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# PROPERTIES OF GENERALIZED SET-VALUED STOCHASTIC INTEGRALS

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**Abstract**

The paper is devoted to properties of generalized set-valued stochastic integrals defined in [10]. These integrals generalize set-valued stochastic integrals defined by E.J. Jung and J.H. Kim in the paper [4]. Up to now we were not able to construct any example of set-valued stochastic processes, different on a singleton, having integrably bounded set-valued integrals defined in [4]. It was shown by M. Michta (see [11]) that in the general case set-valued stochastic integrals defined by E.J. Jung and J.H. Kim, are not integrably bounded. Generalized set-valued stochastic integrals, considered in the paper, are in some non-trivial cases square integrably bounded and can be applied in the theory of stochastic differential equations with set-valued solutions.

**Keywords:** set-valued mappings, set-valued integrals, set-valued stochastic processes.

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## 1. Introduction

The paper is devoted to properties of generalized set-valued stochastic integrals, defined in the author paper [10] for a nonempty subsets of the space $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ of all square integrable $\mathbb{F}$-nonanticipative matrix-valued processes. For a given $m$-dimensional $\mathbb{F}$-Brownian motion $B = (B_t)_{t\geq 0}$ defined on a filtered probability space $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, P)$ and a nonempty subset $\mathcal{G}$ of the space $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$, a generalized set-valued stochastic integral $\int_0^t \mathcal{G} dB_\tau$ is understood as an $\mathcal{F}_t$-measurable set-valued random variable with values in the $d$-dimensional Euclidean space $\mathbb{R}^d$ and subtrajectory integrals $S_{\mathcal{F}_t}(\int_0^t \mathcal{G}dB_\tau)$ equal to $\overline{\operatorname{dec}}\,\mathcal{J}_t(\mathcal{G})$. By $\mathcal{J}_t$ we denote the Itô isometry defined on the space $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ by setting $\mathcal{J}_t(g) = \int_0^t g_\tau dB_\tau$ for every $g \in \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$. Subtrajectory integrals $S_{\mathcal{F}_t}(\int_0^t \mathcal{G}dB_\tau)$ of $\int_0^t \mathcal{G}dB_\tau$ is defined as a set of all $\mathcal{F}_t$-measurable and square integrable selectors of $\int_0^t \mathcal{G}dB_\tau$. It will be also denoted by $S_t(\int_0^t \mathcal{G}dB_\tau)$. In particular, if $\mathcal{G}$ is a nonempty decomposable subset of $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ then $\int_0^t \mathcal{G}dB_\tau = \int_0^t G_\tau dB_\tau$, where $G = (G_t)_{t\geq 0}$ is an $\mathbb{F}$-nonanticipative set-valued process such that $S_{\mathbb{F}}(G) = \mathrm{cl}_{\mathbb{L}}(\mathcal{G})$, where $S_{\mathbb{F}}(G) = \{g \in \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m}) : g_t(\omega) \in G_t(\omega)$ for a.e. $(t,\omega) \in \mathbb{R}^+ \times \Omega\}$. Set-valued stochastic integrals of the form $\int_0^t G_\tau dB_\tau$ have been defined by E.J. Jung and J.H. Kim in the paper [4], basing on the definition of set-valued functional stochastic integrals defined in the author papers [5] and [6] (see also [8]).

The generalized set-valued stochastic integrals, presented in this paper have better properties than set-valued stochastic integrals defined in the paper [4]. In particular, they are integrably bounded for some subsets of the space $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$. Integrable boundedness of set-valued stochastic integrals, defined by E.J. Jung and J.H. Kim, has been investigated by the author of the present paper (see [7–9]) without positive results. We were not able to present any example of multifunction, different on a singleton, having integrably bounded set-valued integral defined in [4]. Unfortunately, the result dealing with integrable boundedness of multifunctions with finite representations Castaing, presented in [9] is not true. The problem was also considered by M. Michta, who has showed (see [11]) that in the general case set-valued integrals, defined by E.J. Jung and J.H. Kim, are not integrably bounded.

Apart from the extension of the definition of set-valued stochastic integrals, we shall also extend the definition of the set-valued conditional expectation. It will be defined for every nonempty subsets of the space $\mathbb{L}(\Omega, \mathcal{F}, P, \mathbb{R}^d)$ and called a generalized set-valued conditional expectation. The present paper is organized as follows. Section 2 contains basic notions of the theory of set-valued stochastic processes, whereas Section 3 is devoted to properties of generalized set-valued stochastic integrals. Integrable boundedness of generalized set-valued stochastic integrals is considered in Section 4. Basic properties of indefinite generalized set-valued stochastic integrals are contained in the last Section of the paper.

Let $(X,\rho)$ be a metric space and denote by $\mathrm{Cl}(X)$ a space of all nonempty closed subsets of $X$. For every $A, C \in \mathrm{Cl}(X)$ let $\overline{h}(A,C) = \sup\{d(a,C) : a \in A\}$, where $d(a,C) = \inf\{\rho(a,c) : c \in C\}$. The Hausdorff distance $h(A,C)$ between $A, C \in \mathrm{Cl}(X)$ is defined by $h(A,C) = \max\{\overline{h}(A,C), \overline{h}(C,A)\}$. Given a sequence $(A_n)_{n\geq 1} \subset \mathrm{Cl}(X) \cup \{\emptyset\}$ by $\underline{\mathrm{Lim}}\, A_n$ and $\overline{\mathrm{Lim}}\, A_n$ we denote its Kuratowski limits inferior and superior defined by $\underline{\mathrm{Lim}}\, A_n = \{x \in X : \lim d(x, A_n) = 0\}$ and

$\overline{\mathrm{Lim}}\, A_n = \{x \in X : \underline{\lim}\, d(x, A_n) = 0\}$, respectively. It can be verified (see [2, 3]) that $\underline{\mathrm{Lim}}\, A_n = \{x \in X : x = \lim x_n, x_n \in A_n, n \geq 1\}$ and $\overline{\mathrm{Lim}}\, A_n = \{x \in X : x = \lim x_{n_k}, x_{n_k} \in A_{n_k}, n_1 < n_2 < \cdots < n_k < \cdots\}$. Immediately from the above definions we get $\underline{\mathrm{Lim}}\, A_n \subset \overline{\mathrm{Lim}}\, A_n$. We call a sequence $(A_n)_{n\geq 1}$ convergent in the Kuratowski sense to $A \in Cl(X) \cup \{\emptyset\}$ if $A = \underline{\mathrm{Lim}}\, A_n = \overline{\mathrm{Lim}}\, A_n$. The limit $A$ is denoted by $\mathrm{Lim}\, A_n$ and said to be Kuratowski's limit of a sequence $(A_n)_{n\geq 1}$. If $A_1 \subset A_2 \subset A_3 \subset \ldots$, then a sequence $(A_n)_{n\geq 1}$ is convergent in the Kuratowski sense and $\mathrm{Lim}\, A_n = \overline{\bigcup_{n\geq 1} A_n}$.

## 2. Set-valued stochastic processes

Throughout the paper we shall deal with a complete filtered probability space $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, P)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. By an $r$-dimensional set-valued random variable we mean a closed valued $\mathcal{F}$-measurable multifunction, i.e., a multifunction $\mathcal{Z} : \Omega \to \mathrm{Cl}(\mathbb{R}^r)$ such that $\{\omega \in \Omega : \mathcal{Z}(\omega) \cap C \neq \emptyset\} \in \mathcal{F}$ for every $C \in \mathrm{Cl}(\mathbb{R}^r)$. A family $G = (G_t)_{t\geq 0}$ of set-valued random variables $G_t : \Omega \to \mathrm{Cl}(\mathbb{R}^r)$ is said to be a set-valued stochastic process defined on $\mathcal{P}_{\mathbb{F}}$. Similarly as in the theory of point valued stochastic processes, a set-valued stochastic process can be defined as a multifunction $G : \mathbb{R}^+ \times \Omega \to \mathrm{Cl}(\mathbb{R}^r)$ such that $G(t, \cdot)$ is a set-valued random variable for every $t \geq 0$. Such defined stochastic process is said to be $\mathbb{F}$-nonanticipative if $G$ is $\beta(\mathbb{R}^+) \otimes \mathcal{F}$-measurable and $G(t, \cdot)$ is $\mathcal{F}_t$-measurable for every $t \geq 0$, where $\beta(\mathbb{R}^+)$ denotes the Borel $\sigma$-algebra on $\mathbb{R}^+$. It is easy to see that the set-valued process $G$ is $\mathbb{F}$-nonanticipative if and only if the multifunction $G : \mathbb{R}^+ \times \Omega \to \mathrm{Cl}(\mathbb{R}^r)$ is $\Sigma_{\mathbb{F}}$-measurable, where $\Sigma_{\mathbb{F}}$ is a $\sigma$-algebra on $\mathbb{R}^+ \times \Omega$ defined by $\Sigma_{\mathbb{F}} = \{A \in \beta(\mathbb{R}^+) \otimes \mathcal{F} : A^t \in \mathcal{F}_t$ for $t \geq 0\}$, where $A^t$ denotes the section of a set $A$ at $t \in \mathbb{R}^+$. For a given $\mathbb{F}$-nonanticipative set-valued stochastic process $G = (G_t)_{t\geq 0}$ with values in $\mathrm{Cl}(\mathbb{R}^{d\times m})$ defined on a filtered probability space $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, P)$, its subtrajectory integrals $S_{\mathbb{F}}(G)$ is defined by $S_{\mathbb{F}}(G) = \{g \in \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m}) : g_t(\omega) \in G_t(\omega)$ for a.e. $(t, \omega) \in \mathbb{R}^+ \times \Omega\}$. If $S_{\mathbb{F}}(G) \neq \emptyset$ then $G$ is said to be Itô integrable.

