Discussiones Mathematicae
Differential Inclusions, Control and Optimization 33 (2013) 149–170
doi:10.7151/dmdico.1149

EXISTENCE RESULTS FOR IMPULSIVE SEMILINEAR FRACTIONAL DIFFERENTIAL INCLUSIONS WITH DELAY IN BANACH SPACES

Hadda Hammouche

Département de Mathématiques
Université de Ghardaia, Algérie
e-mail: h.hammouche@yahoo.fr

Kaddour Guerbati

Département de Mathématiques
Université de Ghardaia, Algérie
e-mail: guerbati_k@yahoo.com

Mouffak Benchohra

Laboratoire de Mathématiques
Université de Sidi Bel Abbès, Algérie
e-mail: benchohra@univ-sba.dz

and

Nadjat Abada

Département de Mathématiques
E.N.S Constantine, Algérie
e-mail: n65abada@yahoo.fr

Abstract

In this paper, we introduce a new concept of mild solution of some class of semilinear fractional differential inclusions of order $0 < \alpha < 1$. Also we establish an existence result when the multivalued function has convex values. The result is obtained upon the nonlinear alternative of Leray-Schauder type.

Keywords: fractional calculus, caputo fractional derivative, multivalued map, differential inclusions, mild solution, fixed point.

2010 Mathematics Subject Classification: 34G10, 26A33, 34A12, 42B05, 34A37, 34G25, 34K30, 34K35, 34K45.

1. Introduction

Our aim in this paper is to study the existence of mild solutions for fractional semilinear differential inclusions of the form:

$$ {}^cD^{\alpha}_{t_k}y(t) - Ay(t) \in F(t, y_t),\ t \in J_k := (t_k, t_{k+1}],\ k = 0, \dots, m \tag{1}$$

$$ \Delta y|_{t=t_k} = I_k(y(t_k^-)),\quad k = 1, \dots, m \tag{2}$$

$$ y(t) = \phi(t) \qquad t \in [-r, 0], \tag{3}$$

where $0 < \alpha < 1$, $F : [0, b] \times D \to \mathcal{P}(E)$ is a given valued multivalued map, $I_k : E \to E$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $\mathcal{P}(E)$ is the collection of all subsets of E, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, where $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$, $D = \{\psi : [-r, 0] \to E, \psi$ continuous everywhere except for a finite number of points at which $\psi(s^-)$ and $\psi(s^+)$ exist and $\psi(s^-) = \psi(s)\}$, $\phi \in D$, $A : D(A) \subset E \to E$ is the generator of an $\alpha-$resolvent operator function ($\alpha-$ROF for short) S_α. For any continuous function y defined on $[-r, b]\backslash\{t_1, t_2, \dots, t_m\}$ and any $t \in J := [0, b]$, we denote by y_t the element of D defined by

$$ y_t(\theta) = y(t + \theta),\quad \theta \in [-r, 0]. $$

Here $y_t(\cdot)$ represents the history of the state from $t - r$, up to the present time t. We assume as usual in the theory of impulsive differential equations that the solution of (1)–(3) is such that at the point of discontinuity t_k satisfies $y(t_k) = y(t_k^-)$.

Fractional differential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering ([4, 14, 20, 22]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [1], Kilbas *et al.* [15], Lakshmikantham *et al.* [16], Miller and Ross [19], Podlubny [20], Samko *et al.* [21], and the papers by Agarwal *et al.* [2], Anguraj *et al.* [3], Belmekki *et al.* [6], Benchohra *et al.* [5, 7, 8] and the references therein.

The Cauchy problem for abstract differential equations involving Riemann-Liouville fractional integral have been considered by several author; see for instance, Cuevas and De Souza [11, 12], Benchohra *et al.* [9] and the reference therein. To our knowledge, there are very few results for impulsive fractional differential inclusions. The results of the present paper extend and complement those obtained in the absence of the impulse functions I_k.

This paper is organized as follow, in Section 2 we introduce some preliminaries that will be used in the sequel, in Section 3 we give a new definition to the mild solution of problem (1)–(3) and we establish our existence result.

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts which are used throughout this paper.

$C[J, E]$ is the Banach space of all continuous functions from J into E with the norm

$$\|u\| = \sup\{|u(t)| : t \in J\}$$

$L^1[J, E]$ denotes the Banach space of measurable functions $u : J \to E$ which are Bochner integrable normed by

$$\|u\|_{L^1} = \int_0^b |u(t)|dt.$$

Let $L^\infty(J, \mathbb{R})$ be the Banach space of measurable functions $u : J \to \mathbb{R}$ which are essentially bounded, equipped with the norm

$$\|y\|_{L^\infty} = \inf\{c > 0 : |u(t)| \leq c, \ a.e. \ t \in J\}.$$

Let $(X, |\cdot|)$ be a normed space. Denote by $\mathcal{P}_{cl}(X), \mathcal{P}_b(X), \mathcal{P}_{cp}(X), \mathcal{P}_c(X), \mathcal{P}_{cl,c}(X)$, $\mathcal{P}_{cp,c}(X)$ the set $\{Y \in \mathcal{P}(X)\}$ such that Y is closed, bounded, compact, convex, closed and convex, compact and convex, respectively. A multivalued function $F : X \to \mathcal{P}(X)$ is called **convex (closed)** valued if $F(x)$ is convex (closed) for all $x \in X$. F is called **bounded** valued on bounded set B if $F(B) = \bigcup_{x \in B} F(x)$ is bounded in X for all $B \in \mathcal{P}_b(X)$ i.e., $\sup_{x \in B}\{\sup\{|u| : u \in F(x)\}\} < \infty$. F is called **upper semi-continuous (u.s.c)** on X if for each $x_0 \in X$ the set $F(x_0)$ is nonempty closed subset of X and if for each open set N of X containing $F(x_0)$, there exists an open neighborhood N_0 of x_0 such that $F(N_0) \subseteq N$. In other words F is u.s.c if the set $F^{-1}(A) = \{x \in X : F(x) \subset A\}$ is open in X for every open set A in X. F is called **compact** if for every M bounded subset of X, $F(M)$ is relatively compact. Finally F is called **completely continuous** if it is upper semi-continuous and compact on X. The following definitions are used in the sequel.

