*Discussiones Mathematicae*
*Differential Inclusions, Control and Optimization 35 (2015) 5–23*
*doi:10.7151/dmdico.1170*

# SPACES OF LIPSCHITZ FUNCTIONS ON METRIC SPACES

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*Dedicated to Zbigniew Semadeni, the founder of Categorical Functional Analysis*

**Abstract**

In this paper the universal properties of spaces of Lipschitz functions, defined over metric spaces, are investigated.

**Keywords:** categories of Lipschitz spaces, Saks spaces, base normed spaces.

**2010 Mathematics Subject Classification:** 46M99, 26A16, 46B40.

## 1. Basic properties of Lipschitz functions

Let $(X, d)$ be a *semimetric space,* i.e., a metric space for which the condition $d(x, y) = 0$ does not imply $x = y$. If there is no confusion, we will omit the semimetric and will only write $X$ instead of $(X, d)$. Moreover, to exclude the trivial case we will assume that $d \equiv 0$ implies $\mathrm{card}(X) = 1$. If several semimetric spaces occur, we will write $(X, d_X)$, i.e., take the space as index.

**Definition 1.1.** If $X, Y$ are (semi) metric spaces with (semi) metrics $d_X, d_Y$, a mapping $f : X \longrightarrow Y$ is called Lipschitz iff there exists an $M \geq 0$ such that

$$(L) \qquad d_X(f(x), f(y)) \leq M d_Y(x, y) \quad \text{for all} \;\; x, y \in X.$$

One puts

$$(1) \qquad L(f) := \inf\{M \mid M \geq 0 \;\text{ and }\; d_Y(f(x), f(y)) \leq M d_X(x, y) \;\text{ for all }\; x, y \in X\}.$$

$L(f)$ is called the Lipschitz constant of $f$. If $L(f) \leq 1$, then $f$ is called a contraction.

Special cases of Definition 1.1 are if $Y$ is a normed or a seminormed linear space and $d_Y$ the metric induced by the norm; in the following we will mostly study the case $Y = \mathbb{R}$ or $Y = \mathbb{C}$. If, for a semimetric space $X$, $\widetilde{d_X}$ denotes the equivalence relation on $X$ induced by the semimetric $d_X$ then $X/_{\widetilde{d_X}}$, the set of equivalence classes, carries a canonical structure of a metric space. A Lipschitz mapping $f : X \longrightarrow Y$ between two semimetric spaces induces a Lipschitz mapping $\hat{f} : X/_{\widetilde{d_X}} \longrightarrow Y/_{\widetilde{d_Y}}$ with $L(f) = L(\hat{f})$. Hence, the theory of Lipschitz mappings between semimetric spaces cannot yield more information than the theory of Lipschitz mappings between metric spaces. This is the reason why, in the following, only Lipschitz mappings on metric spaces are investigated.

**Lemma 1.2.** *Let $A \subset X$ be a non-empty subset of a metric space $(X, d)$ and let*

$$\operatorname{dist}_A : X \longrightarrow \mathbb{R} \quad \textit{with} \quad \operatorname{dist}_A(v) = \inf_{x \in A} d(v, x)$$

*be the distance function of $A$. Then:*

- *the distance function is a Lipschitz function with Lipschitz constant $0 \leq L \leq 1$.*
- *if $A = X$ then $L = 0$ and if $A \neq X$ and there exists a $y \in X \setminus A$ which has a closed point to $A$, i.e., there exists a $z \in A$ with $d(y, z) = \operatorname{dist}_A(y)$, then $L = 1$ and $\operatorname{dist}_A$ is an isometry.*

***Proof.*** Let $x, y \in E$ and $\varepsilon > 0$ be given. By the definition of the infimum there exists a point $z \in A$ with $\operatorname{dist}_A(y) \geq d(y, z) - \varepsilon$. We get

$$\begin{aligned} \operatorname{dist}_A(x) &\leq d(x, z) \leq d(x, y) + d(y, z) \\ &\leq d(x, y) + \operatorname{dist}_A(y) + \varepsilon. \end{aligned}$$

By interchanging $x$ and $y$

$$(2) \qquad |\operatorname{dist}_A(x) - \operatorname{dist}_A(y)| \leq d(x, y)$$

follows. Hence, the distance function is a Lipschitz function with Lipschitz constant $L \leq 1$.

If $A = X$ then $\mathrm{dist}_A = 0$. Now assume that there exists an $y \in X \setminus A$ which has a closed point to $A$, i.e., there exists a $z \in A$ with $d(y,z) = \mathrm{dist}_A(y)$, then

$$|\mathrm{dist}_A(z) - \mathrm{dist}_A(y)| = |0 - \mathrm{dist}_A(y)| = d(z,y),$$

which implies $L = 1$ by (2). ■

**Remark 1.3.** *The Lipschitz functions on $X$ separate points:*
More precisely, let $(X,d)$ be a metric space and let $\{x_1, \ldots, x_n\} \subset X$ be a finite subset of pairwise disjoint points, i.e., $x_i \neq x_j$ for $i \neq j$. Then there exist Lipschitz functions $\varphi_i : X \longrightarrow \mathbb{R}$ such that for $i \in \{1, \ldots, n\}$

$$\varphi_i(x_j) = \begin{cases} 1 & : \quad i = j \\ 0 & : \quad i \neq j\,. \end{cases}$$

***Proof.*** For $i \in \{1, \ldots, n\}$ put $A_i := \{x_1, \ldots, x_n\} \setminus \{x_i\}$ and because of Lemma 1.2 define $\varphi_i : X \longrightarrow \mathbb{R}$ by $\varphi_i(x) = \frac{\mathrm{dist}_{A_i}(x)}{\mathrm{dist}_{A_i}(x_i)}$. Then $\varphi_i$ is a Lipschitz function with these properties. ■

## 2. Spaces of Lipschitz functions

In the following let **Lip** denote the category of metric spaces and Lipschitz maps and define

$$\mathbb{L}\mathrm{ip}(X) := \mathbf{Lip}(X, \mathbb{R}), \quad X \in \mathbf{Lip}.$$

Furthermore, we introduce the following subcategories of **Lip** called $\mathbf{Lip}_1$ which is the subcategory defined by all contractions, $\mathbf{Lip}^\infty$ the subcategory generated by all metric spaces $(X,d)$ of finite diameter, i.e., $\mathrm{diam}(X) = \sup\{d(x,y) \mid x, y \in X\} < \infty$ and finally:

$$\mathbf{Lip}_1^\infty = \mathbf{Lip}_1 \cap \mathbf{Lip}^\infty.$$

For a metric space $X$ consider the space $L^\infty(X)$ of all real valued bounded Borel-measurable functions endowed with the supremum norm $\|\square\|_\infty$. This is a Banach space for any metric space $X$. Put for a metric space $X$

$$Lip(X) := \mathbb{L}\mathrm{ip}(X) \cap L^\infty(X)$$

and take as norm

$$\|f\|_L := \max\{L(f), \|f\|_\infty\}.$$

Note that the same definition makes sense for $f : X \longrightarrow E$ in **Lip** and $E \in$ **Vec**, where **Vec** denotes the category of real normed linear spaces and continuous linear maps and $\mathbf{Vec}_1$ the subcategory defined by linear contractions (For more details see [5] and [9]).