Properties of set-valued random variables and set-valued stochastic processes follow immediately from properties of measurable multifunctions (see [1] and [2]). For a given separable Banach space $\mathcal{X}$ and a $\sigma$-finite complete measure space $(\mathbb{T}, \mathcal{A}, \mu)$, a multifunction $\mathcal{Z} : \mathbb{T} \to \mathrm{Cl}(\mathcal{X})$ is said to be Aumann integrable if its subtrajectory integrals, denoted by $S_{\mathcal{A}}(\mathcal{Z})$ or simply by $S(\mathcal{Z})$ is nonempty. It can be proved (see [1]) that an Aumann integrable multifunction $\mathcal{Z}$ is square integrably bounded if and only if $S(\mathcal{Z})$ is a bounded subset of $\mathbb{L}^2(\mathbb{T}, \mathcal{A}, \mathcal{X})$. It can be verified (see [3], Corollary 3.5 of Chap. 2) that if $\mathcal{Z}$ and $G$ are Aumann integrable and $S(\mathcal{Z}) = S(G)$ then $\mathcal{Z}(t) = G(t)$ for $\mu$-a.e. $t \in \mathbb{T}$. It is clear that

$S(\mathcal{Z})$ is a closed subset of $\mathbb{L}(\mathbb{T}, \mathcal{A}, \mathcal{X})$. It is also decomposable, i.e., for every $A \in \mathcal{A}$ and $u, v \in S(\mathcal{Z})$ one has $1\!\!\mathrm{I}_A u + 1\!\!\mathrm{I}_{\mathbb{T}\setminus A} v \in S(\mathcal{Z})$. If $\mathcal{Z}$ is Aumann integrable then there is a Castaing representation $(z_n)_{n=1}^{\infty}$ of $\mathcal{Z}$ such that $(z_n)_{n=1}^{\infty} \subset S(\mathcal{Z})$, and therefore (see [1], Lemma 1.3), for every $z \in S(\mathcal{Z})$ and $\varepsilon > 0$ there exist a finite $\mathcal{A}$-measurable partition $(A_k)_{k=1}^{N}$ of $\mathbb{T}$ and a family $(z_{n_k})_{k=1}^{N} \subset \{z_n : n \geq 1\}$ such that $\int_{\mathbb{T}} |z - \sum_{k=1}^{N} 1\!\!\mathrm{I}_{A_k} z_{n_k}|^2 d\mu < \varepsilon$. In what follows the family of all finite $\mathcal{A}$-measurable partitions of $\mathbb{T}$ is denoted by $\Pi(\mathbb{T}, \mathcal{A})$. For a given $\Lambda \subset \mathbb{L}(\mathbb{T}, \mathcal{A}, \mathcal{X})$ by $\mathrm{dec}(\Lambda)$ we denote the decomposable hull of $\Lambda$, i.e., the smallest decomposable subset of $\mathbb{L}(\mathbb{T}, \mathcal{A}, \mathcal{X})$ containing $\Lambda$. In a similar way the closed decomposable hull $\overline{\mathrm{dec}}(\Lambda)$ is defined. It can be verified that $\overline{\mathrm{dec}}(\Lambda) = \mathrm{cl}_{\mathbb{L}}[\mathrm{dec}(\Lambda)]$, where the closure is taken in the norm topology of $\mathbb{L}(\mathbb{T}, \mathcal{A}, \mathcal{X})$.

Immediately from the above properties of Aumann integrable multifunctions it follows that if $(z_n)_{n=1}^{\infty} \subset S(\mathcal{Z})$ is a Castaing representation of a multifunction $\mathcal{Z}$, then $S(\mathcal{Z}) = \overline{\mathrm{dec}}\{z_n : n \geq 1\}$. Indeed, it is clear that $\overline{\mathrm{dec}}\{z_n : n \geq 1\} \subset S(\mathcal{Z})$. On the other hand, for every $z \in S(\mathcal{Z})$ and $\varepsilon > 0$ there exist a partition $(A_k)_{k=1}^{N} \in \Pi(\mathbb{T}, \mathcal{A})$ and a family $(z_{n_k})_{k=1}^{N} \subset \{z_n : n \geq 1\}$ such that $\int_{\mathbb{T}} |z - \sum_{k=1}^{N} 1\!\!\mathrm{I}_{A_k} z_{n_k}|^2 d\mu \leq \varepsilon$, which implies that $z \in \overline{\mathrm{dec}}\{z_n : n \geq 1\}$. Thus $S(\mathcal{Z}) = \overline{\mathrm{dec}}\{z_n : n \geq 1\}$. Finally, let us note (see [3], Th. 3.8 of Chap. 2) that a nonempty closed set $\mathcal{K} \subset \mathbb{L}^2(T, \mathcal{A}, \mathcal{X})$ is decomposable if and only if there exists an $\mathcal{A}$-measurable multifunction $F : T \to \mathrm{Cl}(\mathcal{X})$ such that $\mathcal{K} = S(F)$.

For a given integrably bounded set-valued random variable $\mathcal{Z} : \Omega \to \mathrm{Cl}(\mathbb{R}^r)$ and a $\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$ there exists (see [1], Th. 5.1) a unique $\mathcal{G}$-measurable set-valued random variable $E[\mathcal{Z}|\mathcal{G}] : \Omega \to \mathrm{Cl}(\mathbb{R}^r)$ such that $S(E[\mathcal{Z}|\mathcal{G}]) = \mathrm{cl}_{\mathbb{L}}\{E[f|\mathcal{G}] : f \in S(\mathcal{Z})\}$, where the closure is taken in the norm topology of $\mathbb{L}(\Omega, \mathcal{F}, \mathbb{R}^r)$. A set-valued random variable $E[\mathcal{Z}|\mathcal{G}]$ is said to be the set-valued conditional expectation of $\mathcal{Z}$ relative to $\mathcal{G}$. We can extend the above definition to nonempty subsets of the space $\mathbb{L}(\Omega, \mathcal{F}, \mathbb{R}^r)$. Given a nonempty set $\Lambda \subset \mathbb{L}(\Omega, \mathcal{F}, \mathbb{R}^r)$ and a $\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$ the generalized set-valued condition expectation $E[\Lambda|\mathcal{G}]$ of a set $\Lambda$ relative to $\mathcal{G}$ is defined to be an $\mathcal{G}$-measurable set-valued mapping $E[\Lambda|\mathcal{G}] : \Omega \to \mathrm{Cl}(\mathbb{R}^r)$ such that $S(E[\Lambda|\mathcal{G}]) = \overline{\mathrm{dec}}_{\mathcal{G}}\{E[z|\mathcal{G}] : z \in \Lambda\}$, where the decomposable hull is taken with respect to a $\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$. It is clear that if $\Lambda = S(\mathcal{Z})$, where $\mathcal{Z}$ is an Aumann integrable set-valued random variable, then $E[\Lambda|\mathcal{G}] = E[\mathcal{Z}|\mathcal{G}]$ a.s. Indeed, by the above definitions we get $S(E[S(\mathcal{Z})|\mathcal{G}]) = \overline{\mathrm{dec}}_{\mathcal{G}}\{E[z|\mathcal{G}] : z \in S(\mathcal{Z})\} = \mathrm{cl}_{\mathbb{L}}[\mathrm{dec}_{\mathcal{G}}\{E[z|\mathcal{G}] : z \in S(\mathcal{Z})\}] = \mathrm{cl}_{\mathbb{L}}\{E[z|\mathcal{G}] : z \in S(\mathcal{Z})\} = S(E[\mathcal{Z}|\mathcal{G}])$, because the set $\{E[z|\mathcal{G}] : z \in S(\mathcal{Z})\}$ is decomposable with respect to the $\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$. Therefore, $E[S(\mathcal{Z})|\mathcal{G}] = E[\mathcal{Z}|\mathcal{G}]$ a.s.

## 3. Some properties of generalized set-valued integrals

We present here some general properties of generalized set-valued stochastic integrals. In what follows we shall assume that we have given an $m$-dimensional

$\mathbb{F}$-Brownian motion $B = (B_t)_{t\geq 0}$ defined on a complete filtered probability space $\mathcal{P}_{\mathbb{F}}$. Apart from subsets of the space $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ we shall also consider subsets of the space $\mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{R}^d)$ for every $t \geq 0$. The closures of subsets of both $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ and $\mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{R}^d)$ will be denoted by the same way by $\mathrm{cl}_{\mathbb{L}}$.

**Lemma 3.1.** *For every nonempty set $\mathcal{G} \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ and $t \geq 0$ one has*

(i) $\mathcal{J}_t[\mathrm{cl}_{\mathbb{L}}(\mathcal{G})] = \mathrm{cl}_{\mathbb{L}}[\mathcal{J}_t(\mathcal{G})]$,

(ii) $\mathcal{J}_t(\overline{\mathrm{co}}(\mathcal{G})) = \overline{\mathrm{co}}\, \mathcal{J}_t(\mathcal{G})$,

(iii) $\overline{\mathrm{dec}}\{\mathcal{J}_t[\overline{\mathrm{co}}(\mathcal{G})]\} = \overline{\mathrm{co}}[\overline{\mathrm{dec}}\, \mathcal{J}_t(\mathcal{G})]$,

(iv) *if $(\Omega, \mathcal{F}, P)$ is a separable probability space then there is a sequence $(g^n)_{n=1}^{\infty} \subset \mathcal{G}$ such that $\mathrm{cl}_{\mathbb{L}}\, \mathcal{J}_t(\mathcal{G}) = \mathrm{cl}_{\mathbb{L}}\{\mathcal{J}_t(g^n) : n \geq 1\}$ for every $t \geq 0$.*

***Proof.*** (i) By continuity of a mapping $\mathcal{J}_t$ one has $\mathcal{J}_t[\mathrm{cl}_{\mathbb{L}}(\mathcal{G})] \subset \mathrm{cl}_{\mathbb{L}}[\mathcal{J}_t(\mathcal{G})]$. For every $u \in \mathrm{cl}_{\mathbb{L}}[\mathcal{J}_t(\mathcal{G})]$ and every sequence $(g^n)_{n=1}^{\infty} \subset \mathcal{G}$ such that $E|u - \mathcal{J}_t(g^n)|^2 \to 0$ we have $E|\mathcal{J}_t(g^n) - \mathcal{J}_t(g^m)|^2 \to 0$ as $m, n \to \infty$. But $E|\mathcal{J}_t(g^n) - \mathcal{J}_t(g^m)|^2 = E\int_0^t |g^n_\tau - g^m_\tau|^2 d\tau$ for every $t \geq 0$ and every $m, n \geq 1$. Then $(g^n)_{n=1}^{\infty}$ is a Cauchy sequence in the Banach space $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$. Thus there exists $g \in \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ such that $E\int_0^t |g^n_\tau - g_\tau|^2 d\tau \to 0$ as $n \to \infty$, which implies that $g \in \mathrm{cl}_{\mathbb{L}}(\mathcal{G})$ and $E|\mathcal{J}_t(g) - \mathcal{J}_t(g^n)|^2 \to 0$ as $n \to \infty$. Then $\mathcal{J}_t(g) \in \mathcal{J}_t[\mathrm{cl}_{\mathbb{L}}(\mathcal{G})]$ because $\mathcal{J}_t[\mathrm{cl}_{\mathbb{L}}(\mathcal{G})]$ is a closed subset of the space $\mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{R}^d)$. Hence it follows that $u \in \mathcal{J}_t[\mathrm{cl}_{\mathbb{L}}(\mathcal{G})]$ because $u = \mathcal{J}_t(g)$. Then $\mathrm{cl}_{\mathbb{L}}[\mathcal{J}_t(\mathcal{G})] \subset \mathcal{J}_t[\mathrm{cl}_{\mathbb{L}}(\mathcal{G})]$.

