Discussiones Mathematicae
Differential Inclusions, Control and Optimization 35 (2015) 197–207
doi:10.7151/dmdico.1173

BOUNDEDNESS OF SET-VALUED STOCHASTIC INTEGRALS

Michał Kisielewicz

Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra
Prof. Z. Szafrana 4a, 65–516 Zielona Góra, Poland

e-mail: M.Kisielewicz@wmie.uz.zgora.pl

Abstract

The paper deals with integrably boundedness of Itô set-valued stochastic integrals defined by E.J. Jung and J.H. Kim in the paper [4], where has not been proved that this integral is integrably bounded. The problem of integrably boundedness of the above set-valued stochastic integrals has been considered in the paper [7] and the monograph [8], but the problem has not been solved there. The first positive results dealing with this problem due to M. Michta, who showed (see [11]) that there are bounded set-valued $\mathbb{F}$-nonanticipative mappings having unbounded Itô set-valued stochastic integrals defined by E.J. Jung and J.H. Kim. The present paper contains some new conditions implying unboundedness of the above type set-valued stochastic integrals.

Keywords: set-valued mapping, Itô set-valued integral, set-valued stochastic process, integrably boundedness of set-valued integral.

2010 Mathematics Subject Classification: 60H05, 28B20, 47H04.

1. Introduction

The paper is devoted to the integrably boundedness problem of Itô set-valued stochastic integrals defined by E.J. Jung and J.H. Kim in the paper [4] as some set-valued random variables. The first Itô set-valued stochastic integrals, defined as subsets of the spaces $\mathbb{L}^2(\Omega, \mathbb{R}^n)$ and $\mathbb{L}^2(\Omega, \mathcal{X})$, have been considered by F. Hiai and M. Kisielewicz (see [1, 5, 6]), where $\mathcal{X}$ is a Hilbert space. Unfortunately, such defined integrals do not admit their representations by set-valued random variables with values in $\mathbb{R}^n$ and $\mathcal{X}$, because they are not decomposable subsets of $\mathbb{L}^2(\Omega, \mathbb{R}^n)$ and $\mathbb{L}^2(\Omega, \mathcal{X})$, respectively. J.Jung and J.H. Kim (see [4]) defined the Itô set-valued stochastic integral as a set-valued random variable determined by a closed decomposable hull of the set-valued stochastic functional integral defined in [5]. Unfortunately, the proof of the result dealing with integrably boundedness of such integrals, presented in the paper [4] is not correct. Later on, integrably boundedness of such defined set-valued stochastic integrals has been considered in the paper [7] and the monograph [9]. However the problem has not been solved there. The first positive results dealing with this problem due to M. Michta, who showed in [11], that there are bounded set-valued $\mathbb{F}$-nonanticipative mappings having unbounded Itô set-valued stochastic integrals defined by E.J. Jung and J.H. Kim. The present paper contains some new conditions implying unboudednes of the above type set-valued stochastic integrals.

In what follows we shall assume that we have given a filtered probability space $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, P)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ satisfying the usual conditions and such that there are real stochasticaly independet $\mathbb{F}$-Brownian motions $B^j = (B_t^j)_{0 \le t \le T}$, $j = 1, 2, \ldots, m$, defined on $\mathcal{P}_{\mathbb{F}}$. By $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ we denote the norms of the spaces $\mathbb{L}^2([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ and $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^d)$, respectively, where $\Sigma_{\mathbb{F}}$ denotes the σ-algebra of all $\mathbb{F}$-nonanticipative subsets of $[0,T]\times\Omega$. For a given set $\Lambda \subset \mathbb{L}^2([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ by $\mathrm{dec}_{\Sigma_{\mathbb{F}}}\Lambda$ and $\overline{\mathrm{dec}}_{\Sigma_{\mathbb{F}}}\Lambda$ we denote a decomposable and a closed decomposable hull of Λ, respectively, i.e., the smallest decomposble and closed decomposable set containig Λ. Let us recall that a set $\Lambda \subset \mathbb{L}^2([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ is said to be decomposable if for every $u, v \in \Lambda$ and $D \in \Sigma_{\mathbb{F}}$ one has $\mathbb{I}_D u + \mathbb{I}_{D^\sim} v \in \Lambda$, where $D^\sim = ([0,T]\times\Omega) \setminus D$. In a similar way for a given set $\Lambda \subset \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^d)$ decomposable sets $\mathrm{dec}_{\mathcal{F}}\Lambda$ and $\overline{\mathrm{dec}}_{\mathcal{F}}\Lambda$ are defined.

The Fréchet-Nikodym metric space $(\mathcal{S}_{\mathbb{F}}, \lambda)$ corresponding to a measure space $([0,T]\times\Omega, \Sigma_{\mathbb{F}}, \mu)$, with $\mu = dt \times P$, is defined by $\mathcal{S}_{\mathbb{F}} = \{[C] : C \in \Sigma_{\mathbb{F}}\}$, where $[C] = \{D \in \Sigma_{\mathbb{F}} : \mu(C \triangle D) = 0\}$ for $C \in \Sigma_{\mathbb{F}}$ and $\lambda([A],[B]) = \mu(A\triangle B)$ for $A, B \in \Sigma_{\mathbb{F}}$, $A\triangle B = (A \setminus B) \cup (B \setminus A)$. It is a complete metric space homeomorphic with the set $\mathcal{K}_{\mathbb{F}} \subset \mathbb{L}^2([0,T]\times\Omega, \Sigma_{\mathbb{F}}, \mathbb{R})$ defined by $\mathcal{K}_{\mathbb{F}} = \{\mathbb{I}_D \in \mathbb{L}^2([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}) : D \in \Sigma_{\mathbb{F}}\}$. Indeed, for every $A, B \in \Sigma_{\mathbb{F}}$ we have $\lambda([A],[B]) = \mu(A\triangle B) = \int\int_{A\triangle B} dPdt = E\int_0^T \mathbb{I}_{A\triangle B} dt = E\int_0^T |\mathbb{I}_A - \mathbb{I}_B|^2 dt = \|\mathbb{I}_A - \mathbb{I}_B\|^2$.

Let $\mathcal{J}_T^j$ denotes the isometry on the space $\mathbb{L}^2([0,T]\times\Omega, \Sigma_{\mathbb{F}}, \mathbb{R})$ defined by $\mathcal{J}_T^j(g) = \int_0^T g_t dB_t^j$. It can be proved that $\mathcal{J}_T^j(\mathcal{K}_{\mathbb{F}})$ is not an integrably bounded subset of the space $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R})$ for every $j = 1, 2, \ldots, m$ (see [11], Corollary 3.14). It implies (see [11], Theorem 2.2) that $\overline{\mathrm{dec}}_{\mathcal{F}}\mathcal{J}_T^j(\mathcal{K}_{\mathbb{F}})$ is an unbounded subset of the space $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R})$, where the closure is taken with respect to the norm topology of the space $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R})$. Hence it follows that $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(\mathcal{K}_{\mathbb{F}})$ and $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(\mathrm{co}\,\mathcal{K}_{\mathbb{F}})$ are unbounded subsets of this space, because they contain an unbounded subset $\mathcal{J}_T^j(\mathcal{K}_{\mathbb{F}})$. Therefore, $\overline{\mathrm{co}}[\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(\mathcal{K}_{\mathbb{F}})]$ is unbounded, which by virtue of ([9], Proposition 3.1) implies that $\overline{\mathrm{dec}}_{\mathcal{F}}\mathcal{J}_T^j(\mathrm{co}\,\mathcal{K}_{\mathbb{F}})$ is an unbounded subset of the space $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R})$. Finally let us observe that for every $\alpha>0$ a set $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(\alpha\cdot\mathcal{K}_{\mathbb{F}})$ is an unbounded subset of $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R}^d)$, because $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(\alpha\cdot\mathcal{K}_{\mathbb{F}})=\alpha\cdot\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(\mathcal{K}_{\mathbb{F}})$. Hence the following basic results of the paper follows.