$Lip(X)$ carries two norms $\|\Box\|_\infty$ and $\|\Box\|_L$ and we have obviously

$$(*) \qquad \|\Box\|_\infty \le \|\Box\|_L.$$

$\bigcirc_L(Lip(X))$ denotes the closed unit ball with respect to $\|\Box\|_L$ and $\bigcirc_\infty(Lip(X))$ the closed unit ball with respect to $\|\Box\|_\infty$. Obviously

$$\bigcirc_L(Lip(X)) \subset \bigcirc_\infty(Lip(X)).$$

holds. One defines

$$C(Lip(X)) := \{f \ \mid \ f \in Lip(X), \ f(x) \ge 0 \ \text{ for all } \ x \in X\}.$$

$C(Lip(X))$ is a proper, generating cone in $Lip(X)$, i.e., $Lip(X) = C(Lip(X)) - C(Lip(X))$. That $C(Lip(X))$ is proper is trivial. For every $f \in Lip(X)$ and every $x \in X$ one has

$$(**) \qquad -\|f\|_\infty \le f(x) \le \|f\|_\infty,$$

which implies that $C(Lip(X))$ is generating. Furthermore, $C(Lip(X))$ is closed with respect to $\|\Box\|_\infty$ because $C(Lip(X)) = \bigcap_{x\in X} \{f \in Lip(X) \mid f(x) \ge 0\}$ and, moreover, because of $(*)$ also with respect to $\|\Box\|_L$.

In order to avoid mixing up both normed spaces, let

$$Lip_\infty(X) := \big(Lip(X), \|\Box\|_\infty\big) \qquad \text{and} \qquad Lip_L(X) := \big(Lip(X), \|\Box\|_L\big).$$

Moreover, let us point out that the product, as well as the pointwise maximum and minimum of two bounded Lipschitz functions is again a Lipschitz function and let us denote by $1\!\!1 \in Lip(X)$ the constant function $1\!\!1(x) = 1$, for all $x \in X$. $1\!\!1 \in Lip(X)$ is an *order unit* with respect to $\|\Box\|_\infty$ (not $\|\Box\|_L$).

Furthermore, observe that $Lip_\infty(X)$ is in general not a Banach space. To see this, take $X := [0,1]$ the unit interval with $d(x,y) = |x-y|$. Then the function $f : [0,1] \longrightarrow \mathbb{R}$ with $f(x) := \sqrt{x}$ is not Lipschitz, but it is the uniform limit of the Lipschitz functions $f_n : [0,1] \longrightarrow \mathbb{R}$ with $f_n(x) := \min\{nx, \sqrt{x}\}$, because $\sup_{t\in[0,1]} |f_n(t) - f(t)| = \frac{1}{4n^2}$. But $Lip_L(X)$ is a Banach space ([12] 1.6.2 ).

Let us recall some notations: If $C$ is an arbitrary cone of a vector space $E$, then a convex subset $B \subset C$ is called a *base* of $C$ if every $z \in C \setminus \{0\}$ has a unique representation $z = \lambda b$ with $\lambda > 0$ and $b \in B$. Every cone $C \subset E$ induces a *partial order* $\leq$ by $x \leq y$ if and only if $y - x \in C$. A partial order $\leq$ is called *archimedian* if, for some $y \geq 0$ and all $\lambda > 0$ $x \leq \lambda y$ implies $x \leq 0$. For $a, b \in E$ let $[a, b] := \{z \in E \mid a \leq z \leq b\}$. Moreover, we use the notation $a \vee b = \max\{a, b\}$ if the maximum exists and correspondingly the minimum $a \wedge b = \min\{a, b\}$, and write $|a|$ for $a \vee (-a)$. A norm $\|\square\|$ for $E$ is called a *Riesz norm* if, for all $a, b \in E$, the inequality $|a| \leq |b|$ implies $\|a\| \leq \|b\|$. An element $e \in C$ is called an *order unit* if for every $z \in E$ there exists a $\lambda > 0$ such that $\lambda e \leq z \leq \lambda e$.

**Remark 2.1.** Let $E$ be a vector space and $C \subset E$ a generating cone.

- If $C$ has an order unit $e \in C$ then the function
$$z \mapsto \|z\| = \inf\{\lambda > 0 \mid -\lambda e \leq z \leq \lambda e\}$$
is a norm for $E$, which is called an *order unit norm.*

- If $C$ has a base $B \subset C$ such that the set $S = \text{conv}\,(B \cup -B)$ is order bounded, then the Minkowski functional
$$p(z) = \inf\{\lambda > 0 \mid z \in \lambda S\}$$
is a norm for $E$. We call this norm $p$ a *base norm* and call $E$ a *base normed* space. The base is denoted by $B = \text{Bs}(E)$ and $C = R_+\text{Bs}(E)$ holds with $R_+ = [0, +\infty)$.

For a real vector space $E$ we denote by $E^*$ the *algebraic dual,* that is the vector space of all linear forms from $E$ to $\mathbb{R}$. If $E$ is endowed with a locally convex Hausdorff linear topology $\tau$ then the pair $(E, \tau)$ is called a *locally convex topological vector space* and we denote by $E'$ its *topological dual*, that is the vector space of all continuous linear forms from $E$ to $\mathbb{R}$.

A *Saks space* is a triple $(E, \|\square\|, \tau)$ where $\|\square\|$ is a norm on the real linear topological space $E$ and $\tau$ is a locally convex Hausdorff linear topology $\tau$ on $E$ such that the unit ball $\bigcirc_{\|\square\|}(E)$ is $\tau$-closed and $\tau$-bounded. For any normed vector space $(E, \|\square\|)$ the triple $(E', \|\square\|', \sigma(E', E))$ is a Saks space, where $\|\square\|'$ is the dual norm and $\sigma(E', E)$ the weak-* topology on $E'$.