(ii) By linearity of a mapping $\mathcal{J}_t$ we have $\mathcal{J}_t(\mathrm{co}\,\mathcal{G}) = \mathrm{co}\,\mathcal{J}_t(\mathcal{G})$, which implies that $\mathrm{cl}_{\mathbb{L}}[\mathcal{J}_t(\mathrm{co}\,\mathcal{G})] = \overline{\mathrm{co}}\,\mathcal{J}_t(\mathcal{G})$. Hence, by (i) it follows that $\mathcal{J}_t(\overline{\mathrm{co}}(\mathcal{G})) = \overline{\mathrm{co}}\,[\mathcal{J}_t(\mathcal{G})]$ because $\mathrm{cl}_{\mathbb{L}}[\mathcal{J}_t(\mathrm{co}\,\mathcal{G})] = \mathcal{J}_t(\overline{\mathrm{co}}\,\mathcal{G})$ .

(iii) It is clear that $\mathcal{J}_t[\overline{\mathrm{co}}(\mathcal{G})]\} \subset \overline{\mathrm{co}}[\overline{\mathrm{dec}}\,\mathcal{J}_t(\mathcal{G})]$ because $\mathcal{J}_t(\overline{\mathrm{co}}(\mathcal{G})) = \overline{\mathrm{co}}\,\mathcal{J}_t(\mathcal{G})$. Let us observe that $\overline{\mathrm{co}}\,[\overline{\mathrm{dec}}\,\mathcal{J}_t(\mathcal{G})]$ is a decomposable subset of the space $\mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{R}^d)$. Indeed, by the properties of a set $\overline{\mathrm{dec}}\,\mathcal{J}_t(\mathcal{G})$ there is an $\mathcal{F}_t$-measurable set-valued random variable $F = (F_t)_{t\geq 0}$ with values in $\mathrm{Cl}(\mathbb{R}^d)$ and such that $\overline{\mathrm{dec}}\mathcal{J}_t(\mathcal{G}) = S_t(F)$ for every $t \geq 0$. Then, $\overline{\mathrm{co}}[\overline{\mathrm{dec}}\mathcal{J}_t(\mathcal{G})] = \overline{\mathrm{co}}S_t(F) = S_t(\overline{\mathrm{co}}\,F)$, which implies that $\overline{\mathrm{co}}[\overline{\mathrm{dec}}\,\mathcal{J}_t(\mathcal{G})]$ is decomposable. Therefore, $\overline{\mathrm{dec}}\{\mathcal{J}_t[\overline{\mathrm{co}}(\mathcal{G})]\} \subset \overline{\mathrm{co}}[\overline{\mathrm{dec}}\,\mathcal{J}_t(\mathcal{G})]$. On the other hand we have $\overline{\mathrm{dec}}\,\mathcal{J}_t(\mathcal{G}) \subset \overline{\mathrm{dec}}\{\mathcal{J}_t[\overline{\mathrm{co}}(\mathcal{G})]\}$. By virtue of ([8], Th. 3.3, Chap. 2) $\overline{\mathrm{dec}}\{\mathcal{J}_t[\overline{\mathrm{co}}(\mathcal{G})]\}$ is a convex subset of the space $\mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{R}^d)$. Then $\overline{\mathrm{co}}[\overline{\mathrm{dec}}\,\mathcal{J}_t(\mathcal{G})] \subset \overline{\mathrm{dec}}\,\{\mathcal{J}_t[\overline{\mathrm{co}}(\mathcal{G})]\}$. Thus, $\overline{\mathrm{dec}}\{\mathcal{J}_t[\overline{\mathrm{co}}(\mathcal{G})]\} = \overline{\mathrm{co}}[\overline{\mathrm{dec}}\,\mathcal{J}_t(\mathcal{G})]$.

(iv) By separability of the space $(\Omega, \mathcal{F}, P)$ the space $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ is separable. Then $\mathcal{G}$ with its induced topology is a separable subset of $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$. Thus there is a sequence $(g^n)_{n=1}^{\infty} \subset \mathcal{G}$ such that $\mathcal{G} = \mathrm{cl}_{\mathbf{I}}\{g^n : n \geq 1\}$, where $\mathrm{cl}_{\mathbf{I}}$ denotes the closure in the induced topology of $\mathcal{G}$. But

$\mathrm{cl}_{\mathbb{L}}\{g^n : n \geq 1\} \subset \mathrm{cl}_{\mathbb{L}}\,\mathcal{G}$ and $\mathrm{cl}_{\mathbf{I}}\{g^n : n \geq 1\} = \mathcal{G} \cap \mathrm{cl}_{\mathbb{L}}\{g^n : n \geq 1\}$. Then $\mathrm{cl}_{\mathbb{L}}\,\mathcal{G} = \mathrm{cl}_{\mathbb{L}}[\mathrm{cl}_{\mathbf{I}}\{g^n : n \geq 1\}] = \mathrm{cl}_{\mathbb{L}}[\mathcal{G} \cap \mathrm{cl}_{\mathbb{L}}\{g^n : n \geq 1\}] \subset \mathrm{cl}_{\mathbb{L}}\,\mathcal{G} \cap \mathrm{cl}_{\mathbb{L}}\{g^n : n \geq 1\} = \mathrm{cl}_{\mathbb{L}}\{g^n : n \geq 1\}$. Hence, and (i) it follows that $\mathrm{cl}_{\mathbb{L}}\,\mathcal{J}_t(\mathcal{G}) = \mathrm{cl}_{\mathbb{L}}\{\mathcal{J}_t(g^n) : n \geq 1\}$ for every $t \geq 0$. ∎

**Lemma 3.2.** *If $(\Omega, \mathcal{F}, P)$ is separable, $(X, \rho)$ is a metric space and $\Phi : X \ni x \to \Phi(x) \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ is l.s.c. and such that $\Phi(x)$ is a nonempty closed set for every $x \in X$, then there is a sequence $(g^n)_{n=1}^{\infty}$ of continuous functions $g^n : X \to \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ such that $g^n(x) \in \overline{\mathrm{co}}\,\Phi(x)$ for $n \geq 1$ and $\overline{\mathrm{co}}\,\Phi(x) = \mathrm{cl}_{\mathbb{L}}\{g^n(x) : n \geq 1\}$ for every $x \in X$.*

***Proof.*** The result follows immediately from ([2], Prop. 4.4, Chap. 1), because the space $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ is separable and a set valued mapping $X \ni x \to \overline{\mathrm{co}}\,\Phi(x) \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ satisfies the assumptions of ([2], Prop. 4.4, Chap. 1). ∎

**Lemma 3.3.** *If $\mathcal{G} = \{g^n : n \geq 1\} \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$, then $\overline{\mathrm{dec}}\,\mathcal{J}_t(\mathcal{G}) = \mathrm{Lim}\,\overline{\mathrm{dec}}\,\mathcal{J}_t(\mathcal{G}_p)$, where $\mathcal{G}_p = \{g^1, \ldots, g^p\}$ for $p \geq 1$.*

***Proof.*** Let us observe that $\mathrm{cl}_{\mathbb{L}}\,\mathcal{G} = \mathrm{Lim}\,\mathcal{G}_p$. Indeed, we have $\mathcal{G}_p \subset \mathcal{G}_{p+1}$ for every $p \geq 1$. Therefore, the Kuratowski limit $\mathrm{Lim}\,\mathcal{G}_p$ exists. Furthermore, $\mathcal{G}_p \subset \mathrm{cl}_{\mathbb{L}}\,\mathcal{G}$ for every $p \geq 1$, which implies that $\mathrm{Lim}\,\mathcal{G}_p \subset \mathrm{cl}_{\mathbb{L}}\,\mathcal{G}$, because $\mathrm{Lim}\,\mathcal{G}_p = \mathrm{cl}_{\mathbb{L}}\{\bigcup_{p\geq 1} \mathcal{G}_p\}$, where the closures are taken with respect to the norm topology of $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$. On the other hand, for every $g \in \mathrm{cl}_{\mathbb{L}}\,\mathcal{G}$ there is a subsequence $(g^{n_k})_{k=1}^{\infty}$ of $(g^n)_{n=1}^{\infty}$ such that $g^{n_k} \to g$ as $k \to \infty$. For every $k \geq 1$ there is $p_k \geq 1$ such that $g^{n_k} \in \mathcal{G}_{p_k}$. Then $g \in \overline{\mathrm{Lim}}\,\mathcal{G}_p = \mathrm{Lim}\,\mathcal{G}_p$. Thus $\mathrm{cl}_{\mathbb{L}}\,\mathcal{G} \subset \mathrm{Lim}\,\mathcal{G}_p$.

