EXISTENCE AND ATTRACTIVITY FOR FRACTIONAL ORDER INTEGRAL EQUATIONS IN FRÉCHET SPACES

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Abstract

In this paper, we present some results concerning the existence and the attractivity of solutions for some functional integral equations of Riemann-Liouville fractional order, by using an extension of the Burton-Kirk fixed point theorem in the case of a Fréchet space.

Keywords: functional integral equation, left-sided mixed Riemann-Liouville integral of fractional order, solution, attractivity, Fréchet space, fixed point.

2010 Mathematics Subject Classification: 26A33.

1. Introduction

Fractional integral equations have recently been applied in various areas of engineering, science, finance, applied mathematics, bio-engineering and others. However, many researchers remain unaware of this field. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Abbas et al. [6], Baleanu et al. [11], Kilbas et al. [21], Miller and Ross [23], Lakshmikantham et al. [22], Podlubny...
Recently some interesting results on the attractivity of the solutions of some classes of integral equations have been obtained by Abbas et al. [1, 2, 3, 4, 5, 7], Banaś et al. [12, 13, 14, 15], Darwish et al. [16], Dhage [17, 18, 19], Pachpatte [24, 25] and the references therein. In [9, 10], Avramescu and Vladimirescu studied the existence and the stability of solutions for some classes of integral equations of fractional order.

Motivated by those papers, this article deals with the existence and the attractivity of solutions of two classes of functional integral equations of Riemann-Liouville fractional order. We establish some sufficient conditions for the existence and the attractivity of solutions of the following integral equation of Riemann-Liouville fractional order

\[ u(t, x) = \mu(t, x) + h(t, x, u(t, x)) + f(t, x, I_\theta^r u(t, x), u(t, x)) \]

where \( b > 0, \theta = (0, 0), r = (r_1, r_2), r_1, r_2 \in (0, \infty), \mathbb{R}_+ = [0, \infty), \mu : J \to \mathbb{R}, f : J \times \mathbb{R} \to \mathbb{R}, g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are given continuous functions, \( I_\theta^r \) is the left-sided mixed Riemann-Liouville integral of order \( r \).

Next, we investigate the existence and the attractivity of solutions of the following integral equation of fractional order

\[ u(t, x) = \mu(t, x) + f(t, x, I_\theta^r u(t, x), u(t, x)) \]

\[ + \int_0^t \int_0^x \frac{(t-s)^{r_1-1}(x-y)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(t, x, s, y, u(s, y)) dy ds; \]

where \( \mu, f \) are as in equation (1), \( g : J' \times \mathbb{R} \to \mathbb{R} \) is a given continuous function, \( J' = \{(t, x, s, y) \in J^2 : s \leq t, y \leq x\} \) and \( \Gamma(\cdot) \) is the (Euler’s) Gamma function defined by

\[ \Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}dt; \xi > 0. \]

Our investigations are conducted in Fréchet spaces with an application of the fixed point theorem of Burton-Kirk for the existence of solutions of our equations, and we prove that all solutions are uniformly globally attractive. Also, we present two examples illustrating the applicability of the imposed conditions.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By \( L^1([0, a] \times [0, b]) \), for \( a, b > 0 \), we denote the
space of Lebesgue-integrable functions $u : [0, a] \times [0, b] \to \mathbb{R}$ with the norm

$$\|u\|_1 = \int_0^a \int_0^b |u(t, x)| \, dx \, dt.$$ 

As usual, $AC(J)$ is the space of absolutely continuous functions from $J$ into $\mathbb{R}$, and $C(J)$ is the space of all continuous functions from $J$ into $\mathbb{R}$.