**Proposition 2.2.** *For a metric space $X$ the space $Lip_\infty(X) := \big(Lip(X), \|\square\|_\infty\big)$ endowed with the pointwise order of functions is a regular ordered order unit normed space with the closed and generating order cone $C(Lip(X))$ the order unit $\mathbb{1} \in Lip(X)$ and $\bigcirc_\infty(Lip(X)) = [-\mathbb{1}, \mathbb{1}]$. $Lip_\infty(X)$ is in general not complete.*

***Proof.*** The equation $\bigcirc_\infty(Lip(X)) = [-\mathbb{1}, \mathbb{1}]$ follows from $(**)$. The proof of the remaining assertions is straightforward. ∎

**Proposition 2.3.** *$Lip_L(X)$ is a Banach space and the cone $C(Lip(X))$ is generating and $\|\square\|_L$-closed but $\|\square\|_L$ is not a Riesz-norm with respect to $C(Lip(X))$.*

***Proof.*** The completeness of $Lip_L(X)$ is shown in [12], Proposition 1.6.2, and that $C(Lip(X))$ is a proper generating cone was shown above as was the closedness. ■

**Example 2.4.** Define the function

$$f : \mathbb{R} \longrightarrow \mathbb{R} \qquad \text{given by} \quad f(x) := \begin{cases} 2x & : \quad x \in [-\frac{1}{2}, \frac{1}{2}] \\ 1 & : \quad x \geq \frac{1}{2} \\ -1 & : \quad x \leq -\frac{1}{2} \end{cases} ,$$

then $L(f) = 2$ and $\|f\|_\infty = 1$ which implies $\|f\|_L = 2$, i.e., $\bigcirc_L(Lip(\mathbb{R})) \subsetneq \bigcirc_\infty(Lip(\mathbb{R}))$.

Note furthermore, that if $A \subset Lip(X)$ is $\|\square\|_\infty$-closed, then $A$ is also $\|\square\|_L$-closed, i.e., for the topologies,

$$\tau_{\|\square\|_\infty} \subset \tau_{\|\square\|_L},$$

holds which follows from $(*)$.

**Remark 2.5.** For a metric space $X$, $\left(Lip_L(X), \|\square\|_L, \|\square\|_\infty\right)$ is a Saks space with the topology of $\|\square\|_\infty$, i.e., a $2-$normed linear space in the sense of Semadeni [10], who investigated these spaces which led to the introduction of Saks spaces. $C(Lip(X))$ is $\|\square\|_L$-closed, proper and generating but does not make $Lip_L(X)$ a regular ordered Saks space because $\|\square\|_L$ is not a Riesz norm with respect to $C(Lip(X))$ (see [7]).

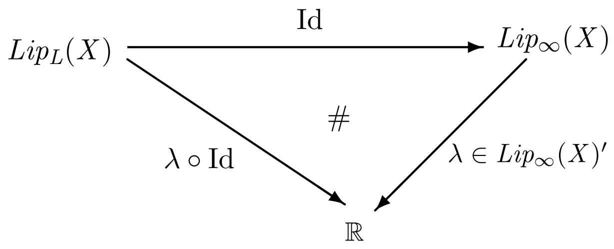
As we have seen, $\|\square\|_\infty \leq \|\square\|_L$ implies $\bigcirc_L(Lip(X)) \subset \bigcirc_\infty(Lip(X))$, i.e., $\bigcirc_L(Lip(X))$ is $\|\square\|_\infty$-bounded and $\bigcirc_\infty(Lip(X))$ is also $\|\square\|_L$-closed.

Additionally it should be noted, for further use, that $C(Lip(X))$ is 1-normal (see [13], Prop. 9.2 (e), p. 86), i.e., $g \leq f \leq h$ implies $\|f\|_\infty \leq \max\{\|g\|_\infty, \|h\|_\infty\}$. Moreover, $Lip_\infty(X)$ is a Stonian vector lattice (see [2] p. 186).

As for any normed linear space $(E, \|\square\|)$, $(E, \|\square\|, \sigma(E, E'))$ is a Saks space, we have the canonical Saks spaces $\left(Lip(X), \|\square\|_\infty, \sigma\left(Lip_\infty(X), Lip_\infty(X)'\right)\right)$ and $\left(Lip(X), \|\square\|_L, \sigma\left(Lip_L(X), Lip_L(X)'\right)\right)$. The first one is even a regularly ordered Saks space because $Lip_\infty(X) \in \mathbf{Vec}_1^+$ (see [7], Example 3.2 iii) ).

## 3. The duals of Lipschitz functions spaces

Now the connection between $Lip_\infty(X)$ and $Lip_L(X)$ will be investigated as well as between their (topological) dual spaces $Lip_\infty(X)'$ and $Lip_L(X)'$ which are both linear subspaces of $Lip(X)^*$. Inequality $(*)$ implies that in the commutative diagram




the identity map is a contraction but not a quasi-isometry( see [12], pp. 3–4).

The dual norm of $Lip_\infty(X)'$ will be denoted by $\|\square\|'_\infty$ and the dual norm of $Lip_L(X)'$ by $\|\square\|'_L$. Now Id induces an injective contraction $\kappa_X : Lip_\infty(X)' \longrightarrow Lip_L(X)'$ with $\kappa_X(\lambda) := \lambda \circ \mathrm{Id}$, which may be considered as an inclusion, hence we often write $\kappa_X(\lambda) := \lambda$.

For $\lambda \in Lip_\infty(X)'$ one has:

$$\begin{aligned} \|\kappa_X(\lambda)\|'_L &= \sup\{|\lambda(f)| \;\mid\; \|f\|_L \le 1\} \\ &\le \sup\{|\lambda(f)| \;\mid\; \|f\|_\infty \le 1\} \quad \text{(because of } (*)) \\ &= \|\lambda\|'_\infty, \end{aligned}$$

i.e., taking $\kappa_X$ as inclusion, then

$$(*') \qquad \|\square\|'_L \le \|\square\|'_\infty.$$

Hence, we have

$$\bigcirc'_\infty(Lip(X) \subset \bigcirc'_L(Lip(X)),$$

i.e., one may regard $Lip_\infty(X)'$ as a subspace of $Lip_L(X)'$.

In the following we will show that in general $Lip_\infty(X)'$ is a proper subspace of $Lip_L(X)'$, $Lip_\infty(X)' \subsetneq Lip_L(X)'$. For this we use the construction of point derivations from D.R. Sherbert [11] (see also [12] Chapter 7) which we briefly outline.