In a similar way we obtain $\mathrm{cl}_{\mathbb{L}}\,\mathcal{J}_t(\mathcal{G}) = \mathrm{Lim}\,\mathcal{J}_t(\mathcal{G}_p)$, which implies that $\overline{\mathrm{dec}}\,\mathcal{J}_t(\mathcal{G}) = \overline{\mathrm{dec}}[\mathrm{Lim}\,\mathcal{J}_t(\mathcal{G}_p)]$. To the end of the proof, we have to verify that $\overline{\mathrm{dec}}[\mathrm{Lim}\,\mathcal{J}_t(\mathcal{G}_p)] = \mathrm{Lim}\,\overline{\mathrm{dec}}[\mathcal{J}_t(\mathcal{G}_p)]$. Let us observe that $\mathrm{Lim}\,\overline{\mathrm{dec}}\{\mathcal{J}_t(\mathcal{G}_p)\} \subset \overline{\mathrm{dec}}\{\mathrm{Lim}\,\mathcal{J}_t(\mathcal{G}_p)\}$, because $\overline{\mathrm{dec}}\{\mathcal{J}_t(\mathcal{G}_p)\} \subset \overline{\mathrm{dec}}\{\mathrm{Lim}\,\mathcal{J}_t(\mathcal{G}_p)\}$ for every $p \geq 1$ and $\overline{\mathrm{dec}}\{\mathrm{Lim}\,\mathcal{J}_t(\mathcal{G}_p)\}$ is a closed subset of $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^d)$. For every $a \in \overline{\mathrm{dec}}\{\mathrm{Lim}\,\mathcal{J}_t(\mathcal{G}_p)\}$ there is a sequence $(a_r)_{r=1}^{\infty}$ of $\mathrm{dec}\{\mathrm{Lim}\,\mathcal{J}_t(\mathcal{G}_p)\} = \mathrm{dec}\{\mathcal{J}_t[\mathrm{Lim}\,\mathcal{G}_p)]\}$ converging to $a$ in the norm topology of $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^d)$. For every $r \geq 1$ there are a partition $(A_k^r)_{k=1}^{N_r} \in \Pi(\Omega, \mathcal{F}_t)$ and a family $(u_k^r)_{k=1}^{N_r} \subset \mathcal{J}_t[\mathrm{Lim}\,\mathcal{G}_p]$ such that $a_r = \sum_{k=1}^{N_r} \mathbb{I}_{A_k^r} u_k^r$. For every $r \geq 1$ and $k = 1, \ldots, N_r$ there is a sequence $(v_p^{k,r})_{p=1}^{\infty}$ such that $v_p^{k,r} \in \mathcal{G}_p$ for every $p \geq 1$ and $\mathcal{J}_t(v_p^{k,r}) \to u_k^r$ in the norm topology of $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^d)$ as $p \to \infty$. Then, for every $r \geq 1$ we have $\sum_{k=1}^{N_r} \mathbb{I}_{A_k^r} \mathcal{J}_t(v_p^{k,r}) \in \mathrm{dec}\{\mathcal{J}_t(\mathcal{G}_p)\}$ for $p \geq 1$ and $\sum_{k=1}^{N_r} \mathbb{I}_{A_k^r} cj_t(v_p^{k,r}) \to \sum_{k=1}^{N_m} \mathbb{I}_{A_k^r} u_k^r = a_r$ as $p \to \infty$. Therefore, $a_r \in \underline{\mathrm{Lim}}[\mathrm{dec}\{\mathcal{J}_t(\mathcal{G}_p)\}] = \mathrm{Lim}[\mathrm{dec}\{\mathcal{J}_t(\mathcal{G}_p)\}]$ for every $r \geq 1$, which implies that $a_r \in \mathrm{Lim}[\overline{\mathrm{dec}}\{\mathcal{J}_t(\mathcal{G}_p)\}]$, because $\mathrm{dec}\mathcal{J}_t(\mathcal{G}_p) \subset \mathrm{dec}\mathcal{J}_t(\mathcal{G}_{p+1})$ for every $p \geq 1$. Hence it follows that $a \in \mathrm{Lim}[\overline{\mathrm{dec}}\mathcal{J}_t(\mathcal{G}_p)]$ for every $a \in \overline{\mathrm{dec}}\{\mathrm{Lim}\,\mathcal{J}_t(\mathcal{G}_p)\}$.

Thus $\overline{\text{dec}}\{\text{Lim}\,\mathcal{J}_t(\mathcal{G}_p)\} \subset \text{Lim}[\overline{\text{dec}}\mathcal{J}_t(\mathcal{G}_p)]$, which implies that $\overline{\text{dec}}\{\text{Lim}\,\mathcal{J}_t(\mathcal{G}_p)\} = \text{Lim}\,\overline{\text{dec}}\{\mathcal{J}_t(\mathcal{G}_p)\}$. ■

We present now the basic properties of generalized set-valued stochastic integrals.

**Theorem 3.4.** *For every nonempty set* $\mathcal{G} \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ *we have* (i) $\int_0^t \text{cl}_{\mathbb{L}}(\mathcal{G})dB_\tau = \int_0^t \mathcal{G}dB_\tau$ *and* (ii) $\int_0^t \text{co}(\mathcal{G})dB_\tau = \overline{\text{co}}\int_0^t \mathcal{G}dB_\tau$ *a.s. for every* $t \geq 0$.

***Proof.*** (i) Immediatelly from (i) of Lemma 3.1 we get $\overline{\text{dec}}\,\mathcal{J}_t[\text{cl}_{\mathbb{L}}(\mathcal{G})] = \overline{\text{dec}}[\mathcal{J}_t(\mathcal{G})]\}$ which, by the definition of generalized set-valued stochastic integrals, implies that $S_t(\int_0^t \text{cl}_{\mathbb{L}}(\mathcal{G})dB_\tau) = S_t(\int_0^t \mathcal{G}dB_\tau)$ for every $t \geq 0$. Then $\int_0^t \text{cl}_{\mathbb{L}}(\mathcal{G})dB_\tau = \int_0^t \mathcal{G}dB_\tau$ a.s. for every $t \geq 0$.

(ii) Similarly, by (iii) of Lemma 3.1 and the definition of generalized set-valued stochastic integrals, we obtain $S_t(\int_0^t \overline{\text{co}}\,(\mathcal{G})dB_\tau) = \overline{\text{co}}\,S_t(\int_0^t \mathcal{G}dB_\tau)$ for every $t \geq 0$. Immediately from (i) it follows that $\int_0^t \overline{\text{co}}\,(\mathcal{G})dB_\tau = \int_0^t \text{co}\,(\mathcal{G})dB_\tau$ a.s. for every $t \geq 0$. Furthermore, $\overline{\text{co}}\,S_t(\int_0^t \mathcal{G}dB_\tau) = S_t(\overline{\text{co}}\int_0^t \mathcal{G}dB_\tau)$. Therefore, $S_t(\int_0^t \text{co}\,(\mathcal{G})dB_\tau) = S_t(\overline{\text{co}}\int_0^t \mathcal{G}dB_\tau)$ for every $t \geq 0$. Thus $\int_0^t \text{co}\,(\mathcal{G})dB_\tau = \overline{\text{co}}\int_0^t \mathcal{G}dB_\tau$ a.s. for every $t \geq 0$. ■

**Theorem 3.5.** *If* $(\Omega, \mathcal{F}, P)$ *is separable then for every nonempty set* $\mathcal{G} \subset \mathbb{L}^2(\mathbb{R}^+\times\Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ *there is a sequence* $(g^n)_{n=1}^\infty \subset \mathcal{G}$ *such that* $(\int_0^t \mathcal{G}dB_\tau)(\omega) = \text{cl}\{(\int_0^t g_\tau^n dB_\tau)(\omega) : n \geq 1\}$ *for every* $t \geq 0$ *and a.e.* $\omega \in \Omega$.

***Proof.*** By (iv) of Lemma 3.1 there is a sequence $(g^n)_{n=1}^\infty \subset \mathcal{G}$ such that $\text{cl}_{\mathbb{L}}\,\mathcal{J}_t(\mathcal{G}) = \text{cl}_{\mathbb{L}}\{\mathcal{J}_t(g^n) : n \geq 1\}$ for every $t \geq 0$, which implies that $\overline{\text{dec}}\,\mathcal{J}_t(\mathcal{G}) = \overline{\text{dec}}\,\{\mathcal{J}_t(g^n) : n \geq 1\}$ for every $t \geq 0$. Let $\Gamma_t(\omega) = \text{cl}\{(\int_0^t g_\tau^n dB_\tau)(\omega) : n \geq 1\}$ for every $t \geq 0$ and $\omega \in \Omega$. We have $S_t(\Gamma_t) = \overline{\text{dec}}\,\{\mathcal{J}_t(g^n) : n \geq 1\}$ for every $t \geq 0$, because $\Gamma_t$ is an Aumann integrably set-valued random variable. On the other hand, by the definition of generalized set-valued stochastic integrals, we have $S_t(\int_0^t \mathcal{G}dB_\tau) = \overline{\text{dec}}\,\mathcal{J}_t(\mathcal{G})$ for every $t \geq 0$. Then $S_t(\int_0^t \mathcal{G}dB_\tau) = S_t(\Gamma_t)$ for every $t \geq 0$, which implies that $(\int_0^t \mathcal{G}dB_\tau)(\omega) = \text{cl}\{(\int_0^t g_\tau^n dB_\tau)(\omega) : n \geq 1\}$ for every $t \geq 0$ and a.e. $\omega \in \Omega$. ■

**Theorem 3.6.** *If* $\mathcal{G} = \{g^n : n \geq 1\} \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$, *then* $\int_0^t \mathcal{G}dB_\tau = \text{Lim}\int_0^t \mathcal{G}_p dB_\tau$ *a.s. for every* $t \geq 0$, *where* $\mathcal{G}_p = \{g^1, \ldots, g^p\}$ *for for* $p \geq 1$.