Definition. A multivalued map $F : J \times D \to \mathcal{P}(E)$ is said to be Carathéodory if

(i) $t \mapsto F(t, u)$ is measurable for each $u \in D$

(ii) $u \mapsto F(t,u)$ is u.s.c. for almost all $t \in J$.

For each $y : [-r,b] \to E$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1(J,E) : v(t) \in F(t,y_t) \text{ a.e. } t \in J\}.$$

The Riemann-Liouville fractional operators are defined as follows (see [19, 20]).

Definition. The fractional integral operator I^α of order $\alpha > 0$ of a continuous function $f(t)$ is given by

$$I_t^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds.$$

We can write $I_t^\alpha f(t) = f(t) * \psi_\alpha(t)$ where $\psi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$ and $\psi_\alpha(t) = 0$ for $t \leq 0$ and $\psi_\alpha(t) \to \delta(t)$ (the delta function) as $\alpha \to 0$ (see [15, 19, 20]).

Definition. The $\alpha-$th Riemann-Liouville fractional-order derivative of f, is defined by:

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s)ds,$$

here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition [15]**.** For a function f given on the interval $[a,b]$, the Caputo fractional-order derivative of order α of f, is defined by

$$({}_{a+}^{c}D_t^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,$$

where $n = [\alpha] + 1$.

Therefore, for $0 < \alpha < 1$, The Caputo's fractional derivative for $t \in [0,b]$ is

$$({}_0^c D_t^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)ds.$$

Definition [10]**.** Let $\alpha > 0$. A function $S_\alpha : \mathbb{R}_+ \to B(X)$ is called an α-resolvent operator(α-ROF) if the following conditions are satisfied:

(a) $S_\alpha(.)$ is strongly continuous on $\mathbb{R}_+$ and $S_\alpha(0) = I$

(b) $S_\alpha(s)S_\alpha(t) = S_\alpha(t)S_\alpha(s)$ for all $s,t \geq 0$

(c) the functional equation

$$S_\alpha(s)I_t^\alpha S_\alpha(t) - I_s^\alpha S_\alpha(s)S_\alpha(t) = I_t^\alpha S_\alpha(t) - I_s^\alpha S_\alpha(s)$$

holds for all $s, t \geq 0$.

The generator A of S_α is defined by:

$$D(A) := \left\{ x \in X : \lim_{t\to 0^+} \frac{S_\alpha(t)x - x}{\psi_{\alpha+1}(t)} \quad exists \right\}$$

and

$$Ax = \lim_{t\to 0^+} \frac{S_\alpha(t)x - x}{\psi_{\alpha+1}(t)}, \quad x \in D(A).$$

Definition. An α-ROF S_α is said to be exponentially bounded if there exist constants $M \geq 0, \omega \geq 0$ such that:

$$\|S_\alpha(t)\| \leq Me^{\omega t}, \qquad t \geq 0,$$

in this case we write $A \in \mathcal{C}_\alpha(M, \omega)$.

Proposition 1. *Let S_α be an α-ROF generated by the operator A. The following assertions hold:*

(a) $S_\alpha(t)D(A) \subset D(A)$ *and* $AS_\alpha(t)x = S_\alpha(t)Ax$ *for all* $x \in D(A)$ *and* $t \geq 0$.

(b) *For all* $x \in X$, $I_t^\alpha S_\alpha(t)x \in D(A)$ *and*

$$S_\alpha(t)x = x + AI_t^\alpha S_\alpha(t)x, \qquad t \geq 0.$$

(c) $x \in D(A)$ *and* $Ax = y$ *if and only if*

$$S_\alpha(t)x = x + AI_t^\alpha S_\alpha(t)x, \qquad t \geq 0.$$

(d) *A is closed, densely defined.*

Proposition 2. *Let $\alpha > 0$. $A \in \mathcal{C}_\alpha(M, \omega)$ if and only if $(\omega^\alpha, \infty) \subset \rho(A)$ and there exists a stongly continuous function $S_\alpha : \mathbb{R}_+ \to B(X)$ such that $\|S_\alpha(t)\| \leq Me^{\omega t}$ and*

$$\int_0^\infty e^{-\lambda t} S_\alpha(t)x dt = \lambda^{\alpha-1} R(\lambda^\alpha, A)x \qquad \lambda > \omega$$

for all $x \in X$. Furthermore, S_α is the α-ROF generated by the operator A.

For more detail see [18].

3. Main result

In order to define the mild solution to the problem (1–3), we shall consider the following space

$$PC = \Big\{ y : [0,b] \to E : y \in C((t_k, t_{k+1}], E);\ k = 0, \dots, m \quad \text{such that} \\ y(t_k^-),\ y(t_k^+) \text{ exist with } y(t_k) = y(t_k^-), k = 1, \dots, m \Big\}$$

which is a Banach space with the norm

$$\|y\|_{PC} := \max\{\|y_k\|_\infty : k = 0, ., ., ., m\}$$

where y_k is the restriction of y to $J_k = [t_k, t_{k+1}]$, $k = 0, \dots, m$.

Set

$$\Omega = \{y : [-r, b] \to E : y \in D \cap PC\}.$$

Also; we introduce the definition of Caputo's derivative in each interval $(t_k, t_{k+1}]$, $k = 0, \dots, m$.

$$({}^c D_{t_k}^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_k}^t (t-s)^{-\alpha} f'(s) ds.$$

Now, we can define a meaning of the mild solution of problem (1–3).