Lemma 1. *If $\alpha>0$ and $h\in\mathbb{L}^2([0,T]\times\Omega,\Sigma_{\mathbb{F}},\mathbb{R})$ are such that $h_t(\omega)\geq\alpha$ for a.e. $(t,\omega)\in[0,T]\times\Omega$ then the set $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h\cdot\mathcal{K}_{\mathbb{F}})$ is an unbounded subset of $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R})$ for every $j=1,2,\ldots,m$.*

Proof. Let $j=1,2,\ldots,m$ be fixed and let us observe that $\mathcal{K}_{\mathbb{F}}=\mathrm{dec}_{\Sigma_{\mathbb{F}}}\{0,1\}$. Hence it follows that $\overline{\mathrm{co}}\,\mathcal{K}_{\mathbb{F}}=\overline{\mathrm{co}}[\mathrm{dec}_{\Sigma_{\mathbb{F}}}\{0,1\}]=\overline{\mathrm{dec}}_{\Sigma_{\mathbb{F}}}[\mathrm{co}\,\{0,1\}]=\overline{\mathrm{dec}}_{\Sigma_{\mathbb{F}}}[\overline{\mathrm{co}}\,\{0,1\}]$. Therefore, $\alpha\cdot\overline{\mathrm{co}}\,\mathcal{K}_{\mathbb{F}}=\overline{\mathrm{dec}}_{\Sigma_{\mathbb{F}}}[\overline{\mathrm{co}}\,\{0,\alpha\}]$ and $h\cdot\overline{\mathrm{co}}\,\mathcal{K}_{\mathbb{F}}=\overline{\mathrm{dec}}_{\Sigma_{\mathbb{F}}}[\overline{\mathrm{co}}\,\{0,h\}]$. But $\overline{\mathrm{co}}\,\{0,\alpha\}=[0,\alpha]\subset[0,h_t(\omega)]=\overline{\mathrm{co}}\,\{0,h_t(\omega)\}$ for a.e. $(t,\omega)\in[0,T]\times\Omega$. Then $\alpha\cdot\overline{\mathrm{co}}\,\mathcal{K}_{\mathbb{F}}=\overline{\mathrm{dec}}_{\Sigma_{\mathbb{F}}}[\overline{\mathrm{co}}\,\{0,\alpha\}]\subset\overline{\mathrm{dec}}_{\Sigma_{\mathbb{F}}}[\overline{\mathrm{co}}\,\{0,h\}]=h\cdot\overline{\mathrm{co}}\mathcal{K}_{\mathbb{F}}$. Therefore, $\overline{\mathrm{dec}}_{\mathcal{F}}\mathcal{J}_T^j(\alpha\cdot\overline{\mathrm{co}}\,\mathcal{K}_{\mathbb{F}})\subset\overline{\mathrm{dec}}_{\mathcal{F}}\mathcal{J}_T^j(h\cdot\overline{\mathrm{co}}\,\mathcal{K}_{\mathbb{F}})$, which implies $\overline{\mathrm{dec}}_{\mathcal{F}}\mathcal{J}_T^j(\alpha\cdot\mathrm{co}\,\mathcal{K}_{\mathbb{F}})\subset\overline{\mathrm{dec}}_{\mathcal{F}}\mathcal{J}_T^j(h\cdot\mathrm{co}\,\mathcal{K}_{\mathbb{F}})$. But $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(\alpha\cdot\mathcal{K}_{\mathbb{F}})$ is an unbounded subset of $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R}^d)$. Therefore, a set $\overline{\mathrm{co}}[\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(\alpha\cdot\mathcal{K}_{\mathbb{F}})]$ is unbounded, which implies that $\overline{\mathrm{dec}}_{\mathcal{F}}\mathcal{J}_T^j(h\cdot\mathrm{co}\,\mathcal{K}_{\mathbb{F}})$ is unbounded, because by ([8], Remark 3.6 of Chap. 2) we have $\overline{\mathrm{dec}}_{\mathcal{F}}\mathcal{J}_T^j(h\cdot\mathrm{co}\,\mathcal{K}_{\mathbb{F}})=\overline{\mathrm{co}}[\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h\cdot\mathcal{K}_{\mathbb{F}})]$. Finally, unboundedness of $\overline{\mathrm{dec}}_{\mathcal{F}}\mathcal{J}_T^j(h\cdot\mathrm{co}\,\mathcal{K}_{\mathbb{F}})$ implies that $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h\cdot\mathcal{K}_{\mathbb{F}})$ is unbounded. ■

Lemma 2. *Let $C\in\Sigma_{\mathbb{F}}$ be a set of positive measure $\mu=dt\times P$ such that $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(\mathbb{I}_C\cdot\mathcal{K}_{\mathbb{F}})$ is unbounded subset of $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R})$. If $\alpha>0$ and $h\in\mathbb{L}^2([0,T]\times\Omega,\Sigma_{\mathbb{F}},\mathbb{R})$ are such that $\mathbb{I}_Ch\geq\mathbb{I}_C\alpha$ and $\mathbb{I}_{C^\sim}h=0$ then $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(\mathbb{I}_Ch\cdot\mathcal{K}_{\mathbb{F}})$ is an unbounded subset of $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R})$, where $C^\sim=([0,T]\times\Omega)\setminus C$.*

Proof. Similarly as in the proof of Lemma 1 we get $\overline{\mathrm{co}}\{0,\alpha\}=[0,\alpha]\subset[0,h_t(\omega)]=\overline{\mathrm{co}}\{0,h_t(\omega)\}$ for every $(t,\omega)\in C$. Then $\overline{\mathrm{co}}\{0,\mathbb{I}_C\alpha\}\subset\overline{\mathrm{co}}\{0,\mathbb{I}_Ch\}$. Therefore, $\mathbb{I}_C\alpha\cdot\overline{\mathrm{co}}\,\mathcal{K}_{\mathbb{F}}=\mathbb{I}_C\alpha\cdot\overline{\mathrm{co}}[\mathrm{dec}_\Sigma\{0,1\}]=\overline{\mathrm{dec}}_\Sigma[\overline{\mathrm{co}}\{0,\mathbb{I}_C\alpha\}]\subset\overline{\mathrm{dec}}_\Sigma[\overline{\mathrm{co}}\{0,\mathbb{I}_Ch\}]=\mathbb{I}_Ch\cdot\overline{\mathrm{co}}\,\mathcal{K}_{\mathbb{F}}$. But $\mathrm{dec}\,\mathcal{J}_T^j(\mathbb{I}_C\alpha\cdot\overline{\mathrm{co}}\,\mathcal{K}_{\mathbb{F}})=\alpha\cdot\mathcal{J}_T^j(\mathbb{I}_C\cdot\overline{\mathrm{co}}\,\mathcal{K}_{\mathbb{F}})$ and $\mathcal{J}_T^j(\mathbb{I}_C\cdot\overline{\mathrm{co}}\,\mathcal{K}_{\mathbb{F}})$ is unbounded. Then $\mathrm{dec}\,\mathcal{J}_T^j(\mathbb{I}_Ch\cdot\overline{\mathrm{co}}\,\mathcal{K}_{\mathbb{F}})$ contains an unbounded subset $\mathcal{J}_T^j(\mathbb{I}_C\alpha\cdot\overline{\mathrm{co}}\,\mathcal{K}_{\mathbb{F}})$. Thus $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(\mathbb{I}_Ch\cdot\overline{\mathrm{co}}\,\mathcal{K}_{\mathbb{F}})$ is an unbounded subset of $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R})$. Hence, similarly as in the proof of Lemma 1, it follows that $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(\mathbb{I}_Ch\cdot\mathcal{K}_{\mathbb{F}})$ is unbounded. ■

