.

Assertion 2. *Operators A_n are of type M.* Indeed, let $(u_k)_{k\in\mathbb{N}}$ be a sequence in X_0 such that

$$\begin{cases} u_k \rightharpoonup u & \text{weakly in } X_0, \\ A_n u_k \rightharpoonup \chi & \text{weakly in } X_0', \\ \limsup\limits_{k\to+\infty} \langle A_n(u_k), u_k\rangle \le \langle \chi, u\rangle. \end{cases} \tag{3.4}$$

Since

$$T_n(b(u))u \ge 0 \qquad \text{and} \qquad T_n(u)u \ge 0,$$

by the Fatou's lemma, we obtain

$$\begin{aligned} &\liminf_{k\to+\infty}\left(\int_\Omega T_n(b(u_k))u_k dx + \int_{\partial\Omega} \lambda T_n(u_k)u_k d\sigma\right) \\ &\ge \int_\Omega T_n(b(u))u dx + \int_{\partial\Omega}\lambda T_n(u)u d\sigma \end{aligned} \tag{3.5}$$

and thanks to the Lebesgue dominated convergence theorem, we get

$$\begin{aligned} &\lim_{k\to+\infty}\left(\int_\Omega T_n(b(u_k))v dx + \int_{\partial\Omega} \lambda T_n(u_k)v d\sigma\right) \\ &= \int_\Omega T_n(b(u))v dx + \int_{\partial\Omega}\lambda T_n(u)v d\sigma, \end{aligned} \tag{3.6}$$

for all v in X_0. Consequently,

$$T_n(b(u_k)) + \lambda T_n(u_k) \rightharpoonup T_n(b(u)) + \lambda T_n(u) \ \text{ weakly in } \ X_0'.$$

Therefore, we deduce that

$$Au_k \rightharpoonup \chi - \big(T_n(b(u)) + \lambda T_n(u)\big) \ \text{ weakly in } \ X_0'.$$

As in Assertion 1 we prove that the operator A is of type M, so we have

$$Au = \chi - \big(T_n(b(u)) + \lambda T_n(u)\big).$$

Thus, it follows that

$$A_n u = \chi.$$

Hence A_n is of type M.

Assertion 3. *Operators A_n are coercive.* Indeed, since

$$T_n(b(u))u + \lambda T_n(u)u \geq 0,$$

then

$$\langle A_n(u), u\rangle \geq \langle A(u), u\rangle. \tag{3.7}$$

According to (1.7), we have

$$\langle A(u), u\rangle \geq C_3 \int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx + \varepsilon \int_\Omega |u|^{p_M(x)} dx. \tag{3.8}$$

Denote

$$\mathcal{I} = \left\{ i \in \{1,\dots,N\} : \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)} \leq 1 \right\} \text{ and } \mathcal{J} = \left\{ i \in \{1,\dots,N\} : \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)} > 1 \right\}.$$

We have

$$\begin{aligned}\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx &= \sum_{i\in\mathcal{I}} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx + \sum_{i\in\mathcal{J}} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx \geq \sum_{i\in\mathcal{I}} \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}^{p_i^+} \\ &\quad + \sum_{i\in\mathcal{J}} \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}^{p_i^-} \geq \sum_{i\in\mathcal{J}} \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}^{p_i^-} \geq \sum_{i\in\mathcal{J}} \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}^{p_m^-} \\ &\geq \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}^{p_m^-} - \sum_{i\in\mathcal{I}} \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}^{p_m^-} \geq \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}^{p_m^-} - N.\end{aligned}$$

Using the convexity of the application $t \in \mathbb{R}^+ \longmapsto t^{p_m^-}$, $p_m^- > 1$, we obtain

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx \geq \frac{1}{N^{p_m^- - 1}} \left(\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)} \right)^{p_m^-} - N.$$

Then

$$\langle A_n(u), u\rangle \geq \frac{C_3}{N^{p_m^- - 1}} \left(\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)} \right)^{p_m^-} + \varepsilon \int_\Omega |u|^{p_M(x)} dx - C_3 N. \tag{3.9}$$

• Assume $|u|_{p_M(\cdot)} > 1$. Then, (2.1) gives $\int_\Omega |u|^{p_M(x)}dx \geq |u|^{p_m^-}_{p_M(\cdot)}$.
So, combining (3.7) and (3.9) we get

$$\begin{aligned}\langle A_n(u), u\rangle &\geq C\left[\left(\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}\right)^{p_m^-} + |u|^{p_m^-}_{p_M(\cdot)}\right] - C_3N \\ &\geq \frac{C}{2^{p_m^- -1}}||u||^{p_m^-}_{\vec{p}(\cdot)} - C_3N, \text{ where } C = \min\left\{\frac{C_3}{N^{p_m^- -1}}; \varepsilon\right\}.\end{aligned}$$

• Assume $|u|_{p_M(\cdot)} \leq 1$. Then, combining (3.7) and (3.9) we get

$$\begin{aligned}\langle A_n(u), u\rangle &\geq C\left[\left(\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}\right)^{p_m^-} + |u|^{p_m^-}_{p_M(\cdot)}\right] - 1 - C_3N + \varepsilon\int_\Omega |u|^{p_M(x)}dx \\ &\geq \frac{C}{2^{p_m^- -1}}||u||^{p_m^-}_{\vec{p}(\cdot)} - 1 - C_3N, \text{ where } C = \min\left\{\frac{C_3}{N^{p_m^- -1}}; 1\right\}.\end{aligned}$$

Consequently, since $p_m^- > 1$, the operator A_n is coercive.