Definition. A function $y \in \Omega$ is called to be mild solution of (1–3) if $y(t) = \phi(t)$ for all $t \in [-r, 0]$, $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1, \dots, m$ and there exists $v(\cdot) \in L^1(J, E)$, such that $v(t) \in F(t, y_t)$, a.e $t \in [0, b]$, and such that y satisfies the following integral equation:

$$(4)\quad y(t) = \begin{cases} S_\alpha(t)\phi(0) + \displaystyle\int_0^t S_\alpha(t-s)v(s)ds & \text{if } t \in [0, t_1], \\ S_\alpha(t-t_k) \displaystyle\prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})\phi(0) \\ + \displaystyle\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j) \times \\ S_\alpha(t_i - s)v(s)ds + \displaystyle\int_{t_k}^t S_\alpha(t-s)v(s)ds \\ + \displaystyle\sum_{i=1}^{k} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j) I_i(y(t_i^-)), & \text{if } t \in (t_k, t_{k+1}]. \end{cases}$$

Let us introduce the following hypotheses:

(H1) assume that A generates a compact α-ROF S_α for $t > 0$ wich is exponentially bounded i.e. There exist constants $M \geq 1, \omega \geq 0$ such that:

$$\|s_\alpha(t)\| \leq Me^{\omega t}, \qquad t \geq 0$$

(H2) $F : J \times D \to \mathcal{P}_{cp,cv}(E)$ is Carathéodory and there exist $p \in L^\infty(J, \mathbb{R})$ and a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that:

$$\|F(t,u)\|_{\mathcal{P}} = \sup\{|v| : v \in F(t,u)\} \leq p(t)\psi(\|u\|_\infty) \quad \text{for all } t \in J,\ u \in D$$

with

$$\int_{C_3}^{\infty} \frac{du}{\psi(u)} du = \infty.$$

Where

$$C_3 = \min\left(C_1, C_2\right).$$

(H3) The functions $I_k : E \to E$ are continuous and there exists a constant $M^* > 0$ such that $|I_k(u)| \leq M^*$ for each $u \in E$, $k = 1, \ldots, m$.

The nonlinear alternative of Leray-Schauder type is used to investigate the existence of solutions of problem (1–3). We need to use the following result due to Lasota and Opial [17].

Lemma 3. *Let E be a Banach space, and F be an L^1-Carathéodory multivalued map with compact convex values, and let $\Gamma : L^1(J, E) \to C(J, E)$ be a linear continuous mapping. Then the operator*

$$\Gamma \circ S_F : C(J, E) \to P_{cp,cv}(C(J, E))$$

is a closed graph operator in $C(J, E) \times C(J, E)$.

Theorem 4. *Under assumptions* (H1)–(H3) *the IVP* (1–3) *has at least one mild solution on* $[-r, b]$.

Proof. Transform the problem (1–3) into a fixed point problem. Consider the multivalued operator: $N : \Omega \to \mathcal{P}(\Omega)$ defined by $N(y) = \{h \in \Omega\}$ such that

$$h(t)=\begin{cases}\phi(t), & \text{if } t\in[-r,0],\\ S_\alpha(t)\phi(0)+\displaystyle\int_0^t S_\alpha(t-s)v(s)ds & \text{if } t\in[0,t_1],\\ S_\alpha(t-t_k)\displaystyle\prod_{i=1}^{k}S_\alpha(t_i-t_{i-1})\phi(0) & \\ +\displaystyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}S_\alpha(t-t_k)\prod_{j=i}^{k-1}S_\alpha(t_{j+1}-t_j) & \\ \quad\times S_\alpha(t_i-s)v(s)ds+\displaystyle\int_{t_k}^{t}S_\alpha(t-s)v(s)ds & \\ +\displaystyle\sum_{i=1}^{k}S_\alpha(t-t_k)\prod_{j=i}^{k-1}S_\alpha(t_{j+1}-t_j)I_i(y(t_i^-)), & \text{if } t\in(t_k,t_{k+1}].\end{cases}$$

Clearly, the fixed points of N are solutions to (1–3).

We shall show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type [13]. The proof will be given in several steps.

Step 1. *$N(y)$ is convex for each $y\in\Omega$.*

Let $h_1,h_2\in N(y)$, then there exist $v_1,v_2\in S_{F,y}$ such that for each $t\in J$

$$h_i=\begin{cases}S_\alpha(t)\phi(0)+\displaystyle\int_0^t S_\alpha(t-s)v_i(s)ds & \text{if } t\in[0,t_1],\\ S_\alpha(t-t_k)\displaystyle\prod_{i=1}^{k}S_\alpha(t_i-t_{i-1})\phi(0) & \\ +\displaystyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}S_\alpha(t-t_k)\prod_{j=i}^{k-1}S_\alpha(t_{j+1}-t_j) & \\ \quad\times S_\alpha(t_i-s)v_i(s)ds+\displaystyle\int_{t_k}^{t}S_\alpha(t-s)v_i(s)ds & \\ +\displaystyle\sum_{i=1}^{k}S_\alpha(t-t_k)\prod_{j=i}^{k-1}S_\alpha(t_{j+1}-t_j)I_i(y(t_i^-)), & \text{if } t\in(t_k,t_{k+1}].\end{cases}\qquad i=1,2$$

Let $0 \le \sigma \le 1$. Then for each $t \in J$ we have:

$$(\sigma h_1 - (1-\sigma)h_2)(t) = \begin{cases} \displaystyle\int_0^t S_\alpha(t-s)\left[\sigma v_1(s) - (1-\sigma)v_2\right] ds & \text{if } t \in [0,t_1], \\ \displaystyle\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j) & \\ \quad \times S_\alpha(t_i - s)\left[\sigma v_1(s) - (1-\sigma)v_2\right] ds & \\ \displaystyle + \int_{t_k}^{t} S_\alpha(t-s)\left[\sigma v_1(s) - (1-\sigma)v_2\right] ds & \text{if } t \in (t_k, t_{k+1}]. \end{cases}$$

Since $S_{F,y}$ is convex (because F has convex values), we have $\sigma h_1 - (1-\sigma)h_2 \in N(y)$.

Step 2. *N maps bounded sets into bounded sets in Ω.*

Let $B_q = \{y \in \Omega : \|y\|_\infty \le q\}$, $q > 0$ a bounded set in Ω. It is equivalent to show that there exists a positive constant l such that for each $y \in B_q$ we have $\|N(y)\| \le l$.