Let us recall (see [4]) that for an m-dimensional $\mathbb{F}$-Brownian motion $B=(B^1,\ldots,B^m)$ defined on $\mathcal{P}_{\mathbb{F}}$ and a given $\mathbb{F}$-nonanticipative set-valued process $\Phi=(\Phi_t)_{0\leq t\leq T}$ defined on $\mathcal{P}_{\mathbb{F}}$ with values in the space $\mathrm{Cl}(\mathbb{R}^{d\times m})$ of all nonempty closed subsets of $\mathbb{R}^{d\times m}$, a set-valued stochastic integral $\int_0^t\Phi_\tau B_\tau$ is defined on $[0,t]\subset[0,T]$ to be a set-valued random variable such that a set $S_{\mathcal{F}_t}(\int_0^t\Phi_\tau dB_\tau)$ of all $\mathcal{F}_t$-measurable selectors of $\int_0^t\Phi_\tau dB_\tau$ covers with a closed decomposable hull $\overline{\mathrm{dec}}_{\mathcal{F}}\mathcal{J}_T^j(S_{\mathbb{F}}(\Phi))$ of the set $\mathcal{J}_t(S_{\mathbb{F}}(\Phi))$, where $S_{\mathbb{F}}(\Phi))$ denotes the set of all square integrable $\mathbb{F}$-nonanticipative selectors of Φ and $\mathcal{J}_t(f)(\omega) = \int_0^t f_\tau(\omega)dB_\tau$ for every $\omega \in \Omega$ and $f \in S_{\mathbb{F}}(\Phi)$. A set-valued stochastic integral $\int_0^t \Phi_\tau dB_\tau$ is said to be integrably bounded if there exists a square integrably bounded random variable $m : \Omega \to \mathbb{R}^+$ such that $\rho(\int_0^t \Phi_\tau dB_\tau, \{0\}) \leq m$ a.s., where ρ is the Hausdorff metric defined on the space $\mathrm{Cl}(\mathbb{R}^d)$ of all nonempty closed subsets of $\mathbb{R}^d$. Immediately from ([2], Theorem 3.2) it follows that $\int_0^t \Phi_\tau dB_\tau$ is integrably bounded if and only if $S_{\mathcal{F}_t}(\int_0^t \Phi_\tau dB_\tau)$ is a bounded subset of the space $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^d)$, which by virtue of the above definition of $\int_0^t \Phi_\tau dB_\tau$ is equivalent to boundedness of the set $\overline{\mathrm{dec}}_{\mathcal{F}}\mathcal{J}_t(S_{\mathbb{F}}(\Phi))$. But $\sup\{E|u|^2 : u \in \overline{\mathrm{dec}}_{\mathcal{F}}\mathcal{J}_t(S_{\mathbb{F}}(\Phi))\} = \sup\{E|u|^2 : u \in \mathrm{dec}_{\mathcal{F}}\mathcal{J}_t(S_{\mathbb{F}}(\Phi))\}$. Therefore, $\int_0^t \Phi_\tau dB_\tau$ is integrably bounded if and only if $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_t(S_{\mathbb{F}}(\Phi))$ is a bounded subset of the space $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^d)$.

The idea of the proof of the main result of the paper is based on the following properties of measurable and integrably bounded multifunctions. For a given square integrably bounded $\mathbb{F}$-nonanticipative set-valued multifunction $G : [0,T] \times \Omega \to \mathrm{Cl}(\mathbb{R}^{d\times m})$ such that $G_t(\omega) = \mathrm{cl}\{g_t^n(\omega) : n \geq 1\}$ for $(t,\omega) \in [0,T] \times \Omega$ we have that $S_{\mathbb{F}}(G) = \overline{\mathrm{dec}}_{\Sigma_{\mathbb{F}}}\{g^n : n \geq 1\}$ (see [8], Remark 3.6 of Chap. 2). Hence, it follows that for every arbitrarily taken $f, g \in \{g^n : n \geq 1\}$ and a multifunction $\Phi : [0,T] \times \Omega \to \mathrm{Cl}(\mathbb{R}^{d\times m})$ defined by $\Phi_t(\omega) = \{f_t(\omega), g_t(\omega)\}$ for $(t,\omega) \in [0,T] \times \Omega$, we have $S_{\mathbb{F}}(\Phi) = \overline{\mathrm{dec}}_{\Sigma_{\mathbb{F}}}\{f, g\} \subset \overline{\mathrm{dec}}_{\Sigma_{\mathbb{F}}}\{g^n : n \geq 1\} = S_{\mathbb{F}}(G)$. Then $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T(S_{\mathbb{F}}(\Phi)) \subset \mathrm{dec}_{\mathcal{F}}\mathcal{J}_T(S_{\mathbb{F}}(G))$. Therefore, for the proof that $\sup\{E|u|^2 : u \in \mathrm{dec}_{\mathcal{F}}\mathcal{J}_T(S_{\mathbb{F}}(G))\} = \infty$ it is enough only to verify that there are $f, g \in \{g^n : n \geq 1\}$ such that $\sup\{E|u|^2 : u \in \mathrm{dec}_{\mathcal{F}}\mathcal{J}(S_{\mathbb{F}}(\Phi))\} = \infty$.

2. BOUNDEDNESS OF $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T(h \cdot \mathcal{K}_{\mathbb{F}})$ FOR MATRIX-VALUED PROCESSES

We shall consider here properties of the set $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T(h \cdot \mathcal{K}_{\mathbb{F}})$ with $\mathcal{K}_{\mathbb{F}}$ defined above and $h \in \mathbb{L}^2([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$, where $\mathcal{J}_T(h \cdot \mathcal{K}_{\mathbb{F}})$ is defined by vector valued Itô integral of $d \times m$-matrix processes with respect to an m-dimensional $\mathbb{F}$-Brownian motion $B = (B^1, \ldots, B^m)$. Let us recall that for $h \in \mathbb{L}^2([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ and $h^{ij} \in \mathbb{L}^2([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R})$ such that $h_t(\omega) = (h_t^{ij}(\omega))_{d\times m}$ for every $(t,\omega) \in [0,T] \times \Omega$, the norm $\|h\|$ is defined by $\|h\|^2 = E\int_0^T |h_t|^2 dt$, where $|h_t|^2 = \sum_{i=1}^d \sum_{j=1}^m |h_t^{ij}|^2$. By $\Pi(\Omega, \mathcal{F})$ we denote the family of all finite $\mathcal{F}$-measurable partitions of Ω.

We begin with the following lemma.