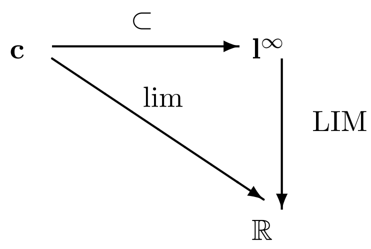
Consider the real Banach space

$$\mathbf{l}^\infty := \{x := (x_n)_{n\in\mathbb{N}} \;\mid\; (x_n)_{n\in\mathbb{N}} \text{ bounded sequence }\}$$

endowed with the supremum norm

$$\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|.$$

Let $\mathbf{c} \subset \mathbf{l}^\infty$ denote the closed subspace of all convergent sequences and let $\lim : \mathbf{c} \longrightarrow \mathbb{R}$ be the continuous linear functional which assigns to every convergent sequence its limit. Consider a norm-preserving Hahn-Banach extension "LIM" of the functional "lim" to $\mathbf{l}^\infty$ :




with the following additional properties:

(i) $\mathrm{LIM}_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} x_{n+1}$,

(ii) $\liminf\limits_{n\to\infty} x_n \leq \mathrm{LIM}_{n\to\infty} x_n \leq \limsup\limits_{n\to\infty} x_n$.

We shall use the the notation $\mathrm{LIM}_{n\to\infty} x_n = \mathrm{LIM}(x)$ for $x := (x_n)_{n\in\mathbb{N}} \in \mathbf{l}^\infty$. These functionals "LIM" are called *translation invariant Banach limits* (see [4], Chapter II.4, Exercise 22).

Now let $(X, d)$ be a metric space and $\Delta := \{(x, x) \in X \times X \mid x \in X\}$ the diagonal of $X \times X$. For a double sequence $w \in ((X \times X) \setminus \Delta)^{\mathbb{N}},\ w := (x_n, y_n)_{n\in\mathbb{N}})$ one defines the sequence

$$T_w(f) := \left\{ \left( \frac{f(y_n) - f(x_n)}{d(y_n, x_n)} \right) \mid n \in \mathbb{N} \right\}, \quad f \in Lip(X).$$

This yields a mapping

$$T_w : Lip(X) \longrightarrow \mathbf{l}^\infty \ \text{ with } \ T_w(f) := \left( \frac{f(y_n) - f(x_n)}{d(y_n, x_n)} \right)_{n\in\mathbb{N}},$$

which satisfies $\|T_w(f)\|_\infty \leq L(f) \leq \|f\|_L$, i.e., a continuous linear map $T_w : Lip_L(X) \longrightarrow \mathbf{l}^\infty$. Hence for any Banach limit LIM the composition $D_w = \mathrm{LIM} \circ T_w$ is a continuous linear functional

$$D_w : Lip_L(X) \longrightarrow \mathbb{R} \ \text{ with } \ D_w(f) = \mathrm{LIM}(T_w(f)),$$

i.e., $D_w \in Lip_L(X)'$.

As the definition of $D_w$ resembles the classical definition of the derivation of functions, one is interested if and under which assumptions $D_w$ is a derivation in the sense of Bourbaki [3] i.e., for $f, g \in Lip(X)$ and $x \in X$,

$$(PR') \qquad D_w(fg) = fD_w(g) + gD_w(f)$$

holds, i.e., in our case, as the left side does depend (directly) on $x \in X$

$$(PR) \qquad D_w(fg) = f(x)D_w(g) + g(x)D_w(f)$$

where the dependence of $D_w(fg)$ on $x \in X$ has to be specified. The answer to this is (see [11], Prop. 8.5)

**Proposition 3.1.** *If $(X, d)$ is a metric space and for $w \in ((X \times X) \setminus \Delta)^{\mathbb{N}}$, $w := (x_n, y_n)_{n\in\mathbb{N}})$ satisfies $x_0 = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$, i.e., $x_0 \in X$ is non-isolated, then $(PR)$ is satisfied and $D_w$ is called a point derivation in $x_0$.*

***Proof.*** If $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \mathbf{l}^\infty$, then, for convergent $(a_n)_{n\in\mathbb{N}}$, with $a = \lim_{n\to\infty} a_n$ one has for any Banach limit LIM

$$\mathrm{LIM}_{n\to\infty}(a_n \cdot b_n - \alpha b_n) = 0,$$

because $|a_n \cdot b_n - \alpha b_n| = |a_n - \alpha||b_n| \le B|a_n - \alpha|$ if $\|(b_n)_{n\in\mathbb{N}}\|_\infty = B$, which implies $\lim_{n\to\infty}(a_n \cdot b_n - \alpha b_n) = \mathrm{LIM}_{n\to\infty}(a_n \cdot b_n - \alpha b_n) = 0$.

Hence

$$\mathrm{LIM}_{n\to\infty}(a_n \cdot b_n) = \alpha \mathrm{LIM}_{n\to\infty} b_n$$

follows. This implies, for $f, g \in Lip(X)$

$$\begin{aligned}
D_w(fg) &= \mathrm{LIM}(T_w(fg)) \\
&= \mathrm{LIM}_{n\to\infty}\left(\frac{(fg)(y_n) - (fg)(x_n)}{d(y_n, x_n)}\right) \\
&= \mathrm{LIM}_{n\to\infty}\left(f(y_n)\frac{g(y_n) - g(x_n)}{d(y_n, x_n)} + g(x_n)\frac{f(y_n) - f(x_n)}{d(y_n, x_n)}\right) \\
&= f(x_0)\mathrm{LIM}_{n\to\infty}\left(\frac{g(y_n) - g(x_n)}{d(y_n, x_n)}\right) + g(x_0)\mathrm{LIM}_{n\to\infty}\left(\frac{f(y_n) - f(x_n)}{d(y_n, x_n)}\right) \\
&= f(x_0)D_w(g) + g(x_0)D_w(f)
\end{aligned}$$

which completes the proof. ∎

In [11], Proposition 8.5, it is shown that for $x_0$ isolated any $D_w \equiv 0$ if $w$ fulfills the condition of Proposition 3.1.

**Proposition 3.2.** *In general, $Lip_\infty(X)'$ is a proper subspace of $Lip_L(X)'$, $Lip_\infty(X)' \subsetneq Lip_L(X)'$.*

***Proof.*** Consider the metric space $X := [0,1]$ with the usual metric $d(x,y) = |x-y|$. Let $D_w \in Lip_L([0,1])'$ be any point derivation for $x = 0$ and define the sequence $\varphi_k$, $k \in \mathbb{N}$, $k > 1$, in $Lip_L(X)$ by

$$\varphi_k(x) := \max\left\{\sqrt{k}x, \frac{\sqrt{k}}{1-k}(x-1)\right\}.$$

Then

$$\max \varphi_k(x) = \varphi_k(\frac{1}{k}) = \frac{1}{\sqrt{k}}.$$

This implies that the sequence $(\varphi_k)_{nk\in\mathbb{N}}$ converges to 0 with respect to $\|\square\|_\infty$. A trivial calculation shows $D_w(\varphi_k) = \sqrt{k}$. Since the constant function 0 has $D_w(0) = 0$ and the sequence $(D_w(\varphi_k))_{k\in\mathbb{N}}$ is unbounded, it follows that $D_w \notin Lip_\infty(X)'$ which completes the proof. ∎

## 4. The space of point functionals

A central role in our investigations plays, for a metric space $X$, $x \in X$, the mapping

$$\delta_X : X \longrightarrow Lip(X)^* \qquad \text{with} \quad \delta_X(x)(f) := f(x), \qquad f \in Lip(X).$$

First note that for every $x \in X$ the linear functional $\delta_X(x) \in Lip(X)^*$ is continuous with respect to both norms $\|\square\|_\infty$ and $\|\square\|_L$. The restriction of $\delta_X$ to $Lip_\infty(X)'$ is denoted by $\delta_X^\infty$ and to $Lip_L(X)'$ by $\delta_X^L$. The upper indicees are omitted if misunderstandings are not possible.