Let $y \in B_q$, then for each $h \in N(y)$, there exists $v \in S_{F,y}$ such that:

$$h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0] \\ \displaystyle S_\alpha(t)\phi(0) + \int_0^t S_\alpha(t-s)v(s)ds & \text{if } t \in [0,t_1], \\ \displaystyle S_\alpha(t-t_k)\prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})\phi(0) & \\ \displaystyle + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j) & \\ \displaystyle \quad \times S_\alpha(t_i - s)v(s)ds + \int_{t_k}^{t} S_\alpha(t-s)v(s)ds & \\ \displaystyle + \sum_{i=1}^{k} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j) I_i(y(t_i^-)), & \text{if } t \in (t_k, t_{k+1}]. \end{cases}$$

Then, for $t \in J$

$$|h(t)| \leq \begin{cases} Me^{\omega t_1}\|\phi(0)\| + Me^{\omega t_1}\int_0^t e^{-\omega s}\|v(s)\|ds & \text{if } t \in [0, t_1], \\ Me^{\omega(t-t_k)}\prod_{i=1}^{k} Me^{\omega(t_i - t_{i-1})}\|\phi(0)\| & \\ + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} Me^{\omega(t-t_k)}\prod_{j=i}^{k-1} Me^{\omega(t_{j+1}-t_j)} & \\ \times Me^{\omega(t_i - s)}\|v(s)\|ds + \int_{t_k}^{t} Me^{\omega(t-s)}\|v(s)\|ds & \\ + \sum_{i=1}^{k-1} Me^{\omega(t-t_k)}\prod_{j=i}^{k-1} Me^{\omega(t_{j+1}-t_j)}\|I_i(y(t_i^-))\|, & \text{if } t \in (t_k, t_{k+1}], \end{cases}$$

which gives

$$|h(t)| \leq \begin{cases} Me^{\omega t_1}\|\phi(0)\| + Me^{\omega t_1}\psi(q)\int_0^t e^{-\omega s}p(s)ds & \\ = l_1 & \text{if } t \in [0, t_1], \\ M^k e^{\omega(t_{k+1})}\|\phi(0)\| & \\ + \sum_{i=1}^{k} M^{k-i+2}e^{\omega(2t_k - t_{i-1})}\psi(q)\int_{t_{i-1}}^{t_i} e^{\omega(-s)}p(s)ds & \\ + Me^{\omega(t_{k+1})}\psi(q)\int_{t_k}^{t} e^{\omega(-s)}p(s)ds & \\ + \sum_{i=1}^{k} M^{k-i+1}M^* e^{\omega(t_{k+1}-t_{i-1})} & \\ = l_k & \text{if } t \in (t_k, t_{k+1}]. \end{cases}$$

Step 3. *N maps bounded sets into equicontinuous sets of Ω.*

Let τ_1, $\tau_2 \in J$ with $\tau_1 < \tau_2$, let B_q be a bounded set in Ω as in Step 2, and let $y \in B_q$and $h \in N(y)$. Then, if $\epsilon > 0$ with $\epsilon < \tau_1 < \tau_2$

$$|h(\tau_2)-h(\tau_1)| \leq \begin{cases} \|S_\alpha(\tau_2)-S_\alpha(\tau_1)\|\|\phi(0)\| \\ + \int_0^{\tau_1-\epsilon} \|S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)\||v(s)|ds \\ + \int_{\tau_1-\epsilon}^{\tau_1} \|S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)\||v(s)|ds \\ + \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2-s)\||v(s)|ds \\ \qquad\qquad\qquad \text{if} \quad \tau_1,\tau_2\in[0,t_1], \end{cases}$$

and

$$|h(\tau_2)-h(\tau_1)| \leq \begin{cases} \|S_\alpha(\tau_2-t_k)-S_\alpha(\tau_1-t_k)\| \prod_{i=1}^{k} S_\alpha(t_i-t_{i-1})\|\phi(0)\| \\ + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} \|S_\alpha(\tau_2-t_k)-S_\alpha(\tau_1-t_k)\| \\ \quad \times \prod_{j=i}^{k-1} \|S_\alpha(t_{j+1}-t_j)\|\|S_\alpha(t_i-s)\||v(s)|ds \\ + \int_{t_k}^{\tau_2} S_\alpha(\tau_2-s)v(s)ds - \int_{t_k}^{\tau_1} S_\alpha(\tau_1-s)v(s)ds \\ + \sum_{i=1}^{k} \|S_\alpha(\tau_2-t_k)-S_\alpha(\tau_1-t_k)\| \prod_{j=i}^{k-1} \|S_\alpha(t_{j+1}-t_j)\| \\ \quad \times \|I_i(y(t_i^-))\|, \\ \qquad\qquad\qquad \text{if} \quad \tau_1,\tau_2\in(t_k,t_{k+1}]. \end{cases}$$

It gives

$$|h(\tau_2)-h(\tau_1)| \leq \begin{cases} \|S_\alpha(\tau_2)-S_\alpha(\tau_1)\|\|\phi(0)\| \\ + \psi(q)\int_0^{\tau_1-\epsilon} \|S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)\|p(s)ds \\ + \psi(q)\int_{\tau_1-\epsilon}^{\tau_1} \|S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)\|p(s)ds \\ + Me^{\omega\tau_2}\psi(q)\int_{\tau_1}^{\tau_2} e^{-\omega s}p(s)ds \\ \qquad\qquad\qquad \text{if} \quad \tau_1,\tau_2\in[0,t_1], \end{cases}$$

and

$$|h(\tau_2)-h(\tau_1)| \leq \begin{cases} \|S_\alpha(\tau_2-t_k)-S_\alpha(\tau_1-t_k)\| \prod_{i=1}^{k} S_\alpha(t_i-t_{i-1})\|\phi(0)\| \\ + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} \|S_\alpha(\tau_2-t_k)-S_\alpha(\tau_1-t_k)\| \\ \quad \times \prod_{j=i}^{k-1}\|S_\alpha(t_{j+1}-t_j)\|\|S_\alpha(t_i-s)\|\|v(s)|ds \\ + \psi(q)\int_{t_k}^{\tau_1-\epsilon}\|S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)\|p(s)ds \\ + \psi(q)\int_{\tau_1-\epsilon}^{\tau_1}\|S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)\|p(s)ds \\ + M\psi(q)e^{\omega\tau_2}\int_{\tau_1}^{\tau_2} e^{-\omega s}p(s)ds \\ + \sum_{i=1}^{k}\|S_\alpha(\tau_2-t_k)-S_\alpha(\tau_1-t_k)\|\prod_{j=i}^{k-1}\|S_\alpha(t_{j+1}-t_j)\| \\ \quad \times \|I_i(y(t_i^-))\|, \\ \qquad\qquad\qquad\qquad \text{if } \tau_1,\tau_2\in(t_k,t_{k+1}]. \end{cases}$$

