Discussiones Mathematicae
Differential Inclusions, Control and Optimization 34 (2014) 207–218
doi:10.7151/dmdico.1161

AN EXISTENCE THEOREM FOR FRACTIONAL HYBRID DIFFERENTIAL INCLUSIONS OF HADAMARD TYPE

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Abstract

This paper studies the existence of solutions for fractional hybrid differential inclusions of Hadamard type by using a fixed point theorem due to Dhage. The main result is illustrated with the aid of an example.

Keywords: Hadamard fractional derivative, hybrid differential inclusions, Diriclet boundary conditions, existence, fixed point.

2010 Mathematics Subject Classification: 34A60, 34A08, 34B18.

1. Introduction

Fractional calculus, in view of its numerous applications in technical and applied sciences, has attracted the attention of many researchers. The nonlocal nature of a fractional-order operator together with its ability to trace the hereditary properties of the underlying process/phenomea has helped to improve the

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mathematical modelling of many real world phenomena involving integer-order operators. Examples include signal processing, control theory, bioengineering and biomedical, viscoelasticity, finance, stochastic processes, wave and diffusion phenomena, plasma physics, social sciences, etc. ([1]–[5]). Much of the work [6]–[19] on the topic deals with the governing equations involving Riemann-Liouville and Caputo type fractional derivatives. Another kind of fractional derivative is Hadamard type which was introduced in 1892 [20]. This derivative differs from aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains logarithmic function of arbitrary exponent. A detailed description of Hadamard fractional derivative and integral can be found in [2, 21, 22, 23, 24, 25].

Hybrid fractional differential equations constitutes another interesting class of problems. For some recent work on the topic, we refer to [26]–[31] and the references cited therein.

In this paper, we introduce a new concept of fractional hybrid differential inclusions of Hadamard type. Precisely we investigate the existence of solutions for the following problem

$$\left\{ \begin{array}{l} {}_HD^{\alpha}\left(\dfrac{x(t)}{f(t,x(t))}\right) \in F(t,x(t)), \quad 1 \le t \le T, \quad 0 < \alpha \le 1, \\ \qquad\qquad\qquad {}_HJ^{1-\alpha}x(t)|_{t=1} = \eta, \end{array} \right. \tag{1}$$

where ${}_HD^{\alpha}$ is the Hadamard fractional derivative, $f \in C([1,T]\times\mathbb{R}, \mathbb{R}\setminus\{0\})$, $F : [1,T]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, ${}_HJ^{(\cdot)}$ is the Hadamard fractional integral and $\eta \in \mathbb{R}$.

The paper is organized as follows: Section 2 contains some preliminary facts that we need in the sequel. In Section 3, we present the main existence result for the given problem whose proof is based on a fixed point theorem due to Dhage.

2. PRELIMINARIES

Let $C([1,T],\mathbb{R})$ denote the Banach space of all continuous real valued functions defined on $[1,T]$ with the norm $\|x\| = \sup\{|x(t)| : t \in [1,T]\}$. For $t \in [1,T]$, we define $x_r(t) = (\log t)^r x(t), r \ge 0$. Let $C_r([1,T],\mathbb{R})$ be the space of all continuous functions x such that $x_r \in C([1,T],\mathbb{R})$ which is indeed a Banach space endowed with the norm $\|x\|_C = \sup\{(\log t)^r|x(t)| : t \in [1,T]\}$.

Let $0 \le \gamma < 1$ and $C_{\gamma,\log}[a,b]$ denote the weighted space of continuous functions defined by

$$C_{\gamma,\log}[a,b] = \left\{g(t) : (\log t)^{\gamma}g(t) \in C[a,b], \ \|y\|_{C_{\gamma,\log}} = \|(\log t)^{\gamma}g(t)\|_C\right\}.$$

In the following we denote $\|y\|_{C_{\gamma,\log}}$ by $\|y\|_C$.

Theorem 1. *Let $\alpha > 0$, $n = -[-\alpha]$ and $0 \leq \gamma < 1$. Let G be an open set in $\mathbb{R}$ and let $f : (a, b] \to \mathbb{R}$ be a function such that: $f(x, y) \in C_{\gamma,\log}[a, b]$ for any $y \in G$, then the following problem*

$$ {}_H D^\alpha y(t) = f(t, y(t)), \quad \alpha > 0, \tag{2}$$

$$ {}_H D^{\alpha-k} y(a+) = b_k, \quad b_k \in \mathbb{R}, \quad (k = 1, \ldots, n,\ n = -[-\alpha]), \tag{3}$$

satisfies the following Volterra integral equation:

$$\begin{aligned} y(t) \ &= \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)} \left(\log \frac{t}{a}\right)^{\alpha - j} \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, y(s)) \frac{ds}{s}, \quad t > a > 0, \end{aligned} \tag{4}$$

i.e., $y(t) \in C_{n-\alpha,\log}[a, b]$ satisfies the relations (2)–(3) *if and only if it satisfies the Volterra integral equation* (4).