**Proposition 4.1.** *Let $(X,d)$ be a metric space. Then*

(a) *For all $x \in X$, $\|\delta_X^\infty(x)\|'_\infty = \|\delta_X^L(x)\|'_L = 1$ holds.*

(b) *$\delta_X^L : X \longrightarrow Lip_L(X)'$ is a contraction.*

(c) *$\delta_X$ is injective and $\delta_X(X)$ is a linearly independent set.*

***Proof.*** (a): For both norms $\|\Box\|_\infty$ and $\|\Box\|_L$ one has

$$\begin{aligned}\|\delta_X^L(x)(f)\|_L' &= \sup\{|\delta_X(x)| \mid f \in \bigcirc_L(Lip(X))\} \\ &= \sup\{|f(x)| \mid f \in \bigcirc_L(Lip(X))\} \le \sup\{|f(x)| \mid \|f\|_\infty \le 1\} \\ &= \|\delta_X^\infty(x)\|_\infty' \le \max\{\|f\|_\infty, 1\} \le \max\{\|f\|_L, 1\} \le 1.\end{aligned}$$

Hence, as $\|\mathbb{1}(x)\| = 1$, $\|\delta_X(x)\|_L' = \|\delta_X(x)\|_\infty' = 1$ follows.

(b): One has

$$\begin{aligned}\|\delta_X^L(x) - \delta_X^L(y)\|_L' &= \sup\left\{|\delta_X^L(x)(f) - \delta_X^L(y)(f)| \mid \|f\|_L \le 1\right\} \\ &= \sup\{|f(x) - f(y)| \mid \|f\|_L \le 1\} \\ &\le \sup\{\|f\|_L d(x,y) \mid \|f\|_L \le 1\} = d(x,y).\end{aligned}$$

To prove (c) let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ and $x_1, x_2, \ldots, x_n \in X$ be given with $x_i \neq x_k$ for $i \neq k$, and assume that $\sum_{j=1}^n \alpha_j \delta_X(x_j) = 0$ holds. By Remark 1.3 there exist $\varphi_1, \varphi_2, \ldots, \varphi_n \in Lip(X)$ with

$$\varphi_i(x_j) = \delta_{ij} = \begin{cases} 1 & : \quad i = j \\ 0 & : \quad i \neq j \end{cases}$$

Now $\sum_{j=1}^n \alpha_j \delta_X(x_j)(\varphi_i) = 0$ yields $\alpha_i = 0$. ∎

Next we consider the linear subspace generated by the point functionals

$$D(X) := \langle \delta_X \rangle := \left\{ \lambda = \sum_{i=1}^n \alpha_i \delta_X(x_i) \mid x_1, \ldots, x_n \in X,\ \alpha_1, \ldots, \alpha_n \in \mathbb{R}, n \in \mathbb{N} \right\}$$

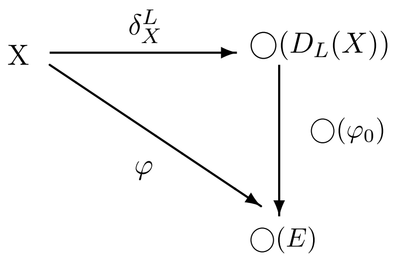
of $Lip(X)^*$.

**Remark 4.2.** By $D_L(X)$ we denote the space $D(X)$ endowed with the dual norm $\|\Box\|_L'$ and by $D_\infty(X)$ we denote the space $D(X)$ endowed with the dual norm $\|\Box\|_\infty'$. The canonical injections are $\delta_X^L$ and $\delta_X^\infty$. At the beginning of Section 2 it was already pointed out, that a normed linear space $E$ and a metric space $X$ with metric $d$, the notions Lipschitz function for a mapping $f : X \longrightarrow E$ and $\|f\|_\infty$ as well $\|f\|_L$ are defined analogously to the case $E = \mathbb{R}$. Hence, for $\delta_X^\infty$ and $\delta_X^L$ one may try to compute these norms. As $\delta_X^\infty$ is not Lipschitz only $\delta_X^L$ remains. For the sake of brevity we denote the $\infty$-norm of $\delta_X^L : X \longrightarrow Lip_L(X)'$ with $\|\delta_X^L\|_\infty$ and get from Proposition 4.1

$$\|\delta_X^L\|_\infty = \sup\left\{ \|\delta_X(x)\|'_L \;\mid\; x \in X \right\} = 1,$$

which implies $\|\delta_X^L\|_L = 1$ for the $L$-norm, as $L\left(\delta_X^L\right) \leq 1$.

**Theorem 4.3.** *Let $X$ be a metric space, $E \in \mathbf{Vec}_1$ with norm $\|\Box\|$ and $\varphi : X \longrightarrow E$, $\varphi \in \mathbf{Lip}$ with $\|\varphi\|_L \leq 1$. Then there exists a unique $\varphi_0 : D_L(X) \longrightarrow E$ in $\mathbf{Vec}_1$ with $\varphi = \bigcirc(\varphi_0)\delta_X^L$ and $\|\varphi_0\| = \|\varphi\|_L$ such that*




*commutes.*

***Proof.*** We may assume that $\varphi \neq 0$ because otherwise the statement is trivially true. The assumption $\|\varphi\|_L \leq 1$ implies $\varphi(X) \subset \bigcirc(E)$, hence we may restrict $\varphi$ to $\bigcirc(E)$ in its image domain. As misunderstandings are not possible we will denote the restriction also by $\varphi$. As $\{\delta_X(x) \mid x \in X\}$ is a basis of $D_L(X)$ the linear mapping $\varphi_0 : D(X) \longrightarrow E$ is well defined by

$$\varphi_0\left(\sum_{i=1}^{n} \alpha_i \delta_X(x_i)\right) := \sum_{i=1}^{n} \alpha_i \varphi(x_i)$$

and satisfies $\varphi = \varphi_0 \circ \delta_X$. A routine calculation shows that $\varphi_0$ is a linear mapping. In order to prove $\|\varphi_0\| = \|\varphi\|_L$, let $\lambda \in E'$ with $\lambda \neq 0$, then

$$\|\lambda \circ \varphi\|_L \;\leq\; \|\lambda\|'\|\varphi\|_L \quad \text{and hence} \quad \frac{\lambda\varphi}{\|\lambda\|'\|\varphi\|_L} \in \bigcirc\left(Lip_L(X)\right).$$