***Proof.*** By virtue of Lemma 3.3 we have $\overline{\text{dec}}\,\mathcal{J}_t(\mathcal{G}) = \text{Lim}\,\overline{\text{dec}}\,\mathcal{J}_t(\mathcal{G}_p)$, for every $t \geq 0$, which implies that $S_t(\int_0^t \mathcal{G}dB_\tau) = \text{Lim}\,S_t(\int_0^t \mathcal{G}_p dB_\tau)$ for $t \geq 0$. We shall show now that $\text{Lim}\,S_t(\int_0^t \mathcal{G}_p dB_\tau) = S_t(\text{Lim}\int_0^t \mathcal{G}_p dB_\tau)$ for $t \geq 0$. Indeed, it is clear that $\text{Lim}\,S_t(\int_0^t \mathcal{G}_p dB_\tau) \subset S_t(\text{Lim}\int_0^t \mathcal{G}_p dB_\tau)$ for $t \geq 0$. Let $a \in S_t(\text{Lim}\int_0^t \mathcal{G}_p dB_\tau)$, i.e., let $a \in \text{Lim}\int_0^t \mathcal{G}_p dB_\tau = \underline{\text{Lim}}\int_0^t \mathcal{G}_p dB_\tau$ a.s. We have, $d(a, \int_0^t \mathcal{G}_p dB_\tau) \to 0$

a.s. as $p \to \infty$. Let us observe that the sequence $\{d(a, \int_0^t \mathcal{G}_p dB_\tau)\}_{p=1}^\infty$ is integrably bounded by a function $\varphi := d(a, \int_0^T g_t^1 dB_t)$ because $d(a, \int_0^t \mathcal{G}_{p+1} dB_t) \le d(a, \int_0^t \mathcal{G}_p dB_\tau)$ for every $p \ge 1$. Therefore, $E\{d(a, \int_0^t \mathcal{G}_p dB_\tau)\} \to 0$ as $p \to \infty$. Hence, by ([1], Th. 2.2) it follows that

$$d^2\left[a, S_t\left(\int_0^t \mathcal{G}_p dB_\tau\right)\right]$$

$$= \inf\left\{ |\!| a-u |\!|^2 : u \in S_t\left(\int_0^t \mathcal{G}_p dB_\tau\right)\right\} = E\left\{d^2\left(a, \int_0^t \mathcal{G}_p dB_\tau\right)\right\}$$

for every $p \ge 1$, where $|\!| \cdot |\!|$ denotes the norm of the space $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^d)$. Then $d(a, S_t(\int_0^t \mathcal{G}_p dB_\tau)) \to 0$ as $p \to \infty$. Therefore, $a \in \underline{\mathrm{Lim}}\, S_t(\int_0^t \mathcal{G}_p dB_\tau) = \mathrm{Lim}\, S_t(\int_0^t \mathcal{G}_p dB_\tau)$ for every $a \in S_t(\mathrm{Lim} \int_0^t \mathcal{G}_p B_\tau)$. Thus $S_t(\mathrm{Lim} \int_0^t \mathcal{G}_p dB_\tau) \subset \mathrm{Lim} S_t(\int_0^t \mathcal{G}_p dB_\tau)$, which implies that $S_t(\mathrm{Lim} \int_0^t \mathcal{G}_p dB_\tau) = \mathrm{Lim}\, S_t(\int_0^t \mathcal{G}_p dB_\tau)$. Now we have $S_t(\int_0^t \mathcal{G} dB_\tau) = S_t(\mathrm{Lim} \int_0^t \mathcal{G}^p dB_\tau)$ for every $t \ge 0$, which implies that $\int_0^t \mathcal{G} dB_\tau = \mathrm{Lim} \int_0^t \mathcal{G}_p dB_\tau$ a.s. for every $t \ge 0$. ∎

**Corollary 3.1.** *If $(\Omega, \mathcal{F}, P)$ is a separable probability space and $\mathcal{G}$ is a nonempty subset of the space $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ then there exists a sequence $(g^n)_{n=1}^\infty \subset \mathcal{G}$ such that $\int_0^t \mathcal{G} dB_\tau = \mathrm{Lim} \int_0^t \mathcal{G}_p dB_\tau$ a.s. for every $t \ge 0$, where $\mathcal{G}_p = \{g^1, \ldots, g^p\}$ for for $p \ge 1$.*

***Proof.*** By (iv) of Lemma 3.1 there is a sequence $(g^n)_{n=1}^\infty \subset \mathcal{G}$ such that $\mathrm{cl}_{\mathbb{L}}\, \mathcal{J}_t(\mathcal{G}) = \mathrm{cl}_{\mathbb{L}}\{\mathcal{J}_t(g^n) : n \ge 1\}$ for every $t \ge 0$. Therefore, $\overline{\mathrm{dec}}\, \mathcal{J}_t(\mathcal{G}) = \overline{\mathrm{dec}}\, \{\mathcal{J}_t(g^n) : n \ge 1\}$ for every $t \ge 0$, which is equivalent to $S_t(\int_0^t \mathcal{G} dB_\tau) = S_t(\int_0^t \{g^n : n \ge 1\} dB_\tau)$ for every $t \ge 0$. Hence, similarly as in the proof of Theorem 3.6, it follows that $\int_0^t \mathcal{G} dB_\tau = \mathrm{Lim} \int_0^t \mathcal{G}_p dB_\tau$ a.s. for every $t \ge 0$. ∎

**Remark 3.1.** If $\mathcal{G}$ is a nonempty bounded decomposable subset of the space $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ then the last result is true without the assumption that $(\Omega, \mathcal{F}, P)$ is a separable probability space.

***Proof.*** By decomposability of the set $\mathcal{G}$ it follows that there is an $\mathbb{F}$-nonanticipative process $G = (G_t)_{t\ge 0}$ with values in $\mathrm{Cl}(\mathbb{R}^{d\times m})$ such that $\mathrm{cl}_{\mathbb{L}}\, \mathcal{G} = S_{\mathbb{F}}(G)$. By $\Sigma_{\mathbb{F}}$-measurability of $G$ there is a sequence $(g^n)_{n=1}^\infty$, of $\Sigma_{\mathbb{F}}$-measurable processes $g^n = (g_t^n)_{t\ge 0}$, a Castaing representation of $G$, such that $G_t(\omega) = \mathrm{cl}\{g_t^n(\omega) : n \ge 1\}$ for every $(t, \omega) \in \mathbb{R}^+ \times \Omega$. We have $(g^n)_{n=1}^\infty \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ because $G$ is square integrably bounded. Therefore, $S_{\mathbb{F}}(G) = \overline{\mathrm{dec}}\{g^n : n \ge 1\}$. Thus there is a sequence $(g^n)_{n=1}^\infty \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ such that $\mathrm{cl}_{\mathbb{L}}\, \mathcal{G} = \overline{\mathrm{dec}}\{g^n : n \ge 1\}$. Hence, similarly as above (see [9], Th. 3.2) we obtain $\int_0^t \mathcal{G} dB_\tau = \mathrm{Lim} \int_0^t \mathrm{dec}\, \mathcal{G}_p dB_\tau$ a.s. for every $t \ge 0$, where $\mathcal{G}_p = \{g^1, \ldots, g^p\}$ for for $p \ge 1$. ∎

**Remark 3.2.** In a similar way as above we can show that if $\mathcal{G} = \{g^n : n \geq 1\} \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$, then $\int_0^t \operatorname{co}\mathcal{G} dB_\tau = \operatorname{Lim} \int_0^t \operatorname{co}\mathcal{G}_p dB_\tau$ a.s. for every $t \geq 0$, where $\mathcal{G}_p = \{g^1, \ldots, g^p\}$ for for $p \geq 1$.

We shall show now that for every nonempty set $\mathcal{G} \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^d)$ such that a generalized set-valued stochastic integral $\int_0^t \mathcal{G} dB_\tau$ of $\mathcal{G}$ with respect to a real $\mathbb{F}$-Brownian motion $B = (B_t)_{t\geq 0}$ is square integrably bounded, we have $\sigma(p, \int_0^t \mathcal{G} dB_\tau) = \sup \int_0^t \mathcal{S}(p,\mathcal{G}) dB_\tau$ a.s. for every $p \in \mathbb{R}^d$ and $t \geq 0$. $\sigma(\,\cdot, A)$ denotes the support function of a set $A \subset \mathbb{R}^d$, $\mathcal{S}(p,\mathcal{G}) = \{(p,g) : g \in \mathcal{G}\}$ and $(p,g)$ denotes for every $g \in \mathcal{G}$ a real-valued $\mathbb{F}$-nonanticipative stochastic process defined by the inner product $\langle \cdot,\cdot \rangle$ of $\mathbb{R}^d$ by setting $(p,g)_t(\omega) = \langle p, g_t(\omega)\rangle$ for every $(t,\omega) \in \mathbb{R}^+ \times \Omega$. Let us note that $\int_0^t \mathcal{S}(p,\mathcal{G}) dB_\tau)$ is a closed subset of the real line $\mathbb{R}$.

**Theorem 3.7.** *For every nonempty set $\mathcal{G} \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^d)$ such that a generalized set-valued stochastic integral $\int_0^t \mathcal{G} dB_\tau$ is square integrably bounded for every $t \geq 0$, we have $\sigma(p, \int_0^t \mathcal{G} dB_\tau) = \sup \int_0^t \mathcal{S}(p,\mathcal{G}) dB_\tau$ a.s. for every $p \in \mathbb{R}^d$ and $t \geq 0$.*

***Proof.*** Let $\int_0^t \mathcal{G} dB_\tau$ be square integrably bounded for fixed $t \geq 0$. For every every $p \in \mathbb{R}^d$ and $A \in \mathcal{F}_t$ we have

$$\int_A \sigma\left(p, \int_0^t \mathcal{G} dB_\tau\right) dP = \int_A \sup\left\{\langle p, x\rangle : x \in \int_0^t \mathcal{G} dB_\tau\right\} dP$$

$$= \sup\left\{\int_A \langle p,u\rangle\, dP : u \in S_t\left(\int_0^t \mathcal{G} dB_\tau\right)\right\} = \sup\left\{\int_A \langle p,u\rangle\, dP : u \in \operatorname{dec}\mathcal{J}_t(\mathcal{G})\right\}$$

$$= \sup\left\{\int_A \left\langle p, \sum_{k=1}^N \mathbb{I}_{A_k}\mathcal{J}_t(g^k)\right\rangle dP : (A_k)_{k=1}^N \in \Pi(\Omega,\mathcal{F}_t), (g^k)_{k=1}^N \subset \mathcal{G}\right\}$$

$$= \sup\left\{\int_A \left[\sum_{k=1}^N \mathbb{I}_{A_k}\left\langle p, \mathcal{J}_t(g^k)\right\rangle\right] dP : (A_k)_{k=1}^N \in \Pi(\Omega,\mathcal{F}_t), (g^k)_{k=1}^N \subset \mathcal{G}\right\}$$

$$= \sup\left\{\int_A \left[\sum_{k=1}^N \mathbb{I}_{A_k}\mathcal{J}_t\left(\left\langle p, g^k\right\rangle\right)\right] dP : (A_k)_{k=1}^N \in \Pi(\Omega,\mathcal{F}_t), (g^k)_{k=1}^N \subset \mathcal{G}\right\}$$

$$= \sup\left\{\int_A u\, dP : u \in \operatorname{dec}\mathcal{J}_t\left(\mathcal{S}(p,\mathcal{G})\right)\right\}$$

$$= \sup\left\{\int_A u\, dP : u \in S_t\left(\int_0^t \mathcal{S}(p,\mathcal{G}) dB_\tau\right)\right\} = \int_A \left[\sup \int_0^t \mathcal{S}(p,\mathcal{G}) dB_\tau\right] dP.$$