As $\tau_1 \to \tau_2$ and ϵ becomes sufficiently small, the right-hand side of the above inequality tends to zero, since S_α is a strongly continuous operator and the compactness of S_α for $t>0$ implies the continuity in the uniform operator topology. This proves the equicontinuity for the case where $t \neq t_i, i=1,\ldots,m+1$. It remains to examine the equicontinuity at $t=t_i$. First we prove the equicontinuity at $t=t_i$. We have for some $y \in Bq$, there exists $v \in S_{F,y}$ such that for each $t \in J$:

if $t\in[0,t_1]$,

$$h(t)=S_\alpha(t)\phi(0)+\int_0^t S_\alpha(t-s)v(s)ds$$

if $t\in(t_k,t_{k+1}]$

$$\begin{aligned} h(t) &= S_\alpha(t-t_k)\prod_{i=1}^{k}S_\alpha(t_i-t_{i-1})\phi(0) \\ &+ \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} S_\alpha(t-t_k)\prod_{j=i}^{k-1}S_\alpha(t_{j+1}-t_j)S_\alpha(t_i-s)v(s)ds \\ &+ \int_{t_k}^{t}S_\alpha(t-s)v(s)ds+\sum_{i=1}^{k}S_\alpha(t-t_k)\prod_{j=i}^{k-1}S_\alpha(t_{j+1}-t_j)I_i(y(t_i^-)). \end{aligned}$$

Fix $\delta_1 > 0$ such that $\{t_k, k \neq l\} \cap [t_l - \delta_1, t_l + \delta_1] = \emptyset$. For $0 < \rho < \delta_1$, we have

$$|h(t_l-\rho)-h(t_l)| \leq \begin{cases} \|S_\alpha(t_l-\rho)-S_\alpha(t_l)\|\|\phi(0)\| \\ +\,\psi(q)\displaystyle\int_0^{t_l-\rho}\|S_\alpha(t_l-\rho-s)-S_\alpha(t_l-s)\|p(s)ds \\ +\,Me^{\omega t_l}\psi(q)\displaystyle\int_{t-\rho}^{t_l}e^{-\omega s}p(s)ds \qquad \text{if } \ t_l-\rho, t_l\in[0,t_1], \end{cases}$$

and

$$|h(t_l-\rho)-h(t_l)| \leq \begin{cases} \|S_\alpha(t_l-\rho-t_k)-S_\alpha(t_l-t_k)\| \\ \displaystyle\prod_{i=1}^{k}\|S_\alpha(t_i-t_{i-1})\|\|\phi(0)\| \\ +\,\psi(q)\displaystyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\|S_\alpha(t_l-\rho-t_k)-S_\alpha(t_l-t_k)\| \\ \quad\times\displaystyle\prod_{j=i}^{k-1}\|S_\alpha(t_{j+1}-t_j)\|\|S_\alpha(t_i-s)\||p(s)|ds \\ +\,\psi(q)\displaystyle\int_{t_k}^{t_l-\rho}\|S_\alpha(t_l-\rho-s)-S_\alpha(t_l-s)\|p(s)ds \\ +\,M\psi(q)e^{\omega t_l}\displaystyle\int_{t_l-\rho}^{t_l}e^{-\omega s}p(s)ds \\ +\displaystyle\sum_{i=1}^{k}\|S_\alpha(t_l-\rho-t_k)-S_\alpha(t_l-t_k)\| \\ \quad\times\displaystyle\prod_{j=i}^{k-1}\|S_\alpha(t_{j+1}-t_j)\|.\|I_i(y(t_i^-))\|, \\ \qquad\qquad\qquad \text{if } \ t_l-\rho, t_l\in(t_k,t_{k+1}]. \end{cases}$$

They tends to zero as $\rho \to 0$.

Define

$$\hat{h}_0(t) = h(t), \quad \text{if } \ t \in [0, t_1]$$

and

$$\hat{h}_i(t) = \begin{cases} h(t), & \text{if } \ t \in (t_i, t_{i+1}] \\ h(t_i^+), & \text{if } \ t = t_i. \end{cases}$$

Next, we prove equicontinuity at $t = t_i^+$. Fix $\delta_2 > 0$ such that $\{t_k, k \neq i\} \cap [t_i - \delta_2, t_i + \delta_2] = \emptyset$. First we study the equicontinuity at $t = 0^+$.

If $t \in [0, t_1]$ we have

$$\hat{h}_1(t) = \begin{cases} h(t), & \text{if } t \in (0, t_1] \\ \phi(0), & \text{if } t = 0. \end{cases}$$

For $0 < \rho < \delta_2$, we have

$$|\hat{h}_1(\rho) - \hat{h}_1(0)| \leq \|\phi(0)\| \|S_\alpha(\rho) - I\| + e^{-\omega\rho}\psi(q)\int_0^\rho e^{-\omega s}p(s)ds.$$

The right hand-side tends to zero as $\rho \to 0$ (I is the unitary operator).