Lemma 3. *For every $h \in \mathbb{L}^2([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ a set $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T(h \cdot \mathcal{K}_{\mathbb{F}})$ is a bounded subset of $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^d)$ if and only if $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j} \cdot \mathcal{K}_{\mathbb{F}})$ is a bounded subset of $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R})$ for every $i = 1, \ldots, d$ and $j = 1, \ldots, m$, where $h^{ij} \in \mathbb{L}^2([0,T] \times$*

$\Omega, \Sigma_{\mathbb{F}}, \mathbb{R})$ *are such that* $h_t(\omega) = (h_t^{ij}(\omega))_{d\times m}$ *for every* $(t,\omega) \in [0,T]\times\Omega$ *and* $i = 1,\ldots,d$ *and* $j = 1,\ldots,m$.

Proof. Let us observe that for every $D \in \Sigma_{\mathbb{F}}$ one has

$$\mathcal{J}_T(\mathbb{I}_D h) = \left(\sum_{j=1}^m \mathcal{J}_T^j(\mathbb{I}_D h^{1,j}),\ldots,\sum_{j=1}^m \mathcal{J}_T^j(\mathbb{I}_D h^{d,j})\right)^*,$$

where $\mathcal{J}_T^j(\mathbb{I}_D h^{i,j}) = \int_0^T \mathbb{I}_D h_t^{i,j} dB_t^j$ for every $D \in \Sigma_{\mathbb{F}}$, $i = 1,\ldots,d$ and $j = 1, \ldots, m$, and u^* denotes the transpose of a matrix $u \in \mathbb{R}^{1\times d}$. Therefore, we have

$$\sup\{E|u|^2 : u \in \operatorname{dec}_{\mathcal{F}}\mathcal{J}_T(h\cdot\mathcal{K}_{\mathbb{F}})\} = \sup\{E|u|^2 : u \in \operatorname{dec}_{\mathcal{F}}\{\mathcal{J}_T(\mathbb{I}_D h) : D\in\Sigma_{\mathbb{F}}\}$$
$$= \sup_{N\ge1}\sup\left\{\sum_{i=1}^d E\sum_{k=1}^N \mathbb{I}_{A_k}\left|\sum_{j=1}^m \mathcal{J}_T^j(\mathbb{I}_{D_k}h^{i,j})\right|^2 : (A_k)_{k=1}^N \in \Pi(\Omega,\mathcal{F}), (D_k)_{k=1}^N\subset\Sigma_{\mathbb{F}}\right\}.$$

Hence it follows that if $\operatorname{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}})$ is a bounded subset of $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R})$ for every $i = 1,\ldots,d$ and $j = 1,\ldots,m$ then there is a positive random variable $\varphi \in \mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R})$ such that $|\mathcal{J}_T^j(\mathbb{I}_{D_k}h^{i,j})| \le \varphi$ a.s. for $(i,j) \in \{1,\ldots,d\}\times\{1,\ldots,m\}$, $N \ge 1$, every $k = 1,\ldots,N$ and every family $(D_k)_{k=1}^N \subset \Sigma_{\mathbb{F}}$. Thus $\sup\{E|u|^2 : u \in \operatorname{dec}_{\mathcal{F}}\mathcal{J}_T(h\cdot\mathcal{K}_{\mathbb{F}})\} \le m^2 d\,\|\varphi\|^2 < \infty$.

Suppose a set $\operatorname{dec}_{\mathcal{F}}\mathcal{J}_T(h\cdot\mathcal{K}_{\mathbb{F}})$ is a bounded subset of $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R}^d)$ and there is $I\times J \subset \{1,\ldots,d\}\times\{1,\ldots,m\}$, such that a set $\operatorname{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}})$ is unbounded for every $(i,j)\in I\times J$ and is bounded for $(i,j)\in\{1,\ldots,d\}\times\{1,\ldots,m\}\setminus(I\times J)$. But boundedness of $\operatorname{dec}_{\mathcal{F}}\mathcal{J}_T(h\cdot\mathcal{K}_{\mathbb{F}})$ implies that $\sum_{j=1}^m \operatorname{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\mathcal{K}_{\mathbb{F}})$ is bounded for every $i = 1,\ldots,d$, which implies that $\sum_{i=1}^d\sum_{j=1}^m \operatorname{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}})$ is bounded. Indeed, suppose $\operatorname{dec}_{\mathcal{F}}\mathcal{J}_T(h\cdot\mathcal{K}_{\mathbb{F}})$ is bounded and there is $\bar{i}\in\{1,\ldots,d\}$ such that we have $\sup\{E|u|^2 : u\in\operatorname{dec}_{\mathcal{F}}\{\sum_{j=1}^m \mathcal{J}_T^j(\mathbb{I}_D h^{\bar{i},j}) : D\in\Sigma_{\mathbb{F}}\} = \infty$. But

$$\sup\left\{E|u|^2 : u\in\operatorname{dec}_{\mathcal{F}}\left\{\sum_{j=1}^m \mathcal{J}_T^j(\mathbb{I}_D h^{\bar{i},j}) : D\in\Sigma_{\mathbb{F}}\right\}\right\}$$
$$= \sup_{N\ge1}\sup\left\{E\sum_{k=1}^N \mathbb{I}_{A_k}\left|\sum_{j=1}^m \mathcal{J}_T^j(\mathbb{I}_{D_k}h^{\bar{i},j})\right|^2 : (A_k)_{k=1}^N\in\Pi(\Omega,\mathcal{F}), (D_k)_{k=1}^N\subset\Sigma_{\mathbb{F}}\right\}$$
$$\le \sup_{N\ge1}\sup\left\{\sum_{i=1}^d E\sum_{k=1}^N \mathbb{I}_{A_k}\left|\sum_{j=1}^m \mathcal{J}_T^j(\mathbb{I}_{D_k}h^{i,j})\right|^2 : (A_k)_{k=1}^N\in\Pi(\Omega,\mathcal{F}), (D_k)_{k=1}^N\subset\Sigma_{\mathbb{F}}\right\}$$
$$= \sup\{E|u|^2 : u\in\operatorname{dec}_{\mathcal{F}}\mathcal{J}_T(h\cdot\mathcal{K}_{\mathbb{F}})\}.$$

Then $\sup\{E|u|^2 : u\in\operatorname{dec}_{\mathcal{F}}\mathcal{J}_T(h\cdot\mathcal{K}_{\mathbb{F}})\} = \infty$. A contradiction. Thus $\operatorname{dec}_{\mathcal{F}}\{\sum_{j=1}^m \mathcal{J}_T^j(\mathbb{I}_D h^{i,j}) : D\in\Sigma_{\mathbb{F}}\} = \sum_{j=1}^m \operatorname{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}})$ is bounded for every $i =$