Therefore, $\sigma(p, \int_0^t \mathcal{G} dB_\tau) = \sup \int_0^t \mathcal{S}(p,\mathcal{G}) dB_\tau$ a.s. for every $p \in \mathbb{R}^d$ and fixed $t \geq 0$. ∎

## 4. Integrable boundedness of generalized set-valued integrals

We present now some results dealing with integrable boundedness of generalized set-valued stochastic integrals. We begin with the following lemma.

**Lemma 4.1.** *For every $m$-dimensional $\mathbb{F}$-Brownian motion $B = (B_t)_{t\geq 0}$ defined on a complete filtered probability space $\mathcal{P}_{\mathbb{F}}$, and every set $\mathcal{G}_p = \{g^i : 1 \leq i \leq p\} \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ one has $E\|\int_0^t \mathcal{G}_p dB_\tau\|^2 \leq p \cdot E\int_0^t \max_{1\leq i\leq p} |g^i_\tau|^2 d\tau < \infty$ for every $t \geq 0$.*

***Proof.*** By virtue of ([1], Th. 2.2) we get

$$E\left\|\int_0^t \mathcal{G}_p dB_\tau\right\|^2 = \sup\left\{E|u|^2 : u \in S_t\Big(\int_0^t \mathcal{G}_p dB_\tau\Big)\right\}$$
$$= \sup\left\{E|u|^2 : u \in \mathrm{dec}\{\mathcal{J}_t(g^1), \ldots, \mathcal{J}_t(g^p)\}\right\}$$
$$= \sup\left\{E\Big[\sum_{j=1}^p \mathbb{I}_{A_j}|\mathcal{J}_t(g^j)|^2\Big] : (A_j)_{j=1}^p \in \Pi(\Omega, \mathcal{F}_t)\right\}$$
$$\leq \sum_{j=1}^p E\int_0^t |g^j_\tau|^2 d\tau \leq p \cdot E\int_0^t \max_{1\leq i\leq p} |g^i_\tau|^2 d\tau < \infty.$$

■

We shall prove now the following theorem dealing with integrable boundedness of generalized set-valued stochastic integrals.

**Theorem 4.2.** *Let $\mathcal{G}$ be a nonempty subset of $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$. If $\int_0^t \mathcal{G} dB_\tau$ is square integrably bounded for every $t \geq 0$ then $\mathbb{I}_{[0,t]}\mathcal{G}$ is a bounded subset of $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ every $t \geq 0$.*

***Proof.*** Let $\int_0^t \mathcal{G} dB_\tau$ be square integrably bounded for fixed $t \geq 0$. Then $S_t(\int_0^t \mathcal{G} dB_\tau)$ is a boundedd subset of $\mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{R}^d)$. Thus there exists $M_t > 0$ such that $E|u|^2 \leq M_t$ for every $u \in S_t(\int_0^t \mathcal{G} dB_\tau)$. But $S_t(\int_0^t \mathcal{G} dB_\tau) = \overline{\mathrm{dec}}\, \mathcal{J}_t(\mathcal{G})$. Therefore, $\mathcal{J}_t(g) \in S_t(\int_0^t \mathcal{G} dB_\tau)$ for every $g \in \mathcal{G}$, which implies that $\|\mathbb{I}_{[0,t]}g\|^2 = E\int_0^t |g_\tau|^2 d\tau = E|\mathcal{J}_t(g)|^2 \leq M_t$ for every $g \in \mathcal{G}$. Thus $\mathbb{I}_{[0,t]}\mathcal{G}$ is a bounded subset of $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$. ■

Let us observe that boundedness of a set $\mathbb{I}_{[0,t]}\mathcal{G}$ is not sufficient for square integrable boundedness of a generalized set-valued integral $\int_0^t \mathcal{G} dB_\tau$. Indeed, by virtue of results of [11] there exists an integrably bounded set-valued $\mathbb{F}$-nonanticipative process $G = (G_t)_{t\geq 0}$ such that $S_{\mathbb{F}}(G)$ is a bounded subset of $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ and $E\|\int_0^t G_\tau dB_\tau\|^2 = \infty$. Taking $\mathcal{G} = S_{\mathbb{F}}(G)$ we obtain for every fixed

$t \geq 0$ a bounded subset $\mathbb{I}_{[0,t]}\mathcal{G}$ of $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ such that $E\| \int_0^t \mathcal{G} dB_\tau \|^2 = \infty$, because $\int_0^t \mathcal{G} dB_\tau = \int_0^t G_\tau dB_\tau$.

We shall present now the following sufficient condition for square integrable boundedness of generalized set-valued stochastic integrals.

**Theorem 4.3.** *Let $\mathcal{G} \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ be a nonempty set and $(g^n)_{n=1}^\infty \subset \mathcal{G}$ a sequence such that $\mathrm{cl}_{\mathbb{L}}\, \mathcal{G} = \mathrm{cl}_{\mathbb{L}}\{g^n : n \geq 1\}$ and $\sum_{n=1}^\infty \|g^n\|^2 < \infty$, where $\|\cdot\|$ is the norm of $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$. Then a generalized set-valued stochastic integral $\int_0^t \mathcal{G} dB_\tau$ is square integrably bounded for every $t \geq 0$.*

***Proof.*** Let $\sum_{n=1}^\infty \|g^n\|^2 < \infty$. Similarly as in the proof of Lemma 4.1, for every $p \geq 1$ we obtain

$$E\left\| \int_0^t \mathcal{G}_p dB_\tau \right\|^2 \leq \sum_{j=1}^p E\int_0^t |g_\tau^j|^2 d\tau \leq \sum_{n=1}^\infty \|g^n\|^2 < \infty,$$

where $\mathcal{G}_p = \{g^1, \ldots, g^p\}$. By Theorem 3.6 we have $\int_0^t \mathcal{G} dB_\tau = \mathrm{Lim} \int_0^t \mathcal{G}_p dB_\tau$ a.s. for every $t \geq 0$. Therefore, for every $t \geq 0$ we get

$$\begin{aligned}
E\left\| \int_0^t \mathcal{G} dB_\tau \right\|^2 &= \sup\left\{ E|u|^2 : u \in S_t\left( \mathrm{Lim} \int_0^t \mathcal{G}_p dB_\tau \right) \right\} \\
&= \sup\left\{ E|u|^2 : u \in \mathrm{Lim}\, S_t\left( \int_0^t \mathcal{G}_p dB_\tau \right) \right\} \\
&= \sup\left\{ E|u|^2 : u \in \bigcup_{p=1}^\infty \left[ S_t\left( \int_0^t \mathcal{G}_p dB_\tau \right) \right] \right\} \leq \sup_{p\geq 1}\left\{ E|u|^2 : u \in S_t\left( \int_0^t \mathcal{G}_p dB_\tau \right) \right\}.
\end{aligned}$$

Hence, by ([1], Th. 2.2) it follows that $S_t(\int_0^t \mathcal{G} dB_\tau)$ is a bounded subset in the space $\mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{R}^d)$. Thus a set-valued integral $\int_0^t \mathcal{G}_\tau dB_\tau$ is square integrably bounded for every $t \geq 0$. ■

**Corollary 4.1.** *For every set $\mathcal{G}_p = \{g^i : 1 \leq i \leq p\} \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ a generalized set-valued stochastic integral $\int_0^t \mathcal{G}_p dB_\tau$ is square integrably bounded for every $t \geq 0$.*

**Remark 4.1.** Immediately from (iii) of Lemma 3.1 and (ii) of Theorem 3.4 it follows that for every nonempty set $\mathcal{G} \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ a generalized set-valued stochastic integral $\int_0^t \mathcal{G} dB_\tau$ is square integrably bounded if and only if $\int_0^t \mathrm{co}\, \mathcal{G} dB_\tau$ is square integrably bounded.

We shall prove now some results dealing with estimations of the Hausdorff distance between generalized set-valued stochastic integrals.