Now we study the equicontinuity at $t = t_i^+$, $i \geq 1$. For $0 < \rho < \delta_2$, we have

$$\begin{aligned} |\hat{h}(t_i + \rho) - \hat{h}(t_i)| \leq\ & \|S_\alpha(\rho) - I\| \prod_{j=1}^{i} \|S_\alpha(t_j - t_{j-1})\| \|\phi(0)\| \\ & + \psi(q) \sum_{l=1}^{i} \int_{t_{l-1}}^{t_l} \|S_\alpha(\rho) - I\| \\ & \quad \times \prod_{j=l}^{i-1} \|S_\alpha(t_{j+1} - t_j)\| \|S_\alpha(t_l - s)\| |p(s)| ds \\ & + M\psi(q)e^{\omega(t_i+\rho)} \int_{t_i}^{t_i+\rho} e^{-\omega s}p(s)ds \\ & + \sum_{l=1}^{i} \|S_\alpha(\rho) - I\| \\ & \quad \times \prod_{j=l}^{i-1} \|S_\alpha(t_{j+1} - t_j)\| . \|I_l(y(t_l^-))\|. \end{aligned}$$

The right hand-side tends to zero as $\rho \to 0$.

The equicontinuity for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq \tau_2$ follows from the uniform continuity of ϕ on the interval $[-r, 0]$. As a consequence of Steps 1 and 2 together with Arzelá-Ascoli theorem it suffices to show that N maps B into a precompact set in E.

Let $0 < t^* < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t^*$. For $y \in B$, we define

$$h_\epsilon(t^*) = \begin{cases} S_\alpha(t^*)\phi(0) + \displaystyle\int_0^{t^*-\epsilon} S_\alpha(t^*-\epsilon-s)v(s)ds & \text{if } t^* \in [0,t_1], \\ S_\alpha(t^*-t_k)\displaystyle\prod_{i=1}^{k} S_\alpha(t_i-t_{i-1})\phi(0) & \\ +\displaystyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} S_\alpha(t^*-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j)\times & \\ S_\alpha(t_i-s)v(s)ds + \displaystyle\int_{t_k}^{t^*-\epsilon} S_\alpha(t^*-\epsilon-s)v(s)ds & \\ +\displaystyle\sum_{i=1}^{k} S_\alpha(t^*-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j)I_i(y(t_i^-)), & \text{if } t^* \in (t_k,t_{k+1}], \end{cases}$$

where $v \in S_{F,y}$. Since $S_\alpha(t^*)$ is a compact operator, the set

$$H^\epsilon(t^*) = \{h_\epsilon(t^*) : \quad h_\epsilon \in N(y)\}$$

is precompact in E for every ϵ, $\quad 0 < \epsilon < t^*$. Moreover, for every $h \in N(y)$ we have

$$|h(t^*) - h_\epsilon(t^*)| \le \begin{cases} \psi(q)\displaystyle\int_0^{t^*-\epsilon} \|S_\alpha(t^*) - S_\alpha(t^*-\epsilon)\|p(s)ds & \\ + M\psi(q)e^{\omega t^*}\displaystyle\int_{t^*-\epsilon}^{t^*} e^{-\omega s}p(s)ds & \text{if } t^* \in [0,t_1], \\ \psi(q)\displaystyle\int_{t_k}^{t^*-\epsilon} \|S_\alpha(t^*) - S_\alpha(t^*-\epsilon)\|p(s)ds & \\ + M\psi(q)e^{\omega t^*}\displaystyle\int_{t^*-\epsilon}^{t^*} e^{-\omega s}p(s)ds & \\ & \text{if } t^* \in (t_k,t_{k+1}]. \end{cases}$$

Therefore, there are precompact sets arbitrarily close to the set $H(t^*) = \{h(t^*) : h \in N(y)\}$. Hence the set $H(t^*) = \{h(t^*) : h \in N(B)\}$ is precompact in E. Hence the operator N is completely continuous.

Step 4. *N has a closed graph.*

Let $y_n \to y_*$, $h_n \in N(y_n)$, and $h_n \to h_*$. We shall show that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that:

$$h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0], \\ S_\alpha(t)\phi(0) + \int_0^t S_\alpha(t-s)v_n(s)ds & \text{if } t \in [0, t_1], \\ S_\alpha(t-t_k)\prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})\phi(0) \\ +\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j) \\ \quad \times S_\alpha(t_i - s)v_n(s)ds + \int_{t_k}^{t} S_\alpha(t-s)v_n(s)ds \\ +\sum_{i=1}^{k} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j)I_i(y(t_i^-)), & \text{if } t \in (t_k, t_{k+1}]. \end{cases}$$

We have to prove that there exists $v_* \in S_{F,y_*}$ such that for each $t \in J$ we have

$$h_*(t) = \begin{cases} S_\alpha(t)\phi(0) + \int_0^t S_\alpha(t-s)v^*(s)ds & \text{if } t \in [0, t_1], \\ S_\alpha(t-t_k)\prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})\phi(0) \\ +\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j) \\ \quad \times S_\alpha(t_i - s)v^*(s)ds + \int_{t_k}^{t} S_\alpha(t-s)v^*(s)ds \\ +\sum_{i=1}^{k} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j)I_i(y(t_i^-)), & \text{if } t \in (t_k, t_{k+1}]. \end{cases}$$

Consider the linear and continuous operator $\mathcal{L} : L^1(J, \mathbb{R}) \to D$ defined by

$$(\mathcal{L}v)(t) = \begin{cases} \int_0^t S_\alpha(t-s)v(s)ds & \text{if } t \in [0, t_1], \\ \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j) \\ \times S_\alpha(t_i - s)v(s)ds + \int_{t_k}^{t} S_\alpha(t-s)v(s)ds & \text{if } t \in (t_k, t_{k+1}]. \end{cases}$$

We have for $t \in [0, t_1]$

$$|(h_n(t) - S_\alpha(t)\phi(0)) - (h_*(t) - S_\alpha(t)\phi(0))| = |h_n(t) - h_*(t)| \\ \leq \|h_n - h_*\|_\infty \to 0, \quad \text{as} \ \ n \mapsto \infty.$$