In particular, if $0 < \alpha \leq 1$, the problem (2)–(3) *is equivalent to the following equation:*

$$ y(t) = \frac{b}{\Gamma(\alpha)} \left(\log \frac{t}{a}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, y(s)) \frac{ds}{s}, \quad s > a > 0. \tag{5}$$

Details can be found in [2].

Some of propositions with the Hadamard calculus (derivative/integral) are formed as follows ([32]).

Proposition 2. If $0 < \alpha < 1$ the following relations hold:

$$ {}_H J^\alpha (\log t)^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu + \alpha)} (\log t)^{\mu+\alpha-1},$$

$$ {}_H D^\alpha (\log t)^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu - \alpha)} (\log t)^{\mu-\alpha-1}.$$

From Theorem 1 we have:

Lemma 3. *Given $y \in C([1, T], \mathbb{R})$, the integral solution of initial value problem*

$$\begin{cases} {}_H D^\alpha \left(\dfrac{x(t)}{f(t, x(t))}\right) = y(t), \quad 1 < t < T, \\ {}_H J^{1-\alpha} x(t)|_{t=1} = \eta, \end{cases} \tag{6}$$

is given by

$$x(t) = f(t,x(t))\left(\frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds\right), \quad t \in [1,T].$$

Let us recall some basic definitions on multi-valued maps [33, 34].

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e., $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [1,T] \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Let $L^1([1,T],\mathbb{R})$ be the Banach space of measurable functions $x : [1,T] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_1^e |x(t)|dt$.

Definition. A multivalued map $F : [1,T]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;

(ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [1,T]$;

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) there exists a function $g \in L^1([1,T],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq g(t)$$

for all $x \in \mathbb{R}$ and for a.e. $t \in [1,T]$.

For each $y \in C([1,T],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([1,T],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [1,T]\}.$$

The following lemma is used in the sequel.

Lemma 4 ([35])**.** *Let X be a Banach space. Let $F : [1,T] \times \mathbb{R} \to \mathcal{P}_{cp,cv}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([1,T],X)$ to $C([1,T],X)$. Then the operator*

$$\Theta \circ S_F : C([1,T],X) \to \mathcal{P}_{cp,cv}(C([1,T],X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([1,T],X) \times C([1,T],X)$.

3. Main result

In the forthcoming analysis, we consider the space $C_{1-\alpha}([1,T],\mathbb{R}) = \{x \in C((1,T],\mathbb{R}) : (\log t)^{1-\alpha}x(t) \in C([1,T],\mathbb{R})\}$ equipped with the norm $\|x\|_C = \sup\{(\log t)^{1-\alpha}|x(t)| : t \in [1,T]\}$. Obviously $(C_{1-\alpha}([1,T],\mathbb{R}), \|x\|_C)$ is a Banach space.

The following fixed point theorem due to Dhage [36] is fundamental in the proof of our main result.

Lemma 5. *Let X be a Banach algebra and let $A : X \to X$ be a single valued and $B : X \to \mathcal{P}_{cp,cv}(X)$ be a multi-valued operator satisfying:*

(a) *A is single-valued Lipschitz with a Lipschitz constant k,*

(b) *B is compact and upper semi-continuous,*

(c) *$2Mk < 1$, where $M = \|B(X)\|$.*

Then either

(i) *the operator inclusion $x \in AxBx$ has a solution, or*

(ii) *the set $\mathcal{E} = \{u \in X | \mu u \in AuBu,\ \mu > 1\}$ is unbounded.*

Theorem 6. *Assume that:*

(H_1) *The function $f : [1,T]\times\mathbb{R} \to \mathbb{R}\backslash\{0\}$ is bounded (i.e., $|f(t,x)| \le K,\ \forall(t,x) \in [1,T]\times\mathbb{R}$) continuous and there exists a bounded function ϕ, with bound $\|\phi\|$, such that $\phi(t) > 0$, a.e $t \in [1,T]$ and*

$$|f(t,x) - f(t,y)| \le \phi(t)|x(t) - y(t)|, \quad \text{a.e. } t \in [1,T] \text{ and for all } x,y \in \mathbb{R};$$

(H_2) *$F : [1,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is L^1-Carathéodory and has nonempty compact and convex values;*

(H_3) $2\|\phi\|\left(\dfrac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1}\frac{g(s)}{s}ds\right) < 1.$

Then the boundary value problem (1) *has at least one solution on* $[1,T]$.