**Theorem 4.3.** *For every $m$-dimensional $\mathbb{F}$-Brownian motion $B = (B_t)_{t\geq 0}$ defined on a complete filtered probability space $\mathcal{P}_{\mathbb{F}}$, and every sets $\{f^i : 1 \leq i \leq p\}$, $\{g^i : 1 \leq i \leq p\} \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ one has*

$$
\begin{aligned}
(4.1)\qquad & Eh^2\left(\int_0^t \{f^i : 1 \leq i \leq p\}dB_\tau, \int_0^t \{g^i : 1 \leq i \leq p\}dB_\tau\right) \\
& \leq p \cdot H^2(\{f^1, \ldots, f^p\}, \{g^1, \ldots, g^p\}),
\end{aligned}
$$

*where $H$ is the Hausdorff metric in $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$.*

***Proof.*** Let $p \geq 1$ be fixed. By virtue of ([1], Th. 2.2) we have

$$
\begin{aligned}
& E\overline{h}^2\left(\int_0^t \{f^i : 1 \leq i \leq p\}dB_\tau, \int_0^t \{g^i : 1 \leq i \leq p\}dB_\tau\right) \\
&= \sup\left\{ \inf\left\{ E|u - v|^2 : v \in S_t\left(\int_0^t \{g^i : 1 \leq i \leq p\}dB_\tau\right)\right\} : \right. \\
&\quad \left. u \in S_t\left(\int_0^t \{f^i : 1 \leq i \leq p\}dB_\tau\right)\right\} \\
&= \sup\{\inf\{E|u - v|^2 : v \in \mathrm{dec}\{\mathcal{J}_t(g^i) : 1 \leq i \leq p\}\} : u \in \mathrm{dec}\{\mathcal{J}_t(f^i) : 1 \leq i \leq p\}\}.
\end{aligned}
$$

For every $u \in \mathrm{dec}\{\mathcal{J}_t(f^i) : 1 \leq i \leq p\}$ one has

$$
\inf\{E|u - v|^2 : v \in \mathrm{dec}\{\mathcal{J}_t(g^i) : 1 \leq i \leq p\}\} \leq \min\{E|u - \mathcal{J}_t(g^i)|^2 : 1 \leq i \leq p\}.
$$

On the other hand, for every $u \in \mathrm{dec}\{\mathcal{J}_t(f^i) : 1 \leq i \leq p\}$ there is $(A_j)_{j=1}^p \in \Pi(\Omega, \mathcal{F}_t)$ such that $u = \sum_{j=1}^p \mathbb{1}_{A_j} \mathcal{J}_t(f^j)$. Then

$$
\begin{aligned}
& E\overline{h}^2\left(\int_0^t \{f^i : 1 \leq i \leq p\}dB_\tau, \int_0^t \{g^i : 1 \leq i \leq p\}dB_\tau\right) \\
& \leq \sup\left\{ \min_{1\leq i\leq p} E\left[\sum_{j=1}^p \mathbb{1}_{A_j} |\mathcal{J}_t(f^j) - \mathcal{J}_t(g^i)|^2\right] : (A_j)_{j=1}^p \in \Pi(\Omega, \mathcal{F}_t)\right\} \\
& \leq p \cdot \max_{1\leq j\leq p}\left\{ \min_{1\leq i\leq p} E|\mathcal{J}_t(f^j) - \mathcal{J}_t(g^i)|^2\right\} \leq p \cdot H^2(\{f^1, \ldots, f^p\}, \{g^1, \ldots, g^p\}).
\end{aligned}
$$

In a similar way we also get $E\overline{h}^2(\int_0^t \{g^i : 1 \leq i \leq p\}dB_\tau, \int_0^t \{f^i : 1 \leq i \leq p\}dB_\tau) \leq p \cdot H^2(\{f^1, \ldots, f^p\}, \{g^1, \ldots, g^p\})$. Then (4.1) is satisfied. ∎

**Corollary 4.2.** *If the assumptions of Theorem* 4.3 *are satified then*

$$Eh^2\left(\int_0^t \operatorname{co}\{f^i : 1 \le i \le p\}dB_\tau, \int_0^t \operatorname{co}\{g^i : 1 \le i \le p\}dB_\tau\right) \tag{4.2}$$

$$\le Eh^2\left(\int_0^t \{f^i : 1 \le i \le p\}dB_\tau, \int_0^t \{g^i : 1 \le i \le p\}dB_\tau\right) \le p \cdot E\int_0^t \max_{1\le j\le p} |f^j_\tau - g^j_\tau|^2 d\tau.$$

**Remark 4.2.** In a similar way we can verify that for every sequences $(f^n)_{n=1}^\infty$, $(g^n)_{n=1}^\infty \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ such that $\sum_{n=1}^\infty \|f^n\|^2 < \infty$, and $\sum_{n=1}^\infty \|g^n\|^2 < \infty$, one has

$$Eh^2\left(\int_0^t \mathcal{F}dB_\tau, \int_0^t \mathcal{G}dB_\tau\right) \le E\int_0^t \sup_{j\ge 1} |f^j_\tau - g^j_\tau|^2 d\tau \tag{4.2}$$

where $\mathcal{F} = \operatorname{co}\{f^n : n \ge 1\}$ and $\mathcal{G} = \operatorname{co}\{g^n : n \ge 1\}$.

## 5. Generalized indefinite set-valued stochastic integrals

Throughout this section we shall consider a nonempty set $\mathcal{G}^p = \overline{\operatorname{co}}\{g^1, \dots, g^p\}$, for a given finite set $\{g^1, \dots, g^p\} \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$. Given an $m$-dimensional $\mathbb{F}$-Brownian motion $B = (B_t)_{t\ge 0}$ defined on $\mathcal{P}_{\mathbb{F}}$, the set-valued $\mathbb{F}$-adapted stochastic process $(\int_0^t \mathcal{G}^p dB_\tau)_{t\ge 0}$ is called the generalized indefinite set-valued stochastic integral of $\mathcal{G}_p$. Immediately from Lemma 4.1 and the properties of the Hausdorff metric $h$ we have $E\|\int_0^t \mathcal{G}^p dB_\tau\|^2 \le p \cdot E\int_0^t \max_{1\le i\le p} |g^i_\tau|^2 d\tau$. We shall prove that the generalized indefinite set-valued stochastic integral $(\int_0^t \mathcal{G}^p dB_\tau)_{t\ge 0}$ is a set-valued submartingale, i.e., such that $\int_0^s \mathcal{G}^p dB_\tau \subset E[\int_0^t \mathcal{G}^p dB_\tau | \mathcal{F}_s]$ a.s. for every $0 \le s < t < \infty$. To do that let us observe (see [7], Th. 3.2) that, by square integrable boundedness of a set-valued generalized set-valued stochastic integral $\int_0^t \mathcal{G}^p dB_\tau$, we have $\int_0^t \mathcal{G}^p dB_\tau = \int_0^s \mathcal{G}^p dB_\tau + \int_s^t \mathcal{G}^p dB_\tau$ a.s. for every $0 \le s < t \le< \infty$.

**Lemma 5.1.** *Let $B = (B_t)_{t\ge 0}$ be an $m$-dimensional $\mathbb{F}$-Brownian motion defined on $\mathcal{P}_{\mathbb{F}}$ and $\mathcal{G}^p \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ be such as above. For every $0 \le s < t < \infty$ one has $\int_0^s \mathcal{G}^p dB_\tau = E[\mathcal{J}_t(\mathcal{G}^p)|\mathcal{F}_s]$ a.s. for every $p \ge 1$.*

***Proof.*** By the definitions of generalized set-valued integrals and the generalized set-valued conditional expectations we get

$$S_{\mathcal{F}_s}\left(\int_0^s \mathcal{G}^p dB_\tau\right) = \overline{\operatorname{dec}}_{\mathcal{F}_s}\{\mathcal{J}_s(g) : g \in \mathcal{G}^p\}$$

$$= \overline{\text{dec}}_{\mathcal{F}_s}\{E[\mathcal{J}_t(g)|\mathcal{F}_s] : g \in \mathcal{G}^p\} = \overline{\text{dec}}_{\mathcal{F}_s}\{E[u|\mathcal{F}_s] : u \in \mathcal{J}_t(\mathcal{G}^p)\}$$

$$= S_{\mathcal{F}_s}(E[\mathcal{J}_t(\mathcal{G}^p)|\mathcal{F}_s]).$$

Then $\int_0^s \mathcal{G}^p dB_\tau = E[\mathcal{J}_t(\mathcal{G}^p)|\mathcal{F}_s]]$ a.s. for every $0 \leq s < t < \infty$. ■

**Corollary 5.1.** *If the assamptions of Lemma* 5.1 *are satisfied then a set-valued stochastic process* $(\int_0^t \mathcal{G}^p dB_\tau)_{t\geq 0}$ *is a set-valued submartingale for every* $p \geq 1$.

***Proof.*** The result follows immediately from Lemma 5.1, an inclusion $E[\mathcal{J}_t(\mathcal{G}^p)|\mathcal{F}_s] \subset E[\overline{\text{dec}}_{\mathcal{F}_t}\{\mathcal{J}_t(\mathcal{G}^p)\}|\mathcal{F}_s]$ for $0 \leq s < t < \infty$ and the equality $S_t(\int_0^t \mathcal{G}^p dB_\tau) = \overline{\text{dec}}_{\mathcal{F}_t}\{\mathcal{J}_t(\mathcal{G}^p)\}$. Indeed, we have

$$E[\overline{\text{dec}}_{\mathcal{F}_t}\{\mathcal{J}_t(\mathcal{G}^p)\}|\mathcal{F}_s] = E\left[S_t\left(\int_0^t \mathcal{G}^p dB_\tau\right)\middle|\mathcal{F}_s\right] = E\left[\int_0^t \mathcal{G}^p dB_\tau\middle|\mathcal{F}_s\right].$$

Then $E[\mathcal{J}_t(\mathcal{G}^p)|\mathcal{F}_s] \subset E[\int_0^t \mathcal{G}^p dB_\tau|\mathcal{F}_s]$, for $0 \leq s < t < \infty$, which implies that

$$S_{\mathcal{F}_s}\left(\int_0^s \mathcal{G}^p dB_\tau\right) = S_{\mathcal{F}_s}(E[\mathcal{J}_t(\mathcal{G}^p)|\mathcal{F}_s]) \subset S_{\mathcal{F}_s}\left(E\left[\int_0^t \mathcal{G}^p dB_\tau\middle|\mathcal{F}_s\right]\right)$$

for every $0 \leq s < t \leq T$. Thus $\int_0^s \mathcal{G}^p dB_\tau \subset E[\int_0^t \mathcal{G}^p dB_\tau|\mathcal{F}_s]$ a.s. for every $0 \leq s < t \leq T$. ■

**Lemma 5.2.** *Let* $B = (B_t)_{t\geq 0}$ *be an* $m$*-dimensional* $\mathbb{F}$*-Brownian motion defined on* $\mathcal{P}_{\mathbb{F}}$*, and* $\mathcal{G}^p \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ *be such that as above. A real-valued process* $(\| \int_0^t \mathcal{G}^p dB_\tau\|)_{t\geq 0}$ *is a positive submartingale.*

***Proof.*** By Corollary 5.1 the set-valued process $(\int_0^t \mathcal{G}^p dB_\tau)_{t\geq 0}$ is a set-valued submartingale. Therefore, $\| \int_0^s \mathcal{G}^p dB_\tau\| \leq \|E[\int_0^t \mathcal{G}^p dB_\tau|\mathcal{F}_s]\|$ a.s. for every $0 \leq s < t < \infty$. By ([1], Th. 5.2) it follows that $\|E[\int_0^t \mathcal{G}^p dB_\tau|\mathcal{F}_s]\| \leq E[\| \int_0^t \mathcal{G}^p dB_\tau\||\mathcal{F}_s]$ a.s. for every $0 \leq s < t < \infty$. Then, $\| \int_0^s \mathcal{G}^p dB_\tau\| \leq E[\| \int_0^t \mathcal{G}^p dB_\tau\||\mathcal{F}_s]$ a.s. for every $0 \leq s < t < \infty$. ■

Immediately from the definition of a generalized set-valued integrals it follows that they are $\mathbb{F}$-adapted. We shall prove now that by the assumptions of Lemma 5.2, they are also continuous. We begin with the following lemma.