From Lemma 3 it follows that $\mathcal{L} \circ S_F$ is a closed graph operator and from the definition of $\mathcal{L}$ one has

$$h_n(t) - S_\alpha(t)\phi(0) \in \mathcal{L} \circ S_{F,y_n}.$$

As $y_n \to y_*$ and $h_n \to h_*$, there is a $v_* \in S_{F,y_*}$ such that

$$h_*(t) - S_\alpha(t)\phi(0) = \int_0^t S_\alpha(t-s)v_*(s)ds.$$

If $t \in (t_k, t_{k+1}]$

$$\begin{aligned} &|(h_n(t) - S_\alpha(t-t_k)\prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})\phi(0) \\ &- \sum_{i=1}^{k} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j) I_i(y(t_i^-))) \\ &- (h_*(t) - S_\alpha(t-t_k)\prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})\phi(0) \\ &- \sum_{i=1}^{k} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j) I_i(y(t_i^-)))| \\ &= |h_n(t) - h_*(t)| \leq \|h_n - h_*\|_\infty \to 0, \quad \text{as} \ \ n \mapsto \infty. \end{aligned}$$

From Lemma 3 it follows that $\mathcal{L} \circ S_F$ is a closed graph operator and from the definition of $\mathcal{L}$ one has

$$\begin{aligned} &h_n(t) - S_\alpha(t-t_k)\prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})\phi(0) \\ &- \sum_{i=1}^{k} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j) I_i(y(t_i^-)) \in \mathcal{L} \circ S_{F,y_n}. \end{aligned}$$

As $y_n \to y_*$ and $h_n \to h_*$, there is a $v_* \in S_{F,y_*}$ such that

$$\begin{aligned} &h_*(t) - S_\alpha(t-t_k)\prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})\phi(0) \\ &- \sum_{i=1}^{k} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j) I_i(y(t_i^-)) \end{aligned}$$

$$= \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j) \times S_\alpha(t_i - s) v^*(s) ds + \int_{t_k}^{t} S_\alpha(t-s) v^*(s) ds.$$

Hence the multivalued operator N is upper semi-continuous.

Step 5. *A priori bounds on solutions.*

Now, it remains to show that the set

$$\mathcal{E} = \{y \in PC(J,E) : y \in \lambda N y, \quad 0 \leq \lambda \leq 1\}$$

is bounded.

Let $y \in \mathcal{E}$ be any element. Then there exists $v \in S_{F,y}$ such that

$$y(t) = \begin{cases} S_\alpha(t)\phi(0) + \int_0^t S_\alpha(t-s)v(s)ds, & \text{if } t \in [0,t_1], \\ S_\alpha(t-t_k)\prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})\phi(0) \\ + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j) \\ \quad \times S_\alpha(t_i-s)v(s)ds + \int_{t_k}^{t} S_\alpha(t-s)v(s)ds \\ + \sum_{i=1}^{k} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j) I_i(y(t_i^-)), & \text{if } t \in (t_k, t_{k+1}]. \end{cases}$$

Then from (H1), (H2), (H3) we obtain

$$\|y(t)\| \leq \begin{cases} Me^{\omega t}\|\phi(0)\| + Me^{\omega t}\int_0^t e^{-\omega s}p(s)\psi(\|y_s\|ds, & \text{if } t \in [0,t_1], \\ M^{k+1}e^{\omega t}\|\phi(0)\| \\ + \sum_{i=1}^{k} M^{k-i+2}e^{\omega t}\int_{t_{i-1}}^{t_i} e^{-\omega s}p(s)\psi(\|y_s\|ds \\ + Me^{\omega t}\int_{t_k}^{t} e^{-\omega s}p(s)\psi(\|y_s\|ds \\ + M^*\sum_{i=1}^{k} M^{k-i+1}e^{\omega(t-t_i)} & \text{if } t \in (t_k, t_{k+1}]. \end{cases}$$

Consider the function $\mu(t)$ defined by

$$\mu(t) = \sup\{|y(s)| : \ r \leq s \leq t\}, \quad 0 \leq t \leq b.$$

Then, we have $\|y_s\| \leq \mu(t)$ for all $t \in J$. Let $t^* \in J$ be such that $\mu(t) = |y(t^*)|$. Then, by the previous inequality we have for $t \in J$

$$\mu(t) \leq \begin{cases} Me^{\omega b}\|\phi(0)\| + Me^{\omega b}\displaystyle\int_0^t e^{-\omega s}p(s)\psi(\mu(s))ds & \text{if } \ t \in [0, t_1], \\ M^{k+1}e^{\omega b}\|\phi(0)\| \\ +\displaystyle\sum_{i=1}^{k} M^{k-i+2}e^{\omega b}\int_{t_{i-1}}^{t_i} e^{-\omega s}p(s)\psi(\mu(s))ds \\ + Me^{\omega b}\displaystyle\int_{t_k}^{t} e^{-\omega s}p(s)\psi(\mu(s))ds \\ + M^*\displaystyle\sum_{i=1}^{k} M^{k-i+1}e^{\omega(b-t_i)} & \text{if } \ t \in (t_k, t_{k+1}]. \end{cases}$$

Set

$$C_1 = Me^{\omega b}\|\phi(0)\|,$$

$$C_2 = M^{m+1}e^{\omega b}\|\phi(0)\| + \sum_{i=1}^{m} M^{m-i+2}e^{\omega b}\int_{t_{i-1}}^{t_i} e^{-\omega s}p(s)\psi(\mu(s))ds + M^*\sum_{i=1}^{m} M^{m-i+1}e^{\omega(b-t_i)}.$$

It follows that

$$\mu(t) \leq \begin{cases} C_1 + Me^{\omega b}\displaystyle\int_0^t e^{-\omega s}p(s)\psi(\mu(s))ds, & \text{if } \ t \in [0, t_1], \\ C_2 + Me^{\omega b}\displaystyle\int_{t_k}^t e^{-\omega s}p(s)\psi(\mu(s))ds \\ & \text{if } \ t \in (t_k, t_{k+1}]. \end{cases}$$

Let us note the right hand-side of the above inequality by $v(t)$ i.e,

$$v(t) = \begin{cases} C_1 + Me^{\omega b}\displaystyle\int_0^t e^{-\omega s}p(s)\psi(\mu(s))ds, & \text{if } \ t \in [0, t_1], \\ C_2 + Me^{\omega b}\displaystyle\int_{t_k}^t e^{-\omega s}p(s)\psi(\mu(s))ds \\ & \text{if } \ t \in (t_k, t_{k+1}]. \end{cases}$$