**Definition 2.1** ([28]). Let $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, $\theta = (0, 0)$ and $u \in L^1([0, a] \times [0, b])$; $a, b > 0$. The left-sided mixed Riemann-Liouville integral of order $r$ of $u$ is defined by

$$(I^\theta_r u)(t, x) = \frac{1}{\Gamma(r_1) \Gamma(r_2)} \int_0^t \int_0^x (t - s)^{r_1 - 1} (x - y)^{r_2 - 1} u(s, y) \, dy \, ds.$$ 

In particular, for almost all $(t, x) \in [0, a] \times [0, b]$,

$$(I^\theta_\sigma u)(t, x) = u(t, x), \quad \text{and} \quad (I^\theta_\sigma u)(t, x) = \int_0^t \int_0^x u(s, y) \, dy \, ds,$$

where $\sigma = (1, 1)$.

For instance, $I^\theta_\sigma u$ exists almost everywhere for all $r_1, r_2 > 0$, when $u \in L^1([0, a] \times [0, b])$. Moreover,

$$(I^\theta_\sigma u)(t, 0) = 0; \quad \text{for a.a. } t \in [0, a]$$

and

$$(I^\theta_\sigma u)(0, x) = 0; \quad \text{for a.a. } x \in [0, b].$$

**Example 2.2.** Let $\lambda, \omega \in (0, \infty)$ and $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$. Then

$$I^\lambda_{\theta} t^{\lambda} x^{\omega} = \frac{\Gamma(1 + \lambda) \Gamma(1 + \omega)}{\Gamma(1 + \lambda + r_1) \Gamma(1 + \omega + r_2)} t^{\lambda + r_1} x^{\omega + r_2}, \quad \text{for a.a. } (t, x) \in [0, a] \times [0, b].$$

Let $X$ be a Fréchet space with a family of semi-norms $\{\| \cdot \|_n\}_{n \in \mathbb{N}}$. We assume that the family of semi-norms $\{\| \cdot \|_n\}$ verifies:

$$\|x\|_1 \leq \|x\|_2 \leq \|x\|_3 \leq ... \quad \text{for every } x \in X.$$

Let $Y \subset X$, we say that $Y$ is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \leq \overline{M}_n \quad \text{for all } y \in Y.$$
To $X$ we associate a sequence of Banach spaces $\{(X^n, \| \cdot \|_n)\}$ as follows: For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim$ defined by: $x \sim_n y$ if and only if $\| x - y \|_n = 0$ for $x, y \in X$. We denote $X^n = (X, \sim, \| \cdot \|_n)$ the quotient space, the completion of $X^n$ with respect to $\| \cdot \|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X$ as follows: For every $x \in X$, we denote $[x]$ the equivalence class of $x$ of subset $X^n$, and we define $Y^n = \{[x] : x \in Y\}$. We denote $\overline{Y^n}$, $\text{int}_n(Y^n)$ and $\partial_n Y^n$, the closure, the interior and the boundary of $Y^n$ with respect to $\| \cdot \|_n$ in $X^n$, respectively. For more information about this subject see [20].

For each $p \in \mathbb{N}$ we consider the following set, $C_p = C([0, p] \times [0, b])$, and we define in $C(J)$ the semi-norms by

$$\|u\|_p = \sup_{(t,x) \in [0,p] \times [0,b]} \|u(t,x)\|.$$ 

Then $C(J)$ is a Fréchet space with the family of semi-norms $\{\|u\|_p\}$.

**Definition 2.3.** Let $X$ be a Fréchet space. A function $N : X \rightarrow X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in [0, 1)$ such that $\|N(u) - N(v)\|_n \leq k_n \|u - v\|_n$ for all $u, v \in X$.

We need the following extension of the Burton-Kirk fixed point theorem in the case of a Fréchet space.

**Theorem 2.4** [8]. Let $(X, \| \cdot \|_n)$ be a Fréchet space and let $A, B : X \rightarrow X$ be two operators such that

(a) $A$ is a compact operator;

(b) $B$ is a contraction operator with respect to a family of seminorms $\{\| \cdot \|_n\}$;

(c) the set $\{x \in X : x = \lambda A(x) + \lambda B \left( \frac{x}{\lambda} \right), \lambda \in (0, 1)\}$ is bounded.

Then the operator equation $A(u) + B(u) = u$ has a solution in $X$.