Proof. Set $X = C_{1-\alpha}([1,T],\mathbb{R})$. Transform the problem (1) into a fixed point problem. Consider the operator $\mathcal{N} : X \to \mathcal{P}(X)$ defined by

$$\mathcal{N}(x) = \Bigg\{ h \in C([1,T],\mathbb{R}) : \ h(t) = f(t,x(t))\left(\frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds\right), \ v \in S_{F,x}\Bigg\}.$$

Now we define two operators $\mathcal{A}_1 : X \to X$ by

$$\mathcal{A}_1 x(t) = f(t,x(t)), \ t \in [1,T], \tag{7}$$

and $\mathcal{B}_1 : X \to \mathcal{P}(X)$ by

$$\mathcal{B}_1(x) = \Bigg\{ h \in C([1,T],\mathbb{R}) : h(t) = \frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds, \ v \in S_{F,x}\Bigg\}. \tag{8}$$

Observe that $\mathcal{N}(x) = \mathcal{A}_1 x \mathcal{B}_1 x$. We shall show that the operators $\mathcal{A}_1$ and $\mathcal{B}_1$ satisfy all the conditions of Lemma 5. For the sake of convenience, we split the proof into several steps.

Step 1. $\mathcal{A}_1$ is a Lipschitz on X, i.e., (a) of Lemma 5 holds.

Let $x, y \in X$. Then by (H_1) we have

$$\begin{aligned} |(\log t)^{1-\alpha}\mathcal{A}_1 x(t) - (\log t)^{1-\alpha}\mathcal{A}_1 y(t)| &= (\log t)^{1-\alpha}|f(t,x(t)) - f(t,y(t))| \\ &\le \phi(t)(\log t)^{1-\alpha}|x(t) - y(t)| \\ &\le \|\phi\|\|x - y\|_C \end{aligned}$$

for all $t \in [1,T]$. Taking the supremum over the interval $[1,T]$, we obtain

$$\|\mathcal{A}_1 x - \mathcal{A}_1 y\|_C \le \|\phi\|\|x - y\|_C$$

for all $x, y \in X$. So $\mathcal{A}_1$ is a Lipschitz on X with Lipschitz constant $\|\phi\|$.

Step 2. The multi-valued operator $\mathcal{B}_1$ is compact and upper semicontinuous on X, i.e., (b) of Lemma 5 holds.

First we show that $\mathcal{B}_1$ has convex values. Let $u_1, u_2 \in \mathcal{B}_1 x$. Then there are $v_1, v_2 \in S_{F,x}$ such that

$$u_i(t) = \frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v_i(s)}{s}ds,$$

$i = 1, 2,\ t \in [1, T]$. For any $\theta \in [0, 1]$, we have

$$\begin{aligned}\theta u_1(t) + (1-\theta)u_2(t) &= \frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{[\theta u_1(s) + (1-\theta)u_2(s)]}{s}ds \\ &= \frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{\overline{v}(s)}{s}ds,\end{aligned}$$

where $\overline{v}(t) = \theta v_1(t) + (1-\theta)v_2(t) \in F(t, x(t))$ for all $t \in [1, T]$. Hence $\theta u_1(t) + (1-\theta)u_2(t) \in \mathcal{B}_1 x$ and consequently $\mathcal{B}_1 x$ is convex for each $x \in X$. As a result $\mathcal{B}_1$ defines a multi valued operator $\mathcal{B}_1 : X \to \mathcal{P}_{cv}(X)$.

Next we show that $\mathcal{B}_1$ maps bounded sets into bounded sets in X. To see this, let Q be a bounded set in X. Then there exists a real number $r > 0$ such that $\|x\| \leq r, \forall x \in Q$.

Now for each $h \in \mathcal{B}_1 x$, there exists a $v \in S_{F,x}$ such that

$$h(t) = \frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds.$$

Then for each $t \in [1, T]$, using (H_2) we have

$$\begin{aligned}(\log t)^{1-\alpha}|\mathcal{B}_1 x(t)| &= \left|\frac{\eta}{\Gamma(\alpha)} + (\log t)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds\right| \\ &\leq \frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s)}{s}ds \\ &\leq \frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1}\frac{g(s)}{s}ds.\end{aligned}$$

This further implies that

$$\|h\|_C \leq \frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds,$$

and so $\mathcal{B}_1(X)$ is uniformly bounded.