For $\xi = \sum_{i=1}^{n} \alpha_i \delta_X^L(x_i) \in D_L(X)$ there exists a $\lambda_\xi \in E'$ with$\|\lambda_\xi\|' = 1$ and $\lambda_\xi\left(\varphi_0(\xi)\right) = \|\varphi_0(\xi)\|_E$. Now:

$$\begin{aligned}
\|\varphi_0(\xi)\|_E \;&=\; \lambda_\xi\left(\varphi_0\left(\sum_{i=1}^{n} \alpha_i \delta_X^L(x_i)\right)\right) = \lambda_\xi\left(\sum_{i=1}^{n} \alpha_i \varphi(x_i)\right) = \sum_{i=1}^{n} \alpha_i \lambda_\xi \circ \varphi(x_i) \\
&\leq\; \|\varphi\|_L \left|\sum_{i=1}^{n} \alpha_i \frac{\lambda_\xi \circ \varphi}{\|\varphi\|_L}(x_i)\right| \;=\; \|\varphi\|_L \left|\left(\sum_{i=1}^{n} \alpha_i \delta_X^L(x_i)\right)\left(\frac{\lambda_\xi \circ \varphi}{\|\varphi\|_L}\right)\right|
\end{aligned}$$

$$= \|\varphi\|_L \sup\left\{|\xi(f)| = |\sum_{i=1}^{n}\alpha_i f(x_i)| \;\Big|\; f \in \bigcirc_L(Lip(X))\right\} \leq \|\varphi\|_L\|\xi\|_L$$

which implies $\|\varphi_0\| \leq \|\varphi\|_L \leq 1$.

On the other hand, $\|\varphi\|_L = \|\varphi_0 \circ \delta_X^L\|_L \leq \|\varphi_0\|\|\delta_X^L\|_L = \|\varphi_0\|$ (because of Remark 4.2) and hence $\|\varphi\|_L = \|\varphi_0\|$ follows. ■

**Remark 4.4.** For a metric space $X$, Theorem 4.3 means that the mapping $\delta_X^L \longrightarrow \bigcirc(D_L(X))$ is universal with respect to all Lipschitz mappings $\varphi : X \longrightarrow E$, $\|\varphi\|_L \leq 1$ where $E$ is a norned real linear space. This is equivalent to the following statement:

The unit ball functor $\bigcirc$ : $\mathbf{Vec}_1 \longrightarrow \mathbf{Lip}_1$ has $D_L$ : $\mathbf{Lip}_1 \longrightarrow \mathbf{Vec}_1$ as a left adjoint.

***Proof.*** The proof is elementary because $\bigcirc$ maps a contraction in $\mathbf{Vec}_1$ to a contraction in $\mathbf{Lip}_1$ and Theorem 4.3 states that $\delta_X^L \longrightarrow \bigcirc(D_L(X))$ is a canonical universal embedding for a metric space into a normed real linear space. ■

**Proposition 4.5.** *Let $(X, d)$ be a metric space. Then $D_\infty(X)$ is a base normed linear space with base* $\mathrm{Bs}\,(D_\infty(X)) = \mathrm{conv}\,(\{\delta_X(x) \mid x \in X\})$, *the convex hull of $\delta_X(X)$, the order cone $C\,(D_\infty(X)) = \mathbb{R}_+\mathrm{Bs}\,(D_\infty(X))$ and the base norm*

$$\|\lambda\|_B = \left\|\sum_{i=1}^{k}\alpha_i\delta_X(x_i)\right\|_B = \sum_{i=1}^{k}|\alpha_i|$$

*if $\lambda = \sum_{i=1}^{k}\alpha_i\delta_X(x_i)$is the representation of $\lambda$ in the basis $\delta_X(X)$.*

***Proof.*** $Lip_\infty(X)$ is an order unit normed linear space (see Proposition 2.2) with order unit $\mathbb{I}$. The order in $Lip_\infty(X)$ is pointwise. Hence, well-known results (cp, e.g. [13], Chapt. 9) imply that $Lip_\infty(X)'$ is a base ordered linear space with cone

$$C(Lip_\infty(X)') := \left\{\lambda \mid \lambda \in Lip_\infty(X)',\ \lambda(f) \geq 0 \text{ for all } f \in C(Lip_\infty(X))\right\}$$

and base

$$\begin{aligned}\mathrm{Bs}\left(Lip_\infty(X)'\right) &= \left\{\lambda \mid f \in Lip_\infty(X)',\ \lambda \geq 0 \text{ and } \lambda(\mathbb{I}) = 1\right\}\\ &= C(Lip(X)') \cap \left\{\lambda \mid \lambda \in Lip_\infty(X)' \text{ and } \lambda(\mathbb{I}) = 1\right\}.\end{aligned}$$

We define

$$C\,(D_\infty(X)) := C\left(Lip_\infty(X)'\right) \cap D_\infty(X) \quad \text{and} \quad B := \mathrm{Bs}\left(Lip_\infty(X)'\right) \cap D_\infty(X).$$

Of course, $B$ is a base set in $D_\infty(X)$ with $B \subset C\,(D_\infty(X))$, hence, $\mathbb{R}_+B \subset C\,(D_\infty(X))$.

If $\lambda := \sum_{i=1}^{n} \alpha_i \delta_X(x_i) \in C\left(D_\infty(X)\right)$, then Remark 1.3 yields at once $\alpha_i \geq 0$, $1 \leq i \leq n$. The converse implication is obviously true. If $\lambda > 0$, then $\|\alpha\| := \sum_{i=1}^{n} \alpha_i > 0$ follows and

$$\lambda = \|\alpha\| \sum_{i=1}^{n} \frac{\alpha_i}{\|\alpha\|} \delta_X(x_i) \tag{1}$$

holds. On the other hand, $\lambda = \sum_{i=1}^{n} \alpha_i \delta_X(x_i) \in B$ is equivalent to $\lambda(\mathbb{1}) = 1$, i.e., $\lambda(\mathbb{1}) = \sum_{i=1}^{n} \alpha_i = 1$. This shows that $B$ is the convex hull of $\{\delta_X(x) \mid x \in X\}$, i.e., $B = \text{conv } \{\delta_X(x) \mid x \in X\}$ and because of (1), $C\left(D_\infty(X)\right) = \mathbb{R}_+ B$, hence $\text{Bs}\left(D_\infty(X)\right) := B$ is a base for $C\left(D_\infty(X)\right)$.