Let $\emptyset \neq \Omega \subset C(J)$, let $G : \Omega \rightarrow \Omega$, and consider the solutions of equation

$$\text{(3)} \quad (Gu)(t,x) = u(t,x).$$

Now we introduce the concept of attractivity of solutions for our equations.

**Definition 2.5** ([12]). Solutions of equation (3) are locally attractive if there exists a ball $B(u_0, \eta)$ in the space $C(J)$ such that, for arbitrary solutions $v = v(t,x)$
and \( w = w(t, x) \) of equation (3) belonging to \( B(u_0, \eta) \cap \Omega \), the equality
\[
\lim_{t \to \infty} \left( v(t, x) - w(t, x) \right) = 0
\]
holds for each \( x \in [0, b] \).

When the limit (4) is uniform with respect to \( B(u_0, \eta) \cap \Omega \), solutions of equation (3) are said to be uniformly locally attractive (or equivalently that solutions of (3) are locally asymptotically stable).

**Definition 2.6 ([12]).** The solution \( v = v(t, x) \) of equation (3) is said to be globally attractive if (4) holds for each solution \( w = w(t, x) \) of (3). If condition (4) is satisfied uniformly with respect to the set \( \Omega \), solutions of equation (3) are said to be globally asymptotically stable (or uniformly globally attractive).

### 3. Existence and attractivity results

Let us start by defining what we mean by a solution of the equation (1) (or (2)).

**Definition 3.1.** A function \( u \in C(J) \) is said to be a solution of (1) (or (2)), if \( u \) satisfies equation (1) (or (2)) on \( J \).

Now, we are concerned with the existence and the uniform global attractivity of solutions for the equation (1). The following hypotheses will be used in the sequel.

\((H_1)\) The function \( \mu \) is continuous and bounded with \( \mu^* = \sup_{(t, x) \in J} |\mu(t, x)| \).

\((H_2)\) The function \( h \) is continuous and there exists \( l \in C(J, \mathbb{R}^+) \) with \( l(t, x) < 1 \) on \( J \) and such that
\[
|h(t, x, u) - h(t, x, v)| \leq l(t, x)|u - v|; \text{ for each } (t, x) \in J \text{ and each } u, v \in \mathbb{R}.
\]

Moreover, assume that the function \( t \to h(t, x, 0) \) is bounded on \( J \) with
\[
h^* = \sup_{(t, x) \in J} h(t, x, 0).
\]

\((H_3)\) The function \( f \) is continuous and there exist functions \( P, Q : J \to \mathbb{R}_+ \) such that
\[
|f(t, x, u, v)| \leq \frac{P(t, x)|u| + Q(t, x)|v|}{1 + |u| + |v|},
\]
for \((t, x) \in J\) and for \(u, v \in \mathbb{R}\). Moreover, assume that

\[
\lim_{{t \to \infty}} P(t, x) = \lim_{{t \to \infty}} Q(t, x) = 0; \text{ for } x \in [0, b].
\]

**Theorem 3.2.** Assume that hypotheses \((H_1) - (H_3)\) hold. Then the equation (1) has at least one solution in the space \(C(J)\). Moreover, solutions of equation (1) are uniformly globally attractive.

**Proof.** Let us define the operators \(A, B : C(J) \to C(J)\) by formulas

\[
(Au)(t, x) = f(t, x, I^r_\theta u(t, x), u(t, x)); \quad (t, x) \in J,
\]

\[
(Bu)(t, x) = \mu(t, x) + h(t, x, u(t, x)); \quad (t, x) \in J.
\]

The problem of finding the solutions of the equation (1) is reduced to finding the solutions of the operator equation \(Au + Bu = u\). We shall show that the operators \(A\) and \(B\) satisfy all the conditions of Theorem 2. The proof will be given in several steps.