**Lemma 5.3.** *Let* $B = (B_t)_{t\geq 0}$ *be an* $m$*-dimensional* $\mathbb{F}$*-Brownian motion,* $T > 0$ *and* $\mathcal{G}^p \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ *be such as above. For every* $\lambda > 0$ *one has*

$$P\left(\left\{\sup_{0\le t\le T}\left\|\int_0^t \mathcal{G}^p dB_\tau\right\|\ge\lambda\right\}\right)\le p/\lambda^2 E\int_0^T \max_{1\le k\le p}|g_\tau^k|^2\tau.$$

***Proof.*** It was proved in Lemma 5.2, that a real-valued stochastic process $(\|\int_0^t \mathcal{G}dB_\tau\|)_{t\ge0}$ is a positive submartingale. Now, the result follows immediately from Chebyshev's and Doob's inequalities, and Lemma 4.1. Indeed, one has

$$P\left(\left\{\sup_{0\le t\le T}\left\|\int_0^t \mathcal{G}^p dB_\tau\right\|\ge\lambda\right\}\right)\le 1/\lambda^2 E\left[\sup_{0\le t\le T}\left\|\int_0^t \mathcal{G}^p dB_\tau\right\|^2\right]$$
$$\le 1/\lambda^2\sup_{0\le t\le T}E\left[\left\|\int_0^t \mathcal{G}^p dB_\tau\right\|^2\right]\le p/\lambda^2 E\int_0^T\max_{1\le k\le p}|g_\tau^k|^2 d\tau.$$

■

We shall prove now that a set-valued stochastic process $(\int_0^t \mathcal{G}^p dB_\tau)_{t\ge0}$ is continuous.

**Theorem 5.4.** *Let $B=(B_t)_{t\ge0}$ be an $m$-dimensional $\mathbb{F}$-Brownian motion defined on $\mathcal{P}_\mathbb{F}$. For every finite set $\{g^1,\dots,g^p\}\subset\mathbb{L}^2(\mathbb{R}^+\times\Omega,\Sigma_\mathbb{F},\mathbb{R}^{d\times m})$ a set-valued stochastic process $(\int_0^t \mathcal{G}^p dB_\tau)_{t\ge0}$ with $\mathcal{G}^p=\overline{\mathrm{co}}\,\{g^1,\dots,g^p\}$ is continuous.*

***Proof.*** Let $\{g^1,\dots,g^p\}\subset\mathbb{L}^2(\mathbb{R}^+\times\Omega,\Sigma_\mathbb{F},\mathbb{R}^{d\times m})$ be given. By the additivity property of generalized set-valued stochastic integrals and Lemma 5.3, for every $t_0\ge0$, $\delta>0$ and $n\ge1$ we obtain

$$P\left[\left\{\sup_{t_0\le t\le t_0+\delta}h\left(\int_0^t\mathcal{G}^p dB_\tau,\int_0^{t_0}\mathcal{G}^p dB_\tau\right)>1/2^n\right\}\right]$$
$$=P\left[\left\{\sup_{t_0\le t\le t_0+\delta}h\left(\int_0^{t_0}\mathcal{G}^p dB_\tau+\int_{t_0}^t\mathcal{G}^p dB_\tau,\int_0^{t_0}\mathcal{G}^p dB_\tau\right)>1/2^n\right\}\right]$$
$$=P\left[\left\{\sup_{t_0\le t\le t_0+\delta}h\left(\int_{t_0}^t\mathcal{G}^p dB_\tau,\{0\}\right)>1/2^n\right\}\right]$$
$$=P\left[\left\{\sup_{t_0\le t\le t_0+\delta}\left\|\int_{t_0}^t\mathcal{G}^p dB_\tau\right\|>1/2^n\right\}\right]\le 2^{2n}p\cdot E\int_{t_0}^{t_0+\delta}\max_{1\le k\le p}|g_\tau^k|^2 d\tau.$$

For every $n\ge1$ there is $\delta_n>0$ such that $E\int_{t_0}^{t_0+\delta_n}\max_{1\le k\le p}|g_\tau^k|^2 d\tau\le 1/(p\cdot2^{3n})$. Therefore, for every $n\ge1$ one has

$$P\left[\left\{\sup_{t_0\le t\le t_0+\delta_n}h\left(\int_0^t\mathcal{G}^p dB_\tau,\int_0^{t_0}\mathcal{G}^p dB_\tau\right)>1/2^n\right\}\right]\le 1/2^n,$$

which implies that $\sum_{n=1}^{\infty} P(\Lambda_n) < \infty$, where

$$\Lambda_n = \left\{\omega \in \Omega : \sup_{t_0 \le t \le t_0+\delta_n} h\left[\left(\int_0^t \mathcal{G}^p dB_\tau\right)(\omega), \left(\int_0^{t_0} \mathcal{G}^p dB_\tau\right)(\omega)\right] > 1/2^n\right\}.$$

Therefore, by the Borel-Cantelli lemma we have $P(\bigcap_{n\ge1}\bigcup_{k\ge n}\Lambda_k) = 0$, which is equivalent to $P(\bigcup_{n\ge1}\bigcap_{k\ge n}\Lambda_k^{\sim}) = 1$, where $\Lambda_k^{\sim} = \Omega \setminus \Lambda_k$. Hence it follows that for every $\omega \in \bigcup_{n\ge1}\bigcap_{k\ge n}\Lambda_k^{\sim}$ there is $n(\omega) \ge 1$ such that for every $k \ge n(\omega)$ one has $\sup_{t_0\le t\le t_0+\delta_k} h[(\int_0^t \mathcal{G}dB_\tau)(\omega), (\int_0^{t_0} \mathcal{G}dB_\tau)(\omega)] \le 1/2^k$, that can be written in the form $\sup_{t_0\le t\le t_0+\delta_k} \| \int_{t_0}^t \mathcal{G}dB_\tau)(\omega)\| \le 1/2^k$. For every $k \ge 1$, every positive $\delta \le \delta_k$ we have

$$\sup_{t_0\le t\le t_0+\delta} \left\|\int_{t_0}^t \mathcal{G}^p dB_\tau\right\| \le \sup_{t_0\le t\le t_0+\delta_k} \left\|\int_{t_0}^t \mathcal{G}^p dB_\tau\right\|,$$

which implies that for a.e. $\omega \in \Omega$ there is $n(\omega) \ge 1$ such that for every $k \ge n(\omega)$ there is $\delta_k > 0$ such that for every $\delta \in (0, \delta_k]$ we have $\sup_{t_0\le t\le t_0+\delta} \| \int_{t_0}^t \mathcal{G}^p dB_\tau)(\omega)\|$ $\le 1/2^k$. Then

$$\lim_{\delta\to0} \sup_{t_0\le t\le t_0+\delta} h\left[\left(\int_0^t \mathcal{G}^p dB_\tau\right)(\omega), \left(\int_0^{t_0} \mathcal{G}^p dB_\tau\right)(\omega)\right] = 0$$

for a.e. $\omega \in \Omega$. In a similar way we obtain

$$\lim_{\delta\to0} \sup_{t_0-\delta\le t\le t_0} h\left[\left(\int_0^t \mathcal{G}^p dB_\tau\right)(\omega), \left(\int_0^{t_0} \mathcal{G}^p dB_\tau\right)(\omega)\right] = 0$$

for a.e. $\omega \in \Omega$, with $0 < \delta < t_0$ for every $t_0 > 0$. Thus the set-valued stochastic process $(\int_0^t \mathcal{G}^p dB_\tau)_{t\ge0}$ is continuous. ∎

**Remark 5.1.** In a similar way we can prove that for a given $m$-dimensional $\mathbb{F}$-Brownian motion $B = (B_t)_{t\ge0}$ defined on $\mathcal{P}_{\mathbb{F}}$ and every sequence $(g^n)_{n=1}^{\infty} \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ such that $\sum_{n=1}^{\infty} \|g^n\|^2 < \infty$, a set-valued stochastic process $(\int_0^t \mathcal{G}dB_\tau)_{t\ge0}$ defined by the set $\mathcal{G} = \text{co}\{g^n : n \ge 1\}$, is a continuous set-valued submartingale.

**Remark 5.2.** Because of unboundedness of set-valued stochastic integrals, defined in the paper [4], we can not consider stochastic differential equations $x_t = x_0 + \int_0^t F(\tau, x_\tau)d\tau + \int_0^t G(\tau, x_\tau)dB_\tau$ with set-valued solutions in the general case, i.e., if $G$ is not a singleton. Instead of, we can consider stochastic differential equation of the form $x_t = x_0 + \int_0^t F(\tau, x_\tau)d\tau + \int_0^t \mathcal{G}_G(x)dB_\tau$, where $\mathcal{G}_G(x) = \text{co}\{g \circ x : g \in \mathcal{G}\}$ for every nonempty set $\mathcal{G}$ of Carathèodory selectors of $G$ such that a set-valued stochastic integral $\int_0^t \mathcal{G}_G(x)dB_\tau$ is square integrably bounded and $(g \circ x)_t(\omega) = g(t, x_t(\omega))$ for every $(t, \omega) \in \mathbb{R}^+ \times \Omega$.

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