Besides, the operators A_n are bounded and hemicontinuous.

Then for any $F_n = \big(T_n(f), T_n(g)\big) \in E' \subset X_0'$, we can deduce the existence of functions $u_n \in X_0$ such that

$$\langle A_n(u_n), v\rangle = \langle F_n, v\rangle \text{ for all } v \in X_0.$$

Namely, every u_n is a weak solution of the approximate problem (3.2). ■

Now we are going to prove that these approximated solutions u_n tend, as n goes to infinity, to a measurable function u which is an entropy solution of the problem (1.1). To start with, we establish some a priori estimates.

Step 2. A priori estimates.

Assume (1.4)–(1.10) and let u_n be a solution of problem (3.2). We have the following results.

Lemma 3.1. *There exists a constant $C_5 > 0$ such that*

$$\int_\Omega |T_k(u_n)|^{p_M^-}dx + \sum_{i=1}^N \int_{\{|u_n|\leq k\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-} dx \leq C_5(k+1). \tag{3.10}$$

Proof. Let us take $T_k(u_n)$ as test function in (3.2). Since

$$\int_\Omega T_n(b(u_n))T_k(u_n)dx + \varepsilon\int_\Omega |u_n|^{p_M(x)-2}u_nT_k(u_n) + \int_{\partial\Omega} \lambda T_n(u_n)T_k(u_n)d\sigma \geq 0,$$

using relation (1.7), we obtain

$$C_3 \sum_{i=1}^{N} \int_{\{|u_n| \le k\}} \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i(x)} dx \le k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big). \tag{3.11}$$

Then, we have

$$\begin{aligned}
\sum_{i=1}^{N} \int_{\{|u_n| \le k\}} \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i^-} dx &= \sum_{i=1}^{N} \int_{\{|u_n| \le k;\ |\frac{\partial u_n}{\partial x_i}| > 1\}} \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i^-} dx \\
&\quad + \sum_{i=1}^{N} \int_{\{|u_n| \le k;\ |\frac{\partial u_n}{\partial x_i}| \le 1\}} \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i^-} dx \\
&\le \sum_{i=1}^{N} \int_{\{|u_n| \le k\}} \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i(x)} dx + N \cdot meas(\Omega) \\
&\le \frac{k}{C_3}\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big) + N \cdot meas(\Omega).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\int_\Omega |T_k(u_n)|^{p_M^-} dx &= \int_{\{|T_k(u_n)| \le 1\}} |T_k(u_n)|^{p_M^-} dx + \int_{\{|T_k(u_n)| > 1\}} |T_k(u_n)|^{p_M^-} dx \\
&\le meas(\Omega) + \int_{\{|T_k(u_n)| > 1\}} k^{p_M^-} dx \le meas(\Omega)(1 + k^{p_M^-}).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\int_\Omega |T_k(u_n)|^{p_M^-} dx + \sum_{i=1}^{N} \int_{\{|u_n| \le k\}} \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i^-} dx \\
&\le meas(\Omega)(1 + N + k^{p_M^-}) + k\frac{1}{C_3}\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big) \le C_5(1 + k),
\end{aligned}$$

where $C_5 = \max\big\{meas(\Omega)(1 + N + k^{p_M^-}); \frac{1}{C_3}\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big)\big\}$. ■

Lemma 3.2. *For any $k > 0$, there exists two constants $C_7 > 0$ and $C_8 > 0$ such that*

(i) $||u_n||_{\mathcal{M}^{q^*}(\Omega)} \le C_7$;

(ii) $\left|\left| \frac{\partial u_n}{\partial x_i} \right|\right|_{\mathcal{M}^{p_i^- q/\bar{p}}(\Omega)} \le C_8, \quad \forall\ i = 1, \ldots, N.$

Proof. (i) is a consequence of Lemmas 2.1 and 3.1.

(ii) • Let $\alpha \geq 1$. For any $k \geq 1$, we have

$$\begin{aligned}\lambda_{\frac{\partial u_n}{\partial x_i}}(\alpha) &= meas\Big(\Big\{x\in\Omega : \Big|\frac{\partial u_n}{\partial x_i}\Big| > \alpha\Big\}\Big) = meas\Big(\Big\{\Big|\frac{\partial u_n}{\partial x_i}\Big| > \alpha; |u_n| \leq k\Big\}\Big)\\ &\quad + meas\Big(\Big\{\Big|\frac{\partial u_n}{\partial x_i}\Big| > \alpha; |u_n| > k\Big\}\Big) \leq \int_{\{|\frac{\partial u_n}{\partial x_i}|>\alpha;\ |u_n|\leq k\}} dx + \lambda_{u_n}(k)\\ &\leq \int_{\{|u_n|\leq k\}} \Big(\frac{1}{\alpha}\Big|\frac{\partial u_n}{\partial x_i}\Big|\Big)^{p_i^-} dx + \lambda_{u_n}(k) \leq \alpha^{-p_i^-} C'k + Ck^{-q^*}.\end{aligned}$$

Then, there exists a positive constant C_6 such that

$$\lambda_{\frac{\partial u_n}{\partial x_i}}(\alpha) \leq C_6\big(k\alpha^{-p_i^-} + k^{-q^*}\big). \tag{3.12}$$

Let us consider the function

$$g : [1,+\infty[\longrightarrow \mathbb{R},\ t \longmapsto g(t) = \frac{t}{\alpha^{p_i^-}} + t^{-q^*}.$$

We have $g'(t) = 0$ for $t = \big(q^*\alpha^{p_i^-}\big)^{\frac{1}{q^*+1}}$. Thus, if we take $k = \big(q^*\alpha^{p_i^-}\big)^{\frac{1}{q^*+1}} \geq 1$ in (3.12) we get

$$\lambda_{\frac{\partial u_n}{\partial x_i}}(\alpha) \leq C_6 k\Big(\frac{q^*+1}{q^*}\frac{1}{\alpha^{p_i^-}}\Big) \leq C_6'\alpha^{-\frac{q^*}{q^*+1}p_i^-} \leq C_6'\alpha^{-p_i^- q/\bar{p}},$$

$\forall\ \alpha \geq 1$, where C_6' is a positive constant.