**Step 1. \(A\) is compact.**

To this aim, we have to prove that \(A\) is continuous and it transforms every bounded set into a relatively compact set. Recall that \(M \subset C(J)\) is bounded if and only if

\[
\forall p \in \mathbb{N}, \exists \ell_p > 0 : \forall u \in M, \|u\|_p \leq \ell_p,
\]

and \(M = \{u(t, x) : (t, x) \in J\} \subset C(J)\) is relatively compact if and only if for any \(p \in \mathbb{N}\), the family \(\{u(t, x) : (t, x) \in [0, p] \times [0, b]\}\) is equicontinuous and uniformly bounded on \([0, p] \times [0, b]\). The proof will be given in several claims.

**Claim 1. \(A\) is continuous.**

Let \(\{u_n\}_{n \in \mathbb{N}}\) be a sequence such that \(u_n \to u\) in \(C(J)\). Then, for each \((t, x) \in J\), we have

\[
| (Au_n)(t, x) - (Au)(t, x) | 
\leq | f(t, x, I^r_\theta u_n(t, x), u_n(t, x)) - f(t, x, I^r_\theta u(t, x), u(t, x)) |.
\]

If \((t, x) \in [0, p] \times [0, b]; \ p \in \mathbb{N}\), then, since \(u_n \to u\) as \(n \to \infty\), then (7) gives

\[
\| A(u_n) - A(u) \|_p \to 0 \quad \text{as } n \to \infty.
\]

**Claim 2. \(A\) maps bounded sets into bounded sets in \(C(J)\).**
Let $M$ be a bounded set in $C(J)$, then, for each $p \in \mathbb{N}$, there exists $\ell_p > 0$, such that for all $u \in C(J)$ we have $\|u\|_p \leq \ell_p$. Then, for arbitrarily fixed $(t, x) \in [0, p] \times [0, b]$ we have

\[|(Au)(t, x)| \leq |f(t, x, I_0^\gamma u(t, x), u(t, x))|\]
\[\leq (P(t, x)|I_0^\gamma u(t, x)| + Q(t, x)|u(t, x)|)\]
\[\times (1 + |I_0^\gamma u(t, x)| + |u(t, x)|)^{-1}\]
\[\leq P(t, x) + Q(t, x)\]
\[\leq P_p + Q_p,
\]
where
\[P_p = \sup_{(t,x) \in [0,p] \times [0,b]} P(t, x) \quad \text{and} \quad Q_p = \sup_{(t,x) \in [0,p] \times [0,b]} Q(t, x).
\]

Thus

\[(8) \quad \|A(u)\|_p \leq P^*_p + Q^*_p := \ell_p.
\]

Claim 3. A maps bounded sets into equicontinuous sets in $C(J)$.

Let $(t_1, x_1), (t_2, x_2) \in [0, p] \times [0, b]$, $t_1 < t_2$, $x_1 < x_2$ and let $u \in M$. Thus we have

\[|(Au)(t_2, x_2) - (Au)(t_1, x_1)|\]
\[\leq |f(t_2, x_2, I_0^\gamma u(t_2, x_2), u(t_2, x_2)) - f(t_1, x_1, I_0^\gamma u(t_1, x_1), u(t_1, x_1))|.
\]

From continuity of $f, I_0^\gamma$ the right-hand side of the above inequality tends to zero as $t_1 \to t_2$, $x_1 \to x_2$. As a consequence of claims 1 to 3 together with the Arzelá-Ascoli theorem, we conclude that $A$ is continuous and compact.

**Step 2.** $B$ is a contraction.

Consider $v, w \in C(J)$. Then, by $(H_2)$, for any $p \in \mathbb{N}$ and each $(t, x) \in [0, p] \times [0, b]$, we have

\[|(Bv)(t, x) - (Bw)(t, x)| \leq l(t, x)|v - w|.
\]

Then

\[\|(Bv) - B(w)\|_p \leq l_p\|v - w\|_p,
\]

where $l_p := \sup_{t,x \in [0,p] \times [0,b]} l(t, x) < 1$. Then $B$ is a contraction.
Step 3. The set \( E := \{ \varphi \in C(J) : \varphi = \lambda A(\varphi) + \lambda B(\frac{\varphi}{1}) \mid \lambda \in (0, \infty) \} \) is bounded.