For a basis representation of $\lambda = \sum_{i=1}^{n} \alpha_i \delta_X(x_i) \in D_\infty(X)$, $\lambda \neq 0$ put

$$\|\alpha^+\| := \sum_{\substack{i=1 \\ \alpha_i \geq 0}}^{n} \alpha_i \quad , \quad \|\alpha^-\| := \sum_{\substack{i=1 \\ \alpha_i < 0}}^{n} \alpha_i.$$

Then

$$\lambda = \|\alpha^+\| \lambda_+ - \|\alpha^-\| \lambda_- \tag{2}$$

with

$$\lambda_+ = \sum_{\substack{i=1 \\ \alpha_i > 0}}^{n} \frac{\alpha_i}{\|\alpha^+\|} \delta_X(x_i)$$

for $\|\alpha^+\| \neq 0$ and $\lambda_+ := 0$ else, and analogously for $\lambda_-$.

As a subset of $\text{Bs}\left(Lip_\infty(X)'\right)$ the set $\text{Bs}\left(D_\infty(X)\right)$ is linearly bounded. Hence the base seminorm induced by $\text{Bs}\left(D_\infty(X)\right)$ is a norm (cp. [13]). We denote this norm by $\|\square\|_0$ for the moment. One of the possible representations for a base norm is:

$$\|\lambda\|_0 = \inf\left\{\beta + \gamma \mid \beta, \gamma \geq 0,\ \lambda = \beta\xi - \gamma\eta,\ \xi, \eta \in \text{Bs}\left(D_\infty(X)\right)\ \right\}.$$

For $\lambda := \sum_{i=1}^{n} \alpha_i \delta_X(x_i)$ (2) implies

$$\|\lambda\|_0 \leq \|\alpha^+\| + \|\alpha^-\| = \sum_{i=1}^{n} |\alpha_i|.$$

Now, let $\lambda = \beta\phi - \gamma\psi$ be a second representation. We may take the union of $\delta_X(x)$, $x \in X$, which appear in the basis representation of $\lambda, \phi$, and $\psi$ and denote it by $\{\delta_X(x_i) \mid x_i \in X,\ 1 \leq i \leq n\}$. Let $\lambda = \sum_{i=1}^{n} \alpha_i \delta_X(x_i)$, $\phi = \sum_{i=1}^{n} \varphi_i \delta_X(x_i)$, and $\psi = \sum_{i=1}^{n} \psi_i \delta_X(x_i)$ where $\varphi_i \geq 0$ and $\psi_i \geq 0$ for $1 \leq i \leq n$. Then

$$\sum_{i=1}^{n} \alpha_i \delta_X(x_i) = \sum_{i=1}^{n} \left(\beta\varphi_i - \gamma\psi_i\right) \delta_X(x_i)$$

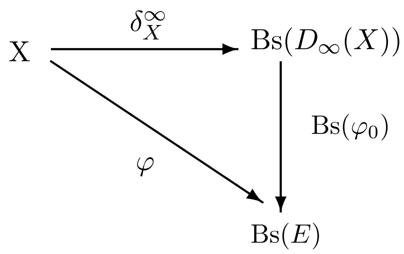
hence $\alpha_i = (\beta\varphi_i - \gamma\psi_i)$ and $|\alpha_i| = |\beta\varphi_i - \gamma\psi_i| \leq \beta\varphi_i + \gamma\psi_i$.

As $\phi, \psi \in \mathrm{Bs}\,(D_\infty(X))$ one has $\sum_{i=1}^n \varphi_i = \sum_{i=1}^n \varphi_i = 1$ and $\sum_{i=1}^n |\alpha_i| \leq \beta + \gamma$ follows such that $\|\lambda\|_B = \sum_{i=1}^n |\alpha_i|$ bas been proved. ∎

In view of Theorem 4.3 it is natural to ask if a similar result can be proved for $\delta_X^\infty : X \longrightarrow \mathrm{Bs}\,(D_\infty(X))$. This is indeed the case as the following Proposition shows.

Let us denote the category of base-normed linear spaces and linear base preserving mappings (which are, by the way contractions) by $\mathbf{BN\text{-}Vec}_1$. If, for $E \in \mathbf{BN\text{-}Vec}_1$, $\mathrm{Bs}\,(E)$ denotes the base of $E$, then

**Proposition 4.6.** *Let $X$ be a metric space, $E \in \mathbf{BN\text{-}Vec}_1$ and $\varphi : X \longrightarrow \mathrm{Bs}(E)$ a Lipschitz mapping. Then there exists a unique $\varphi_0 : D_\infty(X) \longrightarrow E$ in $\mathbf{BN\text{-}Vec}_1$ such that*




*commutes, and $1 = \|\varphi_0\| = \|\varphi\|_\infty$, where $\|\varphi\|_\infty = \sup\{\|\varphi\|_E \mid x \in X\}$.*

***Proof.*** As $\delta_X^\infty(X)$ is a basis of $D_\infty(X)$ the linear mapping $\varphi_0 : D_\infty(X) \longrightarrow E$ is well defined by

$$\varphi_0\left(\sum_{i=1}^n \alpha_i \delta_X(x_i)\right) := \sum_{i=1}^n \alpha_i \varphi(x_i)$$

and makes the above diagram commutative.

The proof of the equality of norms is trivial. As $\varphi_0$ is a contraction $\|\varphi_0 \leq 1$ holds. Also $\|\|\varphi\|_\infty\| = 1$ as $\varphi(X) \subset \mathrm{Bs}(E)$. Moreover

$$1 = \|\varphi(x)\|_E \leq \|\varphi_0\| \|\delta_X^\infty(x)\|'_\infty = \|\varphi_0\| \leq 1,$$

i.e., $\|\varphi_0\| = 1$. ∎

Despite the fact that the Lipschitz constant or the norm $\|\square\|_L$ does not appear explicitly in Proposition 4.6 the result is nonetheless quite interesting for metric spaces.

**Corollary 4.7.** *Let $(X, d_X)$ be a metric space. Then*

(i) $\delta_X^\infty : X \longrightarrow \mathrm{Bs}\,(D_\infty(X))$ *is a universal embedding into the base of the based normed linear space $D_\infty(X)$, i.e. the canonical functor*

$\mathbf{Bs} : \mathbf{BN\text{-}Vec}_1 \longrightarrow \mathbf{Lip}_1$ *has the functor* $\mathbf{D}_\infty : \mathbf{Lip}_1 \longrightarrow \mathbf{BN\text{-}Vec}_1$ *as a left adjoint with the adjunction morphism* $\delta_X^\infty : X \longrightarrow \mathrm{Bs}\,(D_\infty(X))\,.$

(ii) $\delta_X^\infty : X \longrightarrow \mathrm{Bs}\,(D_\infty(X))$ *is a universal contractive embedding into a metric convex module, i.e., if $C$ is a convex module (see [6]) and $f : X \longrightarrow C$ is in* $\mathbf{Lip}_1$, *then there is a unique affine mapping* $f_0 : \mathrm{Bs}\,(D_\infty(X)) \longrightarrow C$ *with* $f = f_0 \circ \delta_X^\infty$.