$1,\dots,d$, which implies that a set $\sum_{i=1}^{d}\sum_{j=1}^{m}\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}})$ is bounded. Let us observe now that

$$\left|\left|\left|\sum_{(i,j)\in I\times J}\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}})\right|\right|\right|^2 \tag{1}$$

$$\le 2\left|\left|\left|\sum_{(i,j)\in(I\times J)^{\sim}}\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}})\right|\right|\right|^2+2\left|\left|\left|\sum_{i=1}^{d}\sum_{j=1}^{m}\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}})\right|\right|\right|^2,$$

where $(I\times J)^{\sim}=\{1,\dots,d\}\times\{1,\dots,m\}\setminus(I\times J)$ and $|||\Lambda|||=\sup\{\|u\|:u\in\Lambda\}$ for $\Lambda\subset\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R})$. Indeed, for every $u\in\sum_{i=1}^{d}\sum_{j=1}^{m}\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}})$ and every $(i,j)\in\{1,\dots,d\}\times\{1,\dots,m\}$ there is $u_{i,j}\in\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}})$ such that $u=\sum_{i=1}^{d}\sum_{j=1}^{m}u_{i,j}$. But $\sum_{i=1}^{d}\sum_{j=1}^{m}u_{i,j}=\sum_{(i,j)\in I\times J}u_{i,j}+\sum_{(i,j)\in(I\times J)^{\sim}}u_{i,j}$. Therefore, $\sum_{(i,j)\in I\times J}u_{i,j}=\sum_{i=1}^{d}\sum_{j=1}^{m}u_{i,j}-\sum_{(i,j)\in(I\times J)^{\sim}}u_{i,j}$, which implies that

$$\sum_{(i,j)\in I\times J}u_{i,j}\in\sum_{i=1}^{d}\sum_{j=1}^{m}\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}})+(-1)\cdot\sum_{(i,j)\in(I\times J)^{\sim}}\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}}).$$

Then for every $v=\sum_{(i,j)\in I\times J}u_{i,j}\in\sum_{(i,j)\in(I\times J)}\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}})$ one has

$$\|v\|^2\le 2\sup\left\{\|u\|^2:u\in\sum_{i=1}^{d}\sum_{j=1}^{m}\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}})\right\}$$
$$+2\sup\left\{\|u\|^2:u\in(-1)\cdot\sum_{(i,j)\in(I\times J)^{\sim}}\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}})\right\}.$$

But $\sup\{\|u\|^2:u\in(-1)\cdot\sum_{(i,j)\in(I\times J)^{\sim}}\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}})\}=\sup\{\|u\|^2:u\in\sum_{(i,j)\in(I\times J)^{\sim}}\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}})\}$. Therefore, from the above inequality, the inequality (1) follows, which implies that $\sum_{(i,j)\in I\times J}\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}})$ is a bounded subset of the space $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R})$. A contradiction. Then boundedness of a set $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T(h\cdot\mathcal{K}_{\mathbb{F}})$ implies that $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^j(h^{i,j}\cdot\mathcal{K}_{\mathbb{F}})$ is a bounded subset of $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R})$ for every $i=1,\dots,d$ and $j=1,\dots,m$. ■

Corollary 1. *For every matrix-valued process* $h=(h^{ij})_{d\times m}\in\mathbb{L}^2([0,T]\times\Omega,\Sigma_{\mathbb{F}},\mathbb{R}^{d\times m})$ *a set* $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T(h\cdot\mathcal{K}_{\mathbb{F}})$ *is an unbounded subset of the space* $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R}^d)$ *if there exist* $(\bar{i},\bar{j})\in\{1,\dots,d\}\times\{1,\dots,m\}$ *and a set* $C\in\Sigma_{\mathbb{F}}$ *of positive measure* $\mu=dt\times P$ *such that* $|(h_t^{\bar{i}\bar{j}}(\omega)|>0$ *for* $(t,\omega)\in C$, $h_t^{\bar{i}\bar{j}}(\omega)=0$ *for* $(t,\omega)\in C^{\sim}$ *and* $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^{\bar{j}}(\mathbb{I}_C h^{\bar{i}\bar{j}}\cdot\mathcal{K}_{\mathbb{F}})$ *is an unbounded subset of the space* $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R})$, *where* $C^{\sim}=[0,T]\times\Omega\setminus C$.

Proof. Immediately from Lemma 3 it follows that a set $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T(h\cdot\mathcal{K}_{\mathbb{F}})$ is unbounded if there exists a pair $(\bar{i},\bar{j})\in\{1,\ldots,d\}\times\{1,\ldots,m\}$ such that a set $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^{\bar{j}}(h^{\bar{i}\bar{j}}\cdot\mathcal{K}_{\mathbb{F}})$ is an unbounded subset of the space $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R})$. It is clear that a set $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^{\bar{j}}(\mathbb{I}_D h^{\bar{i}\bar{j}}\cdot\mathcal{K}_{\mathbb{F}})=\{0\}$ for every $D\in\Sigma_{\mathbb{F}}$ such that $|(h_t^{\bar{i}\bar{j}}(\omega)|=0$ for a.e. $(t,\omega)\in D$ and $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^{\bar{j}}(h^{\bar{i}\bar{j}}\cdot\mathcal{K}_{\mathbb{F}})=\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^{\bar{j}}(\mathbb{I}_{D^\sim}h^{\bar{i}\bar{j}}\cdot\mathcal{K}_{\mathbb{F}})$, where $D^\sim=[0,T]\times\Omega\setminus D$. Then by unboundedness of a set $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^{\bar{j}}(h^{\bar{i}\bar{j}}\cdot\mathcal{K}_{\mathbb{F}})$ there is a set $C\in\Sigma_{\mathbb{F}}$ of positive measure $\mu=dt\times P$ such that $|(h_t^{\bar{i}\bar{j}}(\omega)|>0$ for $(t,\omega)\in C$, $h_t^{\bar{i}\bar{j}}(\omega)=0$ for a.e. $(t,\omega)\in C^\sim$ and $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T^{\bar{j}}(\mathbb{I}_C h^{\bar{i}\bar{j}}\cdot\mathcal{K}_{\mathbb{F}})$ is an unbounded subset of the space $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R})$. ∎

3. UNBOUNDEDNESS OF ITÔ SET-VALUED STOCHASTIC INTEGRALS

Let $B=(B^1,\ldots,B^m)$ be an m-dimensional $\mathbb{F}$-Brownian motion defined on $\mathcal{P}_{\mathbb{F}}$ and $G=(G_t)_{0\le t\le T}$ be an $\mathbb{F}$-nonanticipative square integrably bounded set-valued stochastic process with values in the space $\mathrm{Cl}(\mathbb{R}^{d\times m})$ of all nonempty closed subsets of the space $\mathbb{R}^{d\times m}$. We will show that if G possesses an $\mathbb{F}$-nonanticipative Castaing's representation $(g^n)_{n=1}^\infty$ such that there are $\alpha>0$ and $f,g\in\{g^n:n\ge1\}$ such that there exist $(\bar{i},\bar{j})\in\{1,\ldots,d\}\times\{1,\ldots,m\}$ and real-valued processes $f^{\bar{i}\bar{j}}$ and $g^{\bar{i}\bar{j}}$, elements of matrix-valued processes f and g, respectively such that $|f_t^{\bar{i}\bar{j}}(\omega)-g_t^{\bar{i}\bar{j}}(\omega)|\ge\alpha$ for a.e. $(t,\omega)\in[0,T]\times\Omega$ then a set-valued stochastic integral $\int_0^T G_t dB_t$ is not integrably bounded.

We begin with the following lemmas.