Let \( u \in C(J) \) be such that \( u = \lambda A(u) + \lambda B \left( \frac{u}{\lambda} \right) \) for some \( \lambda \in (0, 1) \). Then, for any \( p \in \mathbb{N} \) and each \((t, x) \in [0, p] \times [0, b]\), we have

\[
|u(t, x)| \leq \lambda|A(u)| + \lambda \left| B \left( \frac{u}{\lambda} \right) \right|
\leq |\mu(t, x)| + P(t, x) + Q(t, x) + h^* + l(t, x)|u(t, x)|
\leq \mu^* + P_p + Q_p + h^* + l_p\|u\|_p.
\]

Thus,

\[
\|u\|_p \leq \frac{\mu^* + P_p + Q_p + h^*}{1 - l_p} =: \ell^*_p.
\]

Hence, the set \( E \) is bounded. As a consequence of Steps 1 and 3 together with Theorem 2, we deduce that \( A + B \) has a fixed point \( u \) in \( C(J) \) which is a solution to equation (1).

Finally, we show the uniform global attractivity of solutions of the equation (1). Let \( u \) and \( v \) be any two solutions of equation (1). Then for each \((t, x) \in J\) we have

\[
|u(t, x) - v(t, x)| \leq |h(t, x, u(t, x)) - h(t, x, v(t, x))| + |f(t, x, I^u_\varphi(t, x), u(t, x)) - f(t, x, I^v_\varphi(t, x), v(t, x))|\]
\[
\leq l(t, x)|u(t, x) - v(t, x)| + 2P(t, x) + 2Q(t, x).
\]

Thus

\[
|u(t, x) - v(t, x)| \leq \frac{2}{1 - l(t, x)}(P(t, x) + Q(t, x)).
\]

By using (9), we deduce that

\[
\lim_{t \to \infty} (u(t, x) - v(t, x)) = 0.
\]

Consequently, equation (1) has a solution and all solutions are uniformly globally attractive.

Now, we are concerned with the existence and the uniform global attractivity of solutions for the equation (2). The following hypotheses will be used in the sequel.

(\( H'_1 \)) The function \( \mu \) is bounded with \( \mu^* = \sup_{(t, x) \in J} |\mu(t, x)| \).

(\( H'_2 \)) There exist continuous functions \( l', k' : J \to \mathbb{R}_+ \) such that
\[ |f(t, x, u_1, u_2) - f(t, x, v_1, v_2)| \leq \frac{l'(t, x)|u_1 - v_1| + k'(t, x)|u_2 - v_2|}{1 + |u_1 - v_1| + |u_2 - v_2|}; \]

for each \((t, x) \in J\) and each \(u_1, u_2, v_1, v_2 \in \mathbb{R}\). Moreover, assume that

\[
\lim_{t \to \infty} l'(t, x) = \lim_{t \to \infty} k'(t, x) = 0; \text{ for } x \in [0, b],
\]

and the function \(t \to f(t, x, 0, 0)\) is bounded on \(J\) with \(f^* = \sup_{(t, x) \in J} f(t, x, 0, 0)\).

\((H'_3)\) There exist continuous functions \(P', Q' : J' \to \mathbb{R}_+\) and such that

\[ |g(t, x, s, y, u)| \leq \frac{P'(t, x, s, y) + Q'(t, x, s, y)|u|}{1 + |u|}; \]

for \((t, x, s, y) \in J', u \in \mathbb{R}\). Moreover, assume that

\[
\lim_{t \to \infty} \int_0^t \frac{P'(t, x, s, y)}{(t - s)^{1 - r_1}} ds = \lim_{t \to \infty} \int_0^t \frac{Q'(t, x, s, y)}{(t - s)^{1 - r_1}} ds = 0.
\]

**Theorem 3.3.** Assume that hypotheses \((H'_1) - (H'_3)\) hold. If

\[
k'_p + \frac{l'_p p^{r_1} b^{r_2}}{\Gamma(1 + r