*If* **Met-Conv** *denotes the category of metric convex modules, or, what is the same, of metric convex subsets of real linear spaces [6] and affine mappings,* $\mathrm{Bs}\,(D_\infty(X))$ *may be regarded as a functor* $\mathbf{BsD}_\infty : \mathbf{Lip}_1 \longrightarrow$ **Met-Conv** *which is left adjoint to the canonical forgetful functor* $\mathfrak{U}$ : **Met-Conv** $\longrightarrow \mathbf{Lip}_1$ *assigning to every* $C \in$ **Met-Conv** *its underlying metric space and* $\delta_X^\infty$ *induces the adjunction morphism.*

***Proof.*** (i) is just a reformulation of Proposition 4.6 (ii) results by straightforward arguments from Proposition 4.6 and the results in §2 of [6]. ∎

Corollary 4.7 (ii) shows an interesting fact, namely the canonical and close connection between metric and convex structures.

## 5. The predual

There is another interesting topology, which was first investigated by R.F. Arens and J. Eells [1], and which will be discussed in this section.

The following proof uses a method completely different from the one used in [1] and is considerably shorter.

Define

$$\beth : Lip_L(X) \longrightarrow D_L'(X) \quad \text{and} \quad \daleth : D_L'(X) \longrightarrow Lip_L(X)$$

by

$$\beth(f)(\lambda) := \lambda(f),\ f \in Lip_L(X),\ \lambda \in D_L'(X)$$

and

$$\daleth(\lambda)(x) := \lambda(\delta_X^L(x)),\ \lambda \in D_L'(X),\ x \in X.$$

**Theorem 5.1.** $\gimel : Lip_L(X) \longrightarrow D'_L(X)$ and $\daleth : D'_L(X) \longrightarrow Lip_L(X)$ are in $\mathbf{Vec}_1$ and

$$\gimel \circ \daleth = id_{D'_L(X)} \quad \text{and} \quad \daleth \circ \gimel = id_{Lip_L(X)},\ x \in X$$

holds. Hence, $\gimel$ and $\daleth$ are isometries.

***Proof.*** Let us denote the norm dual to $\|\Box\|'_L$ of $D_L(X)$ on $D'_L(X)$ by $\|\Box\|^{\#}_L$. For $x, y \in X$ one has if $\lambda \in D'_L(X)$ :

$$\begin{aligned} |\daleth(\lambda)(x) - \daleth(\lambda)(y)| &= |\lambda\left(\delta^L_X(x)\right) - \lambda\left(\delta^L_X(y)\right)| \\ &\leq \|\lambda\|^{\#}_L \|(\delta^L_X(x) - \delta^L_X(y)\|'_L \leq \|\lambda\|^{\#}_L d(x,y) \\ &\quad \text{(because of Proposition 4.1).} \end{aligned}$$

Hence $L(\daleth(\lambda)) \leq \|\lambda\|^{\#}_L$ follows. Moreover

$$|\daleth(\lambda)(x)| = |\lambda\left(\delta^L_X(x)\right)| \leq \|\lambda\|^{\#}_L \|\delta^L_X(x)\|_L = \|\lambda\|^{\#}_L$$

and we have

$$\begin{aligned} \|\daleth(\lambda)\|_\infty &= \sup\{|\daleth(\lambda)(x)| \mid x \in X\} = \sup\{|\lambda\left(\delta^L_X(x)\right)| \mid x \in X\} \\ &\leq \|\lambda\|^{\#}_L \sup\{|\delta^L_X(x)| \mid x \in X\} \leq \|\lambda\|^{\#}_L. \end{aligned}$$

This yields

$$(*) \qquad \|\daleth(\lambda)\|_L = \max\{\|\daleth(\lambda)\|_\infty, L\left(\daleth(\lambda)\right)\} \leq \|\lambda\|^{\#}_L \quad \text{i.e.,} \quad \|\daleth\| \leq 1,$$

$\daleth$ is a contraction.

For $f \in Lip_L(X)$ one gets

$$\begin{aligned} \|\gimel(f)\|^{\#}_L &= \sup\left\{|\gimel(f)(\lambda)| \;\middle|\; \lambda \in D_L(X) \text{ and } \|\lambda\|'_L \leq 1\right\} \\ &= \sup\left\{|\lambda(f)| \;\middle|\; \lambda \in D_L(X) \text{ and } \|\lambda\|'_L \leq 1\right\} \\ &\leq \|\lambda\|'_L \sup\left\{\|f\|_L \;\middle|\; \|\lambda\|'_L \leq 1\right\} \leq \|f\|_L, \end{aligned}$$

which gives

$$(**) \qquad \|\gimel\| = \sup\left\{\|\gimel(f)\|^{\#}_L \;\middle|\; \|f\|_L \leq 1\right\} \leq 1.$$

i.e., $\gimel$ is also a contraction, $\|\gimel\| \leq 1$.

For $\lambda \in D'_L(X)$ and $x \in X$ $\gimel(\daleth(\lambda))\left(\delta^L_X(x)\right) = \delta^L_X(x)\left((\daleth(\lambda)\right) = \daleth(\lambda)(x) = \lambda\left(\delta^L_X(x)\right)$, holds for all $x \in X$, which yields $\gimel\circ\daleth(\lambda) = \lambda$, because $\left\{\delta^L_X(x) \mid x \in X\right\}$ is a basis of $D_L(X)$ and hence

$$\gimel \circ \daleth = id_{D'_L(X)}.$$

Also for $f \in Lip_L(X)$ and any $x \in X$

$$\daleth(\gimel(f))(x) = \gimel(f)\left(\delta^L_X(x)\right) = \delta^L_X(x)(f) = f(x) \quad \text{which results in} \quad \daleth(\gimel)(f) = f$$

and hence

$$\daleth \circ \gimel = id_{Lip_L(X)},$$

i.e., $\daleth$ and $\gimel$ are inverse to each other. This together with $(*)$ and $(**)$ yields the assertion ■

**Remark 5.2.** To show the dependence of $X$, an index $X$ will be added: $\gimel_X$ and $\daleth_X$, because both are natural transformations between interesting functors. The interesting topology mentioned at the beginning of this section is the dual topology $\sigma\left(D'_L(X), D_L(X)\right)$ transferred by $\daleth$ (and $\gimel$) to $Lip_L(X)$.

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Received 31 May 2015