Lemma 4. *Let $G=(G_t)_{0\le t\le T}$ be an $\mathbb{F}$-nonanticipative square integrably bounded set-valued stochastic process with values in the space $\mathrm{Cl}(\mathbb{R}^{d\times m})$ possessing an $\mathbb{F}$-nonanticipative Castaing's representation $(g^n)_{n=1}^\infty$ such that there are $\alpha>0$ and $f,g\in\{g^n:n\ge1\}$ such that there exist $(\bar{i},\bar{j})\in\{1,\ldots,d\}\times\{1,\ldots,m\}$ and real-valued processes $f^{\bar{i}\bar{j}}$ and $g^{\bar{i}\bar{j}}$, elements of matrix-valued processes f and g, respectively and such that $|f_t^{\bar{i}\bar{j}}(\omega)-g_t^{\bar{i}\bar{j}}(\omega)|\ge\alpha$ for a.e. $(t,\omega)\in[0,T]\times\Omega$. There are an $\mathbb{F}$-nonanticipative Castaing's representation $(\widetilde{g}^n)_{n=1}^\infty$ of G and matrix-valued processes $\widetilde{f},\widetilde{g}\in\{\widetilde{g}^n:n\ge1\}$ possessing elements $\widetilde{f}^{\bar{i}\bar{j}}=(\widetilde{f}_t^{\bar{i}\bar{j}})_{0\le t\le T}$ and $\widetilde{g}^{\bar{i}\bar{j}}=(\widetilde{g}_t^{\bar{i}\bar{j}})_{0\le t\le T}$, respectively and such that $\widetilde{f}_t^{\bar{i}\bar{j}}(\omega)-\widetilde{g}_t^{\bar{i}\bar{j}}(\omega)\ge\alpha$ for a.e. $(t,\omega)\in[0,T]\times\Omega$.*

Proof. Let a set-valued process G, a Castaing's representation $(g^n)_{n=1}^\infty$ of G, and $f,g\in\{g^n:n\ge1\}$ be such as above. For simplicity assume that $|f_t^{\bar{i}\bar{j}}(\omega)-g_t^{\bar{i}\bar{j}}(\omega)|\ge\alpha$ is satisfied for every $(t,\omega)\in[0,T]\times\Omega$ and let us denote processes $f^{\bar{i}\bar{j}}$ and $g^{\bar{i}\bar{j}}$ by ϕ, ψ, respectively. We have $\phi_t(\omega)\ne\psi_t(\omega)$ for every $(t,\omega)\in$

$[0,T]\times\Omega$. Let $A = \{(t,\omega)\in[0,T]\times\Omega : \phi_t(\omega) > \psi_t(\omega)\}$. If $A = [0,T]\times\Omega$ then $f_t^{\bar{i}\,\bar{j}}(\omega) - g_t^{\bar{i}\,\bar{j}}(\omega) = |f_t^{\bar{i}\,\bar{j}}(\omega) - g_t^{\bar{i}\,\bar{j}}(\omega)| \geq \alpha$ for every $(t,\omega)\in[0,T]\times\Omega$.

Suppose $0 < \mu(A^{\sim}) < T$, where $A^{\sim} = ([0,T]\times\Omega)\setminus A$, let $\widetilde{\phi} = 1\!\!\mathrm{I}_A\phi + 1\!\!\mathrm{I}_{A^{\sim}}\psi$ and $\widetilde{\psi} = 1\!\!\mathrm{I}_A\psi + 1\!\!\mathrm{I}_{A^{\sim}}\phi$. It is clear that $\{\phi_t(\omega),\psi_t(\omega)\} = \{\widetilde{\phi}_t(\omega),\widetilde{\psi}_t(\omega)\}$ for every $(t,\omega)\in[0,T]\times\Omega$. Furthermore, for every $(t,\omega)\in A$ we have $\widetilde{\phi}_t(\omega) - \widetilde{\psi}_t(\omega) = \phi_t(\omega) - \psi_t(\omega) = |\phi_t(\omega) - \psi_t(\omega)| \geq \alpha$. Similarly for $(t,\omega)\in A^{\sim}$ we get $\widetilde{\phi}_t(\omega) - \widetilde{\psi}_t(\omega) = \psi_t(\omega) - \phi_t(\omega) = -(\phi_t(\omega) - \psi_t(\omega)) = |\phi_t(\omega) - \psi_t(\omega))| \geq \alpha$. Taking $\widetilde{f}^{\bar{i}\,\bar{j}} = \widetilde{\phi}$ and $\widetilde{g}^{\bar{i},\bar{j}} = \widetilde{\psi}$ we obtain $\widetilde{f}_t^{\bar{i}\,\bar{j}}(\omega) - \widetilde{g}_t^{\bar{i}\,\bar{j}}(\omega) \geq \alpha$ for every $(t,\omega)\in[0,T]\times\Omega$. To get a required new Castaing's representation of G we can change in the given above Castaing's representation $(g^n)_{n=1}^{\infty}$ its elements f and g by new matrix-valued functions $\widetilde{f}$ and $\widetilde{g}$ obtained from f and g by changing in matrices f and g theirs elements $f^{\bar{i}\,\bar{j}}$ and $g^{\bar{i}\,\bar{j}}$ by $\widetilde{f}^{\bar{i}\,\bar{j}}$ and $\widetilde{g}^{\bar{i}\,\bar{j}}$, respectively. ■

Lemma 5. *For every $f,g\in \mathbb{L}^2([0,T]\times\Omega,\Sigma_{\mathbb{F}},\mathbb{R}^{d\times m})$ a set-valued stochastic integral $\int_0^T F_t dB_t$ of a multiprocess F defined by $F_t(\omega) = \{f_t(\omega), g_t(\omega)\}$ for $(t,\omega)\in[0,T]\times\Omega$, is square integrably bounded if and only if $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T[(f-g)\cdot\mathcal{K}_{\mathbb{F}}]$ is a bounded subset of the space $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R}^d)$.*

Proof. Let us observe that a set-valued stochastic integral $\int_0^T F_t dB_t$ is square integrably bounded if and only if a set $\overline{\mathrm{dec}}_{\mathcal{F}}\mathcal{J}_T(S_{\mathbb{F}}(F))$ is a bounded subset of the space $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R}^d)$. By ([8], Remark 3.6 of Chap. 2) a set $S_{\mathbb{F}}(F)$ is defined by $S_{\mathbb{F}}(F) = \overline{\mathrm{dec}}_{\Sigma_{\mathbb{F}}}\{f,g\}$. But $\mathrm{dec}_{\Sigma_{\mathbb{F}}}\{f,g\} = \{1\!\!\mathrm{I}_D(f-g)+g : D\in\Sigma_{\mathbb{F}}\} = \{1\!\!\mathrm{I}_D(f-g) : D\in\Sigma_{\mathbb{F}}\}+g = (f-g)\cdot\mathcal{K}_{\mathbb{F}}+g$. Then $\overline{\mathrm{dec}}_{\mathcal{F}}\mathcal{J}_T(S_{\mathbb{F}}(F)) = \overline{\mathrm{dec}}_{\mathcal{F}}\mathcal{J}_T[(f-g)\cdot\mathcal{K}_{\mathbb{F}}] + \mathcal{J}_T(g)$. Thus $\overline{\mathrm{dec}}_{\mathcal{F}}\mathcal{J}_T(S_{\mathbb{F}}(F))$ is bounded if and only if $\overline{\mathrm{dec}}_{\mathcal{F}}\mathcal{J}_T[(f-g)\cdot\mathcal{K}_{\mathbb{F}}]$ is a bounded subset of the space $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R})$. Boundedness of $\overline{\mathrm{dec}}_{\mathcal{F}}\mathcal{J}_T[(f-g)\cdot\mathcal{K}_{\mathbb{F}}]$ is equivalent to boundedness of $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T[(f-g)\cdot\mathcal{K}_{\mathbb{F}}]$. Then a set-valued stochastic integral $\int_0^T F_t dB_t$ is square integrably bounded if and only if $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T[(f-g)\cdot\mathcal{K}_{\mathbb{F}}]$ is a bounded subset of the space $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R}^d)$. ■

Now we prove the main result of the paper.