Then, we have

$$\mu(t) \leq v(t) \quad \text{for all} \;\; t \in J$$

and

$$v(0) = C_1, \qquad\quad v(t_k) = C_2, \quad k = 1, \dots, m.$$

Differentiating both sides of the above equality, we obtain

$$v'(t) = Me^{\omega(b-t)}p(t)\psi(\mu(t))ds, \quad \text{a.e.} \;\; t \in J.$$

Using the nondecreasing character of the function ψ, we have

$$v'(t) \leq Me^{\omega(b-t)}p(t)\psi(v(t)), \quad \text{a.e.} \;\; t \in J,$$

that is

$$\frac{v'(t)}{\psi(v(t))} \leq Me^{\omega(b-t)}p(t), \quad \text{a.e.} \;\; t \in J.$$

Integrating from 0 to t if $t \in [0, t_1]$ and from t_k to t if $t \in (t_k, t_{k+1}]$ we get

$$\int_0^t \frac{v'(s)}{\psi(v(s))}ds \leq Me^{\omega b}\int_0^t e^{-\omega s}p(s)ds, \quad \text{if} \;\; t \in [0, t_1],$$

and

$$\int_{t_k}^t \frac{v'(s)}{\psi(v(s))}ds \leq Me^{\omega b}\int_{t_k}^t e^{-\omega s}p(s)ds, \quad \text{if} \;\; t \in (t_k, t_{k+1}], \quad k = 1, \dots, m.$$

By a change of variables we obtain

$$\int_{t_k}^t \frac{v'(s)}{\psi(v(s))}ds \leq C_1\int_0^t p(s)ds, \qquad k = 0, \dots m.$$

By a change of variables again we get

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq Me^{\omega b}\int_0^t e^{-\omega s}p(s)ds \leq \int_{C_3}^{\infty} \frac{du}{\psi(u)},$$

where

$$C_3 = \min(C_1, C_2).$$

Hence, there exists a constant K such that

$$\mu(t) \leq v(t) \leq K \quad \text{for all} \quad t \in J.$$

Now from the definition of μ it follows that

$$\|y\|_\Omega = \sup_{t\in[-r,b]} |y(t)| \leq \mu(b) \leq K \quad \text{for all} \quad y \in \mathcal{E}.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem of Leray-Schauder, the multivalued operator N has a fixed point y on the interval $[-r, b]$ which is a mild solution of problem (1–3). ∎

Acknowledgements

The authors are grateful to the referee for the helpful remarks.

References

[1] S. Abbas, M. Benchohra and G.M. N'Guérékata, Topics in Fractional Differential Equations, Springer, New York, 2012. doi:10.1007/978-1-4614-4036-9

[2] R.P. Agarwal, M. Benchohra and S. Hamani, *Boundary value problems for fractional differential equations*, Adv. Stud. Contemp. Math. **16** (2008), 181–196.

[3] A. Anguraj, P. Karthikeyan and G.M. N'Guérékata, *Nonlocal Cauchy problem for some fractional abstract differential equations in Banach spaces*, Commun. Math. Anal. **6** (2009).

[4] D. Baleanu, K. Diethelm, E. Scalas and J.J. Trujillo, Fractional Calculus Models and Numerical Methods, World Scientific Publishing, New York, 2012.

[5] M. Belmekki and M. Benchohra, *Existence results for fractional order semilinear functional differential equations*, Proc. A. Razmadze Math. Inst. **146** (2008), 9–20.

[6] M. Belmekki, M. Benchohra and L. Górniewicz, *Semilinear functional differential equations with fractional order and infinite delay,* Fixed Point Th. **9** (2008), 423–439.

[7] M. Benchohra, J.R. Graef and S. Hamani, *Existence results for boundary value problems with nonlinear fractional differential equations*, Appl. Anal. **87** (2008), 851–863. doi:10.1080/00036810802307579

[8] M. Benchohra, J. Henderson, S.K. Ntouyas and A. Ouahab, *Existence results for fractional order functional differential equations with infinite delay*, J. Math. Anal. Appl. **338** (2008), 1340–1350. doi:10.1016/j.jmaa.2007.06.021

[9] M. Benchohra, S. Litimein and G. N'Guérékata, *On fractional integro-differential inclusions with state-dependent delay in Banach spaces*, Appl. Anal. **92** (2013), 335–350. doi:10.1080/00036811.2011.616496

[10] C. Chen and M. Li, *On fractional resolvent operator functions* Semigroup Forum. **80** (2010), 121–142. doi:10.1007/s00233-009-9184-7

[11] C-Cuevas and J-C. de Souza, *S-asymptotically W-periodic solutions of semilinear fractional integro-differential equations*, Appl. Math. Lett. **22** (2009), 865–870. doi:10.1016/j.aml.2008.07.013

[12] C-Cuevas and J-C. de Souza, *Existence of S-asymptotically W-periodic solutions of fractional order functional integro-differential equations with infinite delay*, Nonlinear Anal. **72** (2010), 1680–1689. doi:10.1016/j.na.2009.09.007

[13] A. Granas and J. Dugundji, Fixed Point Theory, Springer-Verlag, New York, 2003. doi:10.1007/978-0-387-21593-8

[14] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[15] A.A. Kilbas, Hari M. Srivastava and Juan J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[16] V. Lakshmikantham, S. Leela and J. Vasundhara, Theory of Fractional Dynamic Systems, Cambridge Academic Publishers, Cambridge, 2009.

[17] A. Lasota and Z. Opial, *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. **13** (1965), 781–786.

[18] M. Li and Q. Zheng, *On spectral inclusions and approximations of $\alpha-$times resolvent families*, Semigroup Forum. **69** (2004), 356–368.

[19] K.S. Miller and B. Ross, An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, 1993.

[20] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[21] S.G. Samko, A.A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives. Theory and Applications, Gordon and Breach, Yverdon, 1993.

[22] V.E. Tarasov, Fractional Dynamics. Applications of Fractional Calculus to Dynamics of Particles, Fields and Media, Springer, Heidelberg, 2010.

Received 21 May 2013