Next we show that $\mathcal{B}_1$ maps bounded sets into equicontinuous sets. Let Q be, as above, a bounded set and $h \in \mathcal{B}_1 x$ for some $x \in Q$. Then there exists a $v \in S_{F,x}$ such that

$$h(t) = \frac{\eta}{\Gamma(\alpha)} (\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds, \ t \in [1, T].$$

Then for any $\tau_1, \tau_2 \in [1, T]$ we have

$$\begin{aligned} & |(\log \tau_2)^{1-\alpha} (\mathcal{B}_1 x)(\tau_2) - (\log \tau_1)^{1-\alpha} (\mathcal{B}_1 x)(\tau_1)| \\ \leq & \left| \int_1^{\tau_2} (\log \tau_2)^{1-\alpha} \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds - \int_1^{\tau_1} (\log \tau_1)^{1-\alpha} \left(\log \frac{\tau_1}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds \right| \\ \leq & \left| \int_1^{\tau_1} \left[(\log \tau_2)^{1-\alpha} \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} - (\log \tau_1)^{1-\alpha} \left(\log \frac{\tau_1}{s}\right)^{\alpha-1} \right] \frac{g(s)}{s} ds \right| \\ & + \left| \int_{\tau_1}^{\tau_2} (\log \tau_2)^{1-\alpha} \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds \right|. \end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in Q$ as $t_2 - t_1 \to 0$. Therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{B}_1 : X \to \mathcal{P}(X)$ is completely continuous.

In our next step, we show that $\mathcal{B}_1$ *has a closed graph.* Let $x_n \to x_*, h_n \in \mathcal{B}_1(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{B}_1$. Associated with $h_n \in \mathcal{B}_1(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [1, T]$,

$$h_n(t) = \frac{\eta}{\Gamma(\alpha)} (\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_n(s)}{s} ds.$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [1, T]$,

$$h_*(t) = \frac{\eta}{\Gamma(\alpha)} (\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds.$$

Let us consider the linear operator $\Theta : L^1([1, T], \mathbb{R}) \to C([1, T], \mathbb{R})$ given by

$$f \mapsto \Theta(v)(t) = \frac{\eta}{\Gamma(\alpha)} (\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds.$$

Observe that

$$\|h_n(t) - h_*(t)\| \ = \ \left\| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{(v_n(s) - v_*(s))}{s} ds \right\| \to 0, \text{as } n \to \infty.$$

Thus, it follows by Lemma 4 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds,$$

for some $v_* \in S_{F,x_*}$.

As a result we have that the operator $\mathcal{B}_1$ is compact and upper semicontinuous operator on X.

Step 3. Now we show that $2Mk < 1$, i.e., (c) of Lemma 5 holds.

This is obvious by (H_3) since we have $M = \|B(X)\| = \sup\{|\mathcal{B}_1 x : x \in X\} \leq \frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds$ and $k = \|\phi\|$.

Thus all the conditions of Lemma 5 are satisfied and a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible.

Let $u \in \mathcal{E}$ be arbitrary. Then we have for $\lambda > 1$, $\lambda u(t) \in \mathcal{A}_1 u(t) \mathcal{B}_1 u(t)$. Then there exists $v \in S_{F,x}$ such that for any $\lambda > 1$, one has

$$u(t) = \lambda^{-1}[f(t, u(t)] \left(\frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \right),$$

for all $t \in [1, T]$. Then we have

$$\begin{aligned} (\log t)^{1-\alpha}|u(t)| \ &\leq \ \lambda^{-1}|f(t, u(t)| \\ &\quad \times \left(\frac{\eta}{\Gamma(\alpha)} + (\log t)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{|v(s)|}{s} ds \right) \\ &\leq \ K \left(\frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds \right) \\ &\leq \ K \left(\frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds \right). \end{aligned}$$

Then we have

$$\|u\|_C \leq K \left(\frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds \right) := M.$$

Thus the condition (ii) of Thorem 5 does not hold. Therefore the operator equation $\mathcal{A}_1 x \mathcal{B}_1 x$ and consequently problem (1) has a solution on $[1, T]$. This completes the proof. ■

Example 7. Consider the initial value problem

$$\begin{cases} {}_HD^{1/2}\left(\dfrac{x(t)}{f(t,x)}\right) \in F(t, x(t)), \quad 1 < t < e, \\ {}_HJ^{1/2}x(t)|_{t=1} = \dfrac{2}{3}, \end{cases} \tag{9}$$

where

$$f(t,x) = \frac{\log t}{2}\tan^{-1}x + \frac{1}{\sqrt{1+t^2}} \text{ and } F(t,x) = \left[\frac{|x|^5}{15(|x|^5+1)}, \frac{|\sin x|}{7(|\sin x|+1)} + \frac{2}{7}\right],$$

and $T = e$. Clearly $\phi(t) = \frac{1}{2}\log t$ with $\|\phi\| = \frac{1}{2}$ (the condition (H_1) holds) and $\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \leq \frac{3}{7} = g(t)$, $x \in \mathbb{R}$. With the given values, the condition (H_3) is clearly satisfied, that is,

$$2\|\phi\|\left(\frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1}\frac{g(s)}{s}ds\right) \simeq 0.859717 < 1.$$

In consequence, the conclusion of Theorem 6 applies to the problem (9).

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Recieved 3 June 2014