Theorem 6. *Let $G = (G_t)_{0\leq t\leq T}$ be an $\mathbb{F}$-nonanticipative square integrably bounded set-valued stochastic process with values in the space $\mathrm{Cl}(\mathbb{R}^{d\times m})$ possessing an $\mathbb{F}$-nonanticipative Castaing's representation $(g^n)_{n=1}^{\infty}$ such that there are $\alpha > 0$ and $f,g\in\{g^n : n\geq 1\}$ such that there are $(\bar{i},\bar{j})\in\{1,\ldots,d\}\times\{1,\ldots,m\}$ and real-valued processes $f^{\bar{i}\,\bar{j}}$, $g^{\bar{i}\,\bar{j}}$, elements of matrix-valued processes f and g, respectively and such that $|f_t^{\bar{i}\,\bar{j}}(\omega) - g_t^{\bar{i}\,\bar{j}}(\omega)| \geq \alpha$ for a.e. $(t,\omega)\in[0,T]\times\Omega$. A set-valued stochastic integral $\int_0^T G_t dB_t$ is not integrably bounded.*

Proof. Let $G = (G_t)_{0\leq t\leq T}$ and $\mathbb{F}$-nonanticipative Castaing's representation $(g^n)_{n=1}^{\infty}$ of G possess properties described above. By virtue of Lemma 4 there is an $\mathbb{F}$-nonanticipative Castaing's representation $(\widetilde{g}^n)_{n=1}^{\infty}$ of G containing processes $\widetilde{f}, \widetilde{g} \in \{\widetilde{g}^n : n \geq 1\}$ and such that there exist real-valued stochastic processes $\widetilde{f}^{\bar{i}\bar{j}} = (\widetilde{f}_t^{\bar{i}\bar{j}})_{0\leq t\leq T}$, $\widetilde{g}^{\bar{i}\bar{j}} = (\widetilde{g}_t^{\bar{i}\bar{j}})_{0\leq t\leq T}$, elements of matrix-valued prosesses $\widetilde{f}, \widetilde{g}$, respectively and such that $\widetilde{f}_t^{\bar{i}\bar{j}}(\omega) - \widetilde{g}_t^{\bar{i}\bar{j}}(\omega) \geq \alpha$ for a.e. $(t, \omega) \in [0, T]$. By virtue of Lemma 1 it follows that $\mathrm{dec}_{\mathcal{F}}\mathcal{J}^{\bar{j}}[(\widetilde{f}^{\bar{i}\bar{j}} - \widetilde{g}^{\bar{i}\bar{j}}) \cdot \mathcal{K}_{\mathbb{F}}]$ is an unbounded subset of $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R})$. Hence, by Lemma 3 it follows that $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T[(\widetilde{f} - \widetilde{g}) \cdot \mathcal{K}_{\mathbb{F}}]$ is an unbounded subset of $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R})$, which by Lemma 5 implies that a set-valued stochastic integral $\int_0^T F_t dB_t$ of a multiprocess F defined by $F_t(\omega) = \{f_t(\omega), g_t(\omega)\}$ for every $(t, \omega) \in [0, T] \times \Omega$ is not integrably bounded. But $\int_0^T F_t dB_t \subset \int_0^T G_t dB_t$ a.s. Therefore, a set-valued stochastic integral $\int_0^T G_t dB_t$ is not integrably bounded. ■

It is natural to expect that a set-valued stochastic integral $\int_0^T G_t dB_t$ is integrably bounded if and only if G possesses an $\mathbb{F}$-nonanticipative Castaing's representation containing only one element. Such result can be obtained if the following hypothesis would be satisfied.

Hypothesis B. *For every set $C \in \Sigma_{\mathbb{F}}$ of positive measure $\mu = dt \times P$ a set $\mathrm{dec}_{\mathcal{F}}\mathcal{J}_T(\mathbb{I}_C \cdot \mathcal{K}_{\mathbb{F}})$ is an unbounded subset of the space $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R})$.*

To obtain such result we begin with the following lemma.

Lemma 7. *Let $h \in \mathbb{L}^2([0, T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R})$ be a non-negative process such that there exists a set $C \in \Sigma_{\mathbb{F}}$ of positive measure $\mu = dt \times P$ such that $h_t(\omega) > 0$ for $(t, \omega) \in C$ and $h_t(\omega) = 0$ for $(t, \omega) \in C^{\sim}$, where $C^{\sim} = ([0, T] \times \Omega) \setminus C$. For every $\varepsilon \in (0, \mu(C))$ there is an $\Sigma_{\mathbb{F}}$-measurable set $C_\varepsilon \subset C$ of positive measure μ and a real number $\alpha_\varepsilon > 0$ such that $h_t(\omega) \geq \alpha_\varepsilon$ for $(t, \omega) \in C_\varepsilon$.*

Proof. Let $C \in \Sigma_{\mathbb{F}}$ be a set of positive measure $\mu = dt \times P$ such that $h_t(\omega) > 0$ for $(t, \omega) \in C$ and $h_t(\omega) = 0$ for $(t, \omega) \in C^{\sim}$. We have $C = \{(t, \omega) \in [0, T \times \Omega : h_t(\omega) > 0\}$. Let $C_m = \{(t, \omega) \in C : h_t(\omega) \geq m\}$ for every $m > 0$. We have $C = \bigcup_{m>0} C_m$ and $C_m \subset C_n$ for $m \geq n$. Put $\widetilde{C}_k = C_{m_k}$, where $m_k = 1/k$. We have $C = \bigcup_{k=1}^{\infty} \widetilde{C}_k$ and $\widetilde{C}_k \subset \widetilde{C}_{k+1}$ for $k \geq 1$. Therefore, $\mu(C) = \lim_{k\to\infty} \mu(\widetilde{C}_k)$. Then for every $\varepsilon \in (0, \mu(C))$ there is k_ε such that $\mu(C) - \mu(\widetilde{C}_{k_\varepsilon}) < \varepsilon$. Thus $\mu(\widetilde{C}_{k_\varepsilon}) > \mu(C) - \varepsilon > 0$. Let $C_\varepsilon = \widetilde{C}_{k_\varepsilon}$ and $\alpha_\varepsilon = 1/k_\varepsilon$. By the definition of a set $\widetilde{C}_{k_\varepsilon}$ we get $1/k_\varepsilon \leq h_t(\omega)$ for $(t, \omega) \in C_\varepsilon$. Then $h_t(\omega) \geq \alpha_\varepsilon$ for $(t, \omega) \in C_\varepsilon$. ■

We can prove now the following result.

Theorem 8. *If the Hypothesis B is satisfied then for every square integrably bounded $\mathbb{F}$-nonanticipative set-valued stochastic process $G = (G_t)_{0\leq t\leq T}$ with values in the space $\mathrm{Cl}(\mathbb{R}^{d\times m})$, a set-valued stochastic integral $\int_0^T G_t dB_t$ is integrably bounded if and only if there is an $\mathbb{F}$-nonanticipative Castaing's representation $(g^n)_{n=1}^{\infty}$ of G such that $\|g^n - g^m\| = 0$ for every $n, m \geq 1$.*

Proof. If $(g^n)_{n=1}^{\infty}$ is an $\mathbb{F}$-nonanticipative Castaing's representation of G such that if $\|g^n - g^m\| = 0$ for every $n, m \geq 1$ then a set-valued stochastic integral $\int_0^T G_t dB_t$ is integrably bounded because in such a case we have $G_t(\omega) = \mathrm{cl}\{g_t(\omega)\}$ for $(t,\omega) \in [0,T] \times \Omega$ with $g \in \mathbb{L}^2([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ and therefore, $\int_0^T G_t dB_t = \int_0^T g_t dB_t$.