• If $0 \leq \alpha < 1$, we have

$$\lambda_{\frac{\partial u_n}{\partial x_i}}(\alpha) = meas\Big(\Big\{\Big|\frac{\partial u_n}{\partial x_i}\Big| > \alpha\Big\}\Big) \leq meas(\Omega) \leq meas(\Omega)\alpha^{-p_i^- q/\bar{p}}.$$

Then

$$\lambda_{\frac{\partial u_n}{\partial x_i}}(\alpha) \leq (C_6' + meas(\Omega))\alpha^{-p_i^- q/\bar{p}}, \quad \forall\ \alpha \geq 0.$$

Therefore, we deduce that there exists a positive constant C_8 such that

$$\Big\|\frac{\partial u_n}{\partial x_i}\Big\|_{\mathcal{M}^{p_i^- q/\bar{p}}(\Omega)} \leq C_8, \quad \forall\ i = 1,\dots,N.$$

∎

Step 3. Existence of entropy solution.

Using Lemma 3.2, we have the following useful lemma (see [5]).

Lemma 3.3. *For $i = 1, \dots, N$, as $n \longrightarrow +\infty$, we have*

$$a_i\Big(x, \frac{\partial u_n}{\partial x_i}\Big) \longrightarrow a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \quad in \ \ L^1(\Omega) \ \ a.e. \ \ x \in \Omega. \tag{3.13}$$

In order to pass to the limit in relation (3.2), we also need the following convergence results which can be proved as in [2] (see also [4, 5]).

Proposition 3.1. *Assume* (1.4)–(1.10). *If $u_n \in W^{1,\vec{p}(\cdot)}(\Omega)$ is a weak solution of* (3.2) *then the sequence $(u_n)_{n\in\mathbb{N}^*}$ is Cauchy in measure. In particular, there exists a measurable function u and a sub-sequence still denoted by u_n such that $u_n \longrightarrow u$ in measure.*

Proposition 3.2. *Assume* (1.4)–(1.10). *If $u_n \in W^{1,\vec{p}(\cdot)}(\Omega)$ is a weak solution of* (3.2) *then*

(i) *there exists $s > 1$ such that $u_n \to u$ a.e. in Ω and $u_n \rightharpoonup u$ in $W^{1,s}(\Omega)$,*

(ii) *for all $i = 1, \dots, N$, $\frac{\partial u_n}{\partial x_i}$ converges strongly in $L^1(\Omega)$. Moreover, $a_i\big(x, \frac{\partial u_n}{\partial x_i}\big)$ converges to $a_i\big(x, \frac{\partial u}{\partial x_i}\big)$ in $L^1(\Omega)$ strongly and in $L^{p_i'(\cdot)}(\Omega)$ weakly for all $i = 1, \dots, N$,*

(iii) *u_n converges to some measurable function v a.e. on $\partial\Omega$.*

We can now pass to the limit in relation (3.2).

Let $\varphi \in W^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and choosing $T_k(u_n - \varphi)$ as test function in (3.2), we get

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u_n}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u_n - \varphi)dx + \int_\Omega T_n(b(u_n))T_k(u_n - \varphi)dx \\ \displaystyle + \varepsilon \int_\Omega |u_n|^{p_M(x)-2} u_n T_k(u_n - \varphi)dx + \lambda \int_{\partial\Omega} T_n(u_n)T_k(u_n - \varphi)d\sigma \\ \displaystyle = \int_\Omega f_n T_k(u_n - \varphi)dx + \int_{\partial\Omega} g_n T_k(u_n - \varphi)d\sigma. \end{cases} \tag{3.14}$$

For the right-hand side of (3.14) we have

$$\begin{aligned} &\int_\Omega f_n T_k(u_n - \varphi)dx + \int_{\partial\Omega} g_n T_k(u_n - \varphi)d\sigma \\ &\longrightarrow \int_\Omega f T_k(u - \varphi)dx + \int_{\partial\Omega} g T_k(u - \varphi)d\sigma, \end{aligned} \tag{3.15}$$

because f_n and g_n converges strongly respectly to f in $L^1(\Omega)$ and g in $L^1(\partial\Omega)$ and $T_k(u_n - \varphi)$ converges weakly-* to $T_k(u - \varphi)$ in $L^\infty(\Omega)$ and a.e. in Ω.

For the first term of (3.14) we have (see [5]):

$$\liminf_{n\to+\infty}\sum_{i=1}^{N}\int_{\Omega}a_i\left(x,\frac{\partial u_n}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u_n-\varphi)dx \geq \sum_{i=1}^{N}\int_{\Omega}a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u-\varphi)dx. \tag{3.16}$$

We now focus our attention on the second and the fourth terms of (3.14). We have

$$T_n(b(u_n))T_k(u_n-\varphi)\longrightarrow b(u)T_k(u-\varphi)\ \text{ a.e. }\ x\in\Omega \tag{3.17}$$

and

$$|T_n(b(u_n))T_k(u_n-\varphi)|\leq k|b(u_n)|. \tag{3.18}$$

We will show that

$$|b(u_n)|\leq ||f_n||_\infty \ \text{ a.e. on }\ \Omega \quad\text{and}\quad |u_n|\leq\frac{1}{\lambda}||g_n||_\infty \ \text{ a.e. on }\ \partial\Omega.$$

Indeed, recall that for any $\delta>0$,

$$H_\delta(s)=\min\left(\frac{s^+}{\delta};1\right)\quad\text{and}\quad sign_0^+(s)=\begin{cases}1 & \text{if } s>0\\ 0 & \text{if } s\leq 0.\end{cases}$$

If γ is a maximal monotone operator defined on $\mathbb{R}$, we denote by γ_0 the main section of γ i.e.,

$$\gamma_0(s)=\begin{cases}\text{minimal absolute value of } \gamma(s) & \text{if } \gamma(s)\neq\emptyset\\ +\infty & \text{if } [s,+\infty)\cap D(\gamma)=\emptyset\\ -\infty & \text{if } (-\infty,s]\cap D(\gamma)=\emptyset.\end{cases}$$

Remark that as δ goes to 0, $H_\delta(s)$ goes to $sign_0^+(s)$.

We take $\varphi=H_\delta(u_n-M)$ as a test function in (3.2) for the weak solution u_n and $M>0$ (a constant to be chosen later), to get

$$\left\{\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}a_i\left(x,\frac{\partial u_n}{\partial x_i}\right)\frac{\partial}{\partial x_i}H_\delta(u_n-M)dx+\varepsilon\int_\Omega|u_n|^{p_M(x)-2}u_nH_\delta(u_n-M)dx\\ &+\int_\Omega T_n\big(b(u_n)\big)H_\delta(u_n-M)dx+\lambda\int_{\partial\Omega}T_n(u_n)H_\delta(u_n-M)d\sigma\\ &=\int_\Omega f_nH_\delta(u_n-M)dx+\int_{\partial\Omega}g_nH_\delta(u_n-M)d\sigma.\end{aligned}\right. \tag{3.19}$$

We have

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i\Big(x,\frac{\partial u_n}{\partial x_i}\Big)\frac{\partial}{\partial x_i}H_\delta(u_n-M)dx\\&=\frac{1}{\delta}\sum_{i=1}^{N}\int_{\left\{\frac{(u_n-M)^+}{\delta}<1\right\}}a_i\Big(x,\frac{\partial u_n}{\partial x_i}\Big)\frac{\partial}{\partial x_i}(u_n-M)^+dx\\&=\frac{1}{\delta}\sum_{i=1}^{N}\int_{\left\{0<u_n-M<\delta\right\}}a_i\Big(x,\frac{\partial u_n}{\partial x_i}\Big)\frac{\partial}{\partial x_i}u_ndx\geq 0\ \text{ according to (1.7)}\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega|u_n|^{p_M(x)-2}u_nH_\delta(u_n-M)dx\\&=\int_{\left\{\frac{(u_n-M)^+}{\delta}<1\right\}}|u_n|^{p_M(x)-2}u_n\frac{(u_n-M)^+}{\delta}dx+\int_{\left\{\frac{(u_n-M)^+}{\delta}\geq1\right\}}|u_n|^{p_M(x)-2}u_ndx\\&\geq\frac{1}{\delta}\int_{\left\{M<u_n<M+\delta\right\}}|u_n|^{p_M(x)-2}u_n(u_n-M)dx\geq 0.\end{aligned}$$

Then, (3.19) gives

$$\begin{aligned}&\int_\Omega T_n\big(b(u_n)\big)H_\delta(u_n-M)dx+\lambda\int_{\partial\Omega}T_n(u_n)H_\delta(u_n-M)d\sigma\\&\leq\int_\Omega f_nH_\delta(u_n-M)dx+\int_{\partial\Omega}g_nH_\delta(u_n-M)d\sigma,\end{aligned}$$

which is equivalent to

$$\text{(3.20)}\qquad\left\{\begin{aligned}&\int_\Omega\Big(T_n\big(b(u_n)\big)-T_n\big(b(M)\big)\Big)H_\delta(u_n-M)dx\\&\quad+\lambda\int_{\partial\Omega}\big(T_n(u_n)-T_n(M)\big)H_\delta(u_n-M)d\sigma\\&\leq\int_\Omega\Big(f_n-T_n\big(b(M)\big)\Big)H_\delta(u_n-M)dx\\&\quad+\int_{\partial\Omega}\big(g_n-\lambda T_n(M)\big)H_\delta(u_n-M)d\sigma.\end{aligned}\right.$$

We obtain the same inequality as in [14] (see (3.4) p. 210). Thus, we get

$$|b(u_n)|\leq||f_n||_\infty\ \text{ a.e. in }\ \Omega\quad\text{and}\quad|u_n|\leq\frac{1}{\lambda}||g_n||_\infty\ \text{ a.e. in }\ \partial\Omega.$$

We use the Lebesgue dominated convergence theorem to get

$$\lim_{n\to+\infty}\int_\Omega T_n(b(u_n))T_k(u_n-\varphi)dx=\int_\Omega b(u)T_k(u-\varphi)dx \tag{3.21}$$

and

$$\lim_{n\to+\infty}\int_{\partial\Omega} T_n(u_n)T_k(u_n-\varphi)d\sigma=\int_{\partial\Omega} uT_k(u-\varphi)d\sigma. \tag{3.22}$$

For the third term of (3.14), we prove that

$$\liminf_{n\to+\infty}\varepsilon\int_\Omega |u_n|^{p_M(x)-2}u_nT_k(u_n-\varphi)dx\geq 0 \ \text{ for } \ \varepsilon\to 0.$$

We have

$$\int_\Omega |u_n|^{p_M(x)-2}u_nT_k(u_n-\varphi)dx=\int_\Omega\Big(|u_n|^{p_M(x)-2}u_n-|\varphi|^{p_M(x)-2}\varphi\Big)T_k(u_n-\varphi)dx$$
$$+\int_\Omega |\varphi|^{p_M(x)-2}\varphi T_k(u_n-\varphi)dx.$$

Since the quantity $\big(|u_n|^{p_M(x)-2}u_n-|\varphi|^{p_M(x)-2}\varphi\big)T_k(u_n-\varphi)$ is nonnegative and for all x in Ω, the application $\xi\longmapsto|\xi|^{p_M(x)-2}\xi$ is continuous, then we get

$$\Big(|u_n|^{p_M(x)-2}u_n-|\varphi|^{p_M(x)-2}\varphi\Big)T_k(u_n-\varphi)\to\Big(|u|^{p_M(x)-2}u-|\varphi|^{p_M(x)-2}\varphi\Big)T_k(u-\varphi)$$

a.e. in Ω. It follows by Fatou's lemma that

$$\begin{cases}\liminf_{n\to+\infty}\int_\Omega\Big(|u_n|^{p_M(x)-2}u_n-|\varphi|^{p_M(x)-2}\varphi\Big)T_k(u_n-\varphi)dx\\ \geq\int_\Omega\Big(|u|^{p_M(x)-2}u-|\varphi|^{p_M(x)-2}\varphi\Big)T_k(u-\varphi)dx.\end{cases} \tag{3.23}$$

We have

$$\int_\Omega\big||\varphi|^{p_M(x)-2}\varphi\big|dx=\int_\Omega|\varphi|^{p_M(x)-1}dx\leq\int_\Omega\Big(||\varphi||_\infty\Big)^{p_M(x)-1}dx$$
$$\leq\int_{\{||\varphi||_\infty\leq 1\}}\Big(||\varphi||_\infty\Big)^{p_M(x)-1}dx+\int_{\{||\varphi||_\infty>1\}}\Big(||\varphi||_\infty\Big)^{p_M(x)-1}dx$$
$$\leq meas(\Omega)+\big(||\varphi||_\infty\big)^{p_M^+-1}meas(\Omega)<+\infty.$$

Hence, $|\varphi|^{p_M(x)-2}\varphi \in L^1(\Omega)$. Since $T_k(u_n - \varphi)$ converges weakly-* to $T_k(u - \varphi)$ in $L^\infty(\Omega)$ and $|\varphi|^{p_M(x)-2}\varphi \in L^1(\Omega)$, then we obtain

$$\lim_{n\to+\infty} \int_\Omega |\varphi|^{p_M(x)-2}\varphi T_k(u_n - \varphi)dx = \int_\Omega |\varphi|^{p_M(x)-2}\varphi T_k(u - \varphi)dx. \tag{3.24}$$

By adding (3.23) and (3.24), we get

$$\liminf_{n\to+\infty} \int_\Omega |u_n|^{p_M(x)-2}u_n T_k(u_n - \varphi)dx \geq \int_\Omega |u|^{p_M(x)-2}u T_k(u - \varphi)dx. \tag{3.25}$$

Since

$$\int_\Omega |u|^{p_M(x)-2}u T_k(u - \varphi)dx \leq k \int_\Omega |u|^{p_M(x)-1}dx < +\infty,$$

thus we get

$$\liminf_{n\to+\infty} \varepsilon \int_\Omega |u_n|^{p_M(x)-2}u_n T_k(u_n - \varphi)dx \geq 0 \quad \text{for} \quad \varepsilon \to 0. \tag{3.26}$$

Thus, combining (3.15), (3.16), (3.21), (3.22) and (3.26) we have

$$\begin{cases} \sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u - \varphi)dx + \int_\Omega b(u)T_k(u - \varphi)dx \\ + \int_{\partial\Omega} \lambda u T_k(u - \varphi)d\sigma \leq \int_\Omega f T_k(u - \varphi)dx + \int_{\partial\Omega} g T_k(u - \varphi)d\sigma. \end{cases} \tag{3.27}$$

It means that, u is an entropy solution of problem (1.1). ■

Now we state the uniqueness result of entropy solution.

Theorem 3.2. *Assume that* (1.4)–(1.10) *hold and let* u *be an entropy solution of* (1.1). *Then,* u *is unique.*

Proof. The proof is done in two steps.

Step 1. A priori estimates.

Assume (1.4)–(1.10), $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$.

Lemma 3.4. *Let* u *be an entropy solution of* (1.1). *Then*

$$\sum_{i=1}^N \int_{\{|u|\leq k\}} \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)} dx \leq \frac{k}{C_3}\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big) \tag{3.28}$$

and there exists a positive constant C_9 *such that*

$$\|b(u)\|_1 \leq C_9 \cdot meas(\Omega) + \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}. \tag{3.29}$$

Proof. Let us take $\varphi = 0$ in the entropy inequality (3.1).

- Since $\int_\Omega b(u)T_k(u)dx + \lambda \int_{\partial\Omega} uT_k(u)d\sigma \geq 0$, by (1.7), then we get (3.28).
- Also, using the fact that

$$\sum_{i=1}^{N} \int_\Omega a_i\big(x, \frac{\partial u}{\partial x_i}\big) \frac{\partial}{\partial x_i} T_k(u)dx + \lambda \int_{\partial\Omega} uT_k(u)d\sigma \geq 0,$$

the relation (3.1) gives

$$\int_\Omega b(u)T_k(u)dx \leq \int_\Omega fT_k(u)dx + \int_{\partial\Omega} gT_k(u)d\sigma. \tag{3.30}$$

Then

$$\int_{\{|u|\leq k\}} b(u)T_k(u)dx + \int_{\{|u|>k\}} b(u)T_k(u)dx \leq k\|f\|_{L^1(\Omega)} + k\|g\|_{L^1(\partial\Omega)},$$

which imply that

$$\int_{\{|u|>k\}} b(u)T_k(u)dx \leq k\|f\|_{L^1(\Omega)} + k\|g\|_{L^1(\partial\Omega)}$$

or

$$\int_{\{u>k\}} b(u)dx + \int_{\{u<-k\}} -b(u)dx \leq \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}.$$

Therefore,

$$\int_{\{|u|>k\}} |b(u)|dx \leq \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}.$$

So, we obtain

$$\begin{aligned}\int_\Omega |b(u)|dx &= \int_{\{|u|\leq k\}} |b(u)|dx + \int_{\{|u|>k\}} |b(u)|dx \\ &\leq \int_{\{|u|\leq k\}} |b(u)|dx + \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}.\end{aligned}$$

Since the function b is nondecreasing, then

$$\int_{\{|u|\leq k\}} |b(u)|dx \leq \max\{b(k); |b(-k)|\} \cdot meas(\Omega).$$

Consequently, there exists a constant $C_9 = \max\{b(k); |b(-k)|\}$ such that

$$||b(u)||_1 \leq C_9 \cdot meas(\Omega) + ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}.$$

□

Lemma 3.5. *If u is an entropy solution of* (1.1), *then there exists a constant D which depends on f, g and Ω and such that*

$$meas\{|u| > k\} \leq \frac{D}{\min(b(k), |b(-k)|)}, \quad \forall\ k > 0 \tag{3.31}$$

and a constant $D' > 0$ which depends on f, g and Ω and such that

$$meas\left\{\left|\frac{\partial u}{\partial x_i}\right| > k\right\} \leq \frac{D'}{k^{\frac{1}{(p_M^-)'}}}, \quad \forall\ k \geq 1. \tag{3.32}$$

Proof. • For any $k > 0$, the relation (3.29) gives

$$\int_{\{|u|>k\}} \min(b(k), |b(-k)|)dx \leq \int_{\{|u|>k\}} |b(u)|dx \leq C_9 \cdot meas(\Omega) + ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}.$$

Therefore,

$$\min(b(k), |b(-k)|) \cdot meas\{|u| > k\} \leq C_9 \cdot meas(\Omega) + ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)} = D;$$

that is

$$meas\{|u| > k\} \leq \frac{D}{\min(b(k), |b(-k)|)}.$$

• See [2] for the proof of (3.32). □

Lemma 3.6. *If u is an entropy solution of* (1.1), *then*

$$\lim_{h\to+\infty} \int_\Omega |f|\chi_{\{|u|>h-t\}}dx + \lim_{h\to+\infty} \int_{\partial\Omega} |g|\chi_{\{|u|>h-t\}}d\sigma = 0, \tag{3.33}$$

where $h > 0$ and $t > 0$.

Proof. Since the function b is surjective, according to (3.31), we have

$$\lim_{h\to+\infty} meas\{|u| > h - t\} = 0.$$

As $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$, it follows by the Lebesgue dominated convergence theorem that

$$\lim_{h\to+\infty} \int_\Omega |f|\chi_{\{|u|>h-t\}}dx + \lim_{h\to+\infty} \int_{\partial\Omega} |g|\chi_{\{|u|>h-t\}}d\sigma = 0.$$

□

Lemma 3.7. *If u is an entropy solution of* (1.1), *then there exists a positive constant K such that*

$$\rho_{p_i'(\cdot)}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-1}\chi_F\right) \le K, \quad \forall\, i = 1, \dots, N, \tag{3.34}$$

where $F = \{h < |u| \le h + k\}$, $h > 0$, $k > 0$.

Proof. Let $\varphi = T_h(u)$ as test function in the entropy inequality (3.1). We get

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i} T_k(u - T_h(u))dx + \int_\Omega b(u)T_k(u - T_h(u))dx \\ \displaystyle + \lambda \int_{\partial\Omega} uT_k(u - T_h(u))d\sigma \le \int_\Omega fT_k(u - T_h(u))dx + \int_{\partial\Omega} gT_k(u - T_h(u))d\sigma. \end{cases}$$

Thus,

$$\sum_{i=1}^N \int_{\{h<|u|\le h+k\}} a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_i}dx \le k||f||_{L^1(\Omega)} + k||g||_{L^1(\partial\Omega)}$$

and using (1.7), we have

$$\int_F \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx \le \frac{k}{C_3}\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big), \quad \forall\, i = 1 \dots, N.$$

Consequently,

$$\rho_{p_i'(\cdot)}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-1}\chi_F\right) \le K, \quad \forall\, i = 1, \dots, N.$$

□

Step 2. Uniqueness of entropy solution.

Let $h > 0$ and u, v be two entropy solutions of (1.1). We write the entropy inequality corresponding to the solution u, with $T_h(v)$ as a test function, and to the solution v, with $T_h(u)$ as a test function. We get

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i} T_k(u - T_h(v))dx + \int_\Omega b(u)T_k(u - T_h(v))dx \\ \displaystyle + \lambda \int_{\partial\Omega} uT_k(u - T_h(v))d\sigma \le \int_\Omega fT_k(u - T_h(v))dx + \int_{\partial\Omega} gT_k(u - T_h(v))d\sigma \end{cases} \tag{3.35}$$

and

(3.36)
$$\begin{cases} \displaystyle\sum_{i=1}^{N} \int_{\Omega} a_i\Big(x, \frac{\partial v}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(v - T_h(u))dx + \int_{\Omega} b(v) T_k(v - T_h(u))dx \\ \displaystyle + \lambda \int_{\partial\Omega} v T_k(v - T_h(u))d\sigma \le \int_{\Omega} f T_k(v - T_h(u))dx + \int_{\partial\Omega} g T_k(v - T_h(u))d\sigma. \end{cases}$$

Upon addition, we get

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \int_{\Omega} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u - T_h(v))dx \\ \displaystyle\quad + \sum_{i=1}^{N} \int_{\Omega} a_i\Big(x, \frac{\partial v}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(v - T_h(u))dx \\ \displaystyle\quad + \int_{\Omega} b(u) T_k(u - T_h(v))dx + \int_{\Omega} b(v) T_k(v - T_h(u))dx \\ \displaystyle\quad + \lambda \int_{\partial\Omega} u T_k(u - T_h(v))d\sigma + \lambda \int_{\partial\Omega} v T_k(v - T_h(u))d\sigma \\ \displaystyle \le \int_{\Omega} f\big[T_k(u - T_h(v)) + T_k(v - T_h(u))\big]dx \\ \displaystyle\quad + \int_{\partial\Omega} g\big[T_k(u - T_h(v)) + T_k(v - T_h(u))\big]d\sigma. \end{cases} \tag{3.37}$$

Define the following sets

$$E_1 = \{|u - v| \le k;\ |v| \le h\};\quad E_2 = E_1 \cap \{|u| \le h\} \ \text{ and }\ E_3 = E_1 \cap \{|u| > h\}.$$

We start with the first integral in (3.37). We have

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\{|u - T_h(v)| \le k\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u - T_h(v))dx \\ &= \sum_{i=1}^{N} \int_{\{|u - T_h(v)| \le k\} \cap \{|v| \le h\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u - T_h(v))dx \\ &+ \sum_{i=1}^{N} \int_{\{|u - T_h(v)| \le k\} \cap \{|v| > h\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u - T_h(v))dx \\ &= \sum_{i=1}^{N} \int_{\{|u - v| \le k\} \cap \{|v| \le h\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial (u - v)}{\partial x_i} dx \end{aligned}$$

$$+\sum_{i=1}^{N}\int_{\{|u-hsign(v)|\leq k\}\cap\{|v|>h\}} a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial u}{\partial x_i}dx$$
$$\geq \sum_{i=1}^{N}\int_{E_1} a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}(u-v)dx$$
$$\geq \sum_{i=1}^{N}\int_{E_2} a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}(u-v)dx+\sum_{i=1}^{N}\int_{E_3} a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}(u-v)dx.$$

Then, we obtain

$$(3.38)\quad \begin{cases} \displaystyle\sum_{i=1}^{N}\int_{\{|u-T_h(v)|\leq k\}} a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}T_k(u-T_h(v))dx \\ \displaystyle\geq \sum_{i=1}^{N}\int_{E_2} a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}(u-v)dx-\sum_{i=1}^{N}\int_{E_3} a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial v}{\partial x_i}dx. \end{cases}$$

According to (1.5) and the Hölder type inequality we have

$$\Big|\sum_{i=1}^{N}\int_{E_3} a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial v}{\partial x_i}dx\Big|\leq C_1\sum_{i=1}^{N}\int_{E_3}\Big(j_i(x)+\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)-1}\Big)\Big|\frac{\partial v}{\partial x_i}\Big|dx$$
$$\leq C_1\sum_{i=1}^{N}\Big(|j_i|_{p_i'(\cdot)}+\Big|\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-1}\Big|_{p_i'(\cdot),\{h<|u|\leq h+k\}}\Big)\big|\frac{\partial v}{\partial x_i}\big|_{p_i(\cdot),\{h-k<|v|\leq h\}},$$

where

$$\Big|\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-1}\Big|_{p_i'(\cdot),\{h<|u|\leq h+k\}}=\Big\|\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-1}\Big\|_{L^{p_i'(\cdot)}(\{h<|u|\leq h+k\})}.$$

Thanks to relation (2.1) and Lemma 3.7, the quantity $\Big(|j_i|_{p_i'(\cdot)}+\big|\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-1}\big|_{p_i'(\cdot),\{h<|u|\leq h+k\}}\Big)$ is finite for all $i=1,\dots,N$.

According to Lemma 3.6, the quantity $\big|\frac{\partial v}{\partial x_i}\big|_{p_i(\cdot),\{h-k<|v|\leq h\}}$ converges to zero as h goes to infinity. Consequently, the last integral of (3.38) converges to zero as h goes to infinity. Then

$$(3.39)\quad \begin{aligned} &\sum_{i=1}^{N}\int_{\{|u-T_h(v)|\leq k\}} a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}T_k(u-T_h(v))dx \\ &\geq I_h+\sum_{i=1}^{N}\int_{E_2} a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}(u-v)dx \end{aligned}$$

with $\lim_{h\to+\infty} I_h=0$.

We may adopt the same procedure to treat the second term in (3.37) to obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{|v-T_h(u)|\leq k\}} a_i\big(x,\frac{\partial v}{\partial x_i}\big)\frac{\partial}{\partial x_i}T_k(v-T_h(u))dx\\&\geq J_h-\sum_{i=1}^{N}\int_{E_2} a_i\big(x,\frac{\partial v}{\partial x_i}\big)\frac{\partial}{\partial x_i}(u-v)dx\end{aligned}\tag{3.40}$$

with $\lim_{h\to+\infty} J_h=0$.

For the other terms in the left-hand side of (3.37), we denote

$$K_h=\int_\Omega b(u)T_k(u-T_h(v))dx+\int_\Omega b(v)T_k(v-T_h(u))dx$$

and

$$L_h=\int_{\partial\Omega} uT_k(u-T_h(v))d\sigma+\int_{\partial\Omega} vT_k(v-T_h(u))d\sigma.$$

We have

$$b(u)T_k(u-T_h(v))\to b(u)T_k(u-v)\ \text{ a.e. in }\Omega\ \text{ since }\ h\to+\infty$$

and

$$\big|b(u)T_k(u-T_h(v))\big|\leq k|b(u)|\in L^1(\Omega).$$

Then, by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{h\to+\infty}\int_\Omega b(u)T_k(u-T_h(v))dx=\int_\Omega b(u)T_k(u-v)dx$$

and

$$\lim_{h\to+\infty}\int_\Omega b(v)T_k(v-T_h(u))dx=\int_\Omega b(v)T_k(v-u)dx.$$

Then

$$\lim_{h\to+\infty} K_h=\int_\Omega \big(b(u)-b(v)\big)T_k(u-v)dx.\tag{3.41}$$

In the same way, we get

$$\lim_{h\to+\infty} L_h=\int_{\partial\Omega} \big(u-v\big)T_k(u-v)d\sigma.\tag{3.42}$$

Now, we consider the right-hand side of inequality (3.37). We have

$$\lim_{h\to+\infty} f\Bigg(T_k(u-T_h(v))+T_k(v-T_h(u))\Bigg)=0\ \text{ a.e. in }\ \Omega$$

and

$$\left|f\big(T_k(u - T_h(v)) + T_k(v - T_h(u))\big)\right| \le 2k|f| \in L^1(\Omega).$$

By the Lebesgue dominated convergence theorem, we get

$$\lim_{h\to+\infty} \int_\Omega f\Big(T_k\big(u - T_h(v)\big) + T_k\big(v - T_h(u)\big)\Big)dx = 0. \tag{3.43}$$

Similarly, we have

$$\lim_{h\to+\infty} \int_{\partial\Omega} g\Big(T_k\big(u - T_h(v)\big) + T_k\big(v - T_h(u)\big)\Big)d\sigma = 0. \tag{3.44}$$

After passing to the limit as h goes to $+\infty$ in (3.37) we get

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_{\{|u-v|\le k\}} \Big(a_i\big(x, \frac{\partial u}{\partial x_i}\big) - a_i\big(x, \frac{\partial v}{\partial x_i}\big)\Big)\frac{\partial}{\partial x_i}(u - v)dx \\ \displaystyle + \int_\Omega \big(b(u) - b(v)\big)T_k(u - v)dx + \int_{\partial\Omega} \big(u - v\big)T_k(u - v)d\sigma \le 0. \end{cases} \tag{3.45}$$

Since b, $T_k(\cdot)$ and $a_i(x, \cdot)$ are monotone, then

$$\int_\Omega \big(b(u) - b(v)\big)T_k(u - v)dx = 0, \tag{3.46}$$

$$\int_{\partial\Omega} \big(u - v\big)T_k(u - v)d\sigma = 0 \tag{3.47}$$

and

$$\int_{\{|u-v|\le k\}} \sum_{i=1}^N \Big(a_i\big(x, \frac{\partial u}{\partial x_i}\big) - a_i\big(x, \frac{\partial v}{\partial x_i}\big)\Big)\frac{\partial}{\partial x_i}(u - v)dx = 0. \tag{3.48}$$

According to (1.7), we deduce from (3.48) that

$$u - v = c \text{ a.e. } x \in \Omega \text{ where } c \text{ is a constant.}$$

By (3.47), we deduce that for all $k \in \mathbb{N}^*$ there exists $C_k \subset \partial\Omega$ with $meas(C_k) = 0$ and such that for all $x \in \partial\Omega \setminus C_k$,

$$\big(u(x) - v(x)\big)T_k\big(u(x) - v(x)\big) = 0.$$

Therefore,

$$\big(u(x) - v(x)\big)T_k\big(u(x) - v(x)\big) = 0, \text{ for all } x \in \partial\Omega \setminus \bigcup_{k\in\mathbb{N}^*} C_k. \tag{3.49}$$

So, we get

$$u - v = 0 \quad \text{a.e. on} \quad \partial\Omega.$$

Finally as

$$u - v = c \;\; \text{a.e. in} \;\; \Omega \quad \text{and} \quad u - v = 0 \quad \text{a.e. on} \quad \partial\Omega,$$

it follows that

$$u = v \;\; \text{a.e. in} \;\; \Omega.$$

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