Suppose $(g^n)_{n=1}^{\infty}$ is an $\mathbb{F}$-nonanticipative Castaing's representation of G such that there are $f, g \in \{g^n : n \geq 1\}$ such that $\|f - g\| > 0$ and let $C \in \Sigma_{\mathbb{F}}$ be a set of positive measure $\mu = t \times P$ such that $|f_t(\omega) - g_t(\omega)| > 0$ for a.e. $(t,\omega) \in C$ and $|f_t(\omega) - g_t(\omega)| = 0$ for a.e. $(t,\omega) \in C^{\sim}$. Similarly as in the proof of Lemma 4 we can select an $\mathbb{F}$-nonanticipative Castaing's representation $(\widetilde{g}^n)_{n=1}^{\infty}$ of G such that there are $\widetilde{f}, \widetilde{g} \in \{\widetilde{g}^n : n \geq 1\}$ having elements $\widetilde{f}^{i,j}$ and $\widetilde{g}^{i,j}$ such that $\widetilde{f}_t^{i,j}(\omega) - \widetilde{g}_t^{i,j}(\omega) > 0$ for a.e. $(t,\omega) \in C$ and $\widetilde{f}_t^{i,j}(\omega) - \widetilde{g}_t^{i,j}(\omega) = 0$ for a.e. $(t,\omega) \in C^{\sim}$. By virtue of Lemma 7 for $\varepsilon > 0$ there is $\Sigma_{\mathbb{F}}$-measurable set $C_\varepsilon \subset C$ of positive measure $\mu = dt \times P$ and a real number $\alpha_\varepsilon > 0$ such that $\mathbb{I}_{C_\varepsilon} \widetilde{h}^{i,j} \geq \mathbb{I}_{C_\varepsilon} \alpha_\varepsilon$, where $\widetilde{h}^{i,j} = \widetilde{f}^{i,j} - \widetilde{g}^{i,j}$. Let $h^{i,j} = \mathbb{I}_{C_\varepsilon} \widetilde{h}^{i,j} + \mathbb{I}_{C_\varepsilon^{\sim}} \cdot 0$, where $C_\varepsilon^{\sim} = [0,T] \times \Omega \setminus C_\varepsilon$. We have $\mathbb{I}_{C_\varepsilon} h^{i,j} \geq \mathbb{I}_{C_\varepsilon} \alpha_\varepsilon$, $\mathbb{I}_{C_\varepsilon^{\sim}} h^{i,j} = 0$ and $\mathrm{dec}_{\mathcal{F}} \mathcal{J}_T^j(\mathbb{I}_{C_\varepsilon} \cdot \mathcal{K}_{\mathbb{F}})$ is an unbounded subset of the space $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R})$. Therefore, by virtue of Lemma 2 a set $\mathrm{dec}_{\mathcal{F}} \mathcal{J}_T^j(\mathbb{I}_{C_\varepsilon} \widetilde{h}^{i,j} \cdot \mathcal{K}_{\mathbb{F}})$ is an unbounded subset of $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R})$. But $\mathbb{I}_{C_\varepsilon} \widetilde{h}^{i,j} \leq \mathbb{I}_C \widetilde{h}^{i,j}$. Therefore, $[0, \mathbb{I}_{C_\varepsilon} \widetilde{h}^{i,j}] \subset [0, \mathbb{I}_C \widetilde{h}^{i,j}]$, which similarly as in the proof of Lemma 1, implies that $\mathrm{dec}_{\mathcal{F}} \mathcal{J}_T^j(\mathbb{I}_{C_\varepsilon} \widetilde{h}^{i,j} \cdot \mathcal{K}_{\mathbb{F}}) \subset \mathrm{dec}_{\mathcal{F}} \mathcal{J}_T^j(\mathbb{I}_C \widetilde{h}^{i,j} \cdot \mathcal{K}_{\mathbb{F}})$. Therefore, $\mathrm{dec}_{\mathcal{F}} \mathcal{J}_T^j(\mathbb{I}_C \widetilde{h}^{i,j} \cdot \mathcal{K}_{\mathbb{F}})$ is an unbounded subset of the space $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R})$, which by Corollary 1 implies that $\mathrm{dec}_{\mathcal{F}} \mathcal{J}_T^j[(\widetilde{f} - \widetilde{g}) \cdot \mathcal{K}_{\mathbb{F}}]$ is an unbounded subset of the space $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^d)$. Hence by Lemma 5 it folows that a set-valued stochastic integral $\int_0^T F_t dB_t$ of a set-valued process F defined by $F_t(\omega) = \{f_t(\omega), g_t(\omega)\}$ for $(t,\omega) \in [0,T] \times \Omega$ is not square integrably bounded. But $\int_0^T F_t dB_t \subset \int_0^T G_t dB_t$ a.s. Therefore, a set-valued stochastic integral $\int_0^T G_t dB_t$ is not square integrably bounded. ■

References

[1] F. Hiai, *Multivalued stochastic integrals and stochastic inclusions*, Division of Applied Mathematics, Research Institute of Applied Electricity, Sapporo 060 Japan (not published).

[2] F. Hiai and H. Umegaki, *Integrals, conditional expectations, and martingales of multivalued functions*, J. Multivariate Anal. **7** (1977), 149–182. doi:10.1016/0047-259X(77)90037-9

[3] Sh. Hu and N.S. Papageorgiou, Handbook of Multivalued Analysis I (Kluwer Academic Publishers, Dordrecht, London, 1997). doi:10.1007/978-1-4615-6359-4

[4] E.J. Jung and J. H. Kim, *On the set-valued stochastic integrals*, Stoch. Anal. Appl. **21** (2) (2003), 401–418. doi:10.1081/SAP-120019292

[5] M. Kisielewicz, *Set-valued stochastic integrals and stochastic inclusions*, Discuss. Math. Diff. Incl. **15** (1) (1995), 61–74.

[6] M. Kisielewicz, *Set-valued stochastic integrals and stochastic inclusions*, Stoch. Anal. Appl. **15** (5) (1997), 783–800. doi:10.1080/07362999708809507

[7] M. Kisielewicz, *Some properties of set-valued stochastic integrals*, J. Math. Anal. Appl. **388** (2012), 984–995. doi:10.1016/j.jmaa.2011.10.050

[8] M. Kisielewicz, Stochastic Differential Inclusions and Applications (Springer, New York, 2013). doi:10.1007/978-1-4614-6756-4

[9] M. Kisielewicz, *Properties of generalized set-valued stochastic integrals*, Discuss. Math. DICO **34** (1) (2014), 131–147. doi:10.7151/dmdico.1155

[10] M. Kisielewicz and M. Michta, *Integrably bounded set-valued stochastic integrals*, J. Math. Anal. Appl. (submitted to print).

[11] M. Michta, *Remarks on unboundedness of set-valued Itô stochastic integrals*, J. Math. Anal. Appl **424** (2015), 651–663. doi:10.1016/j.jmaa.2014.11.041

Received 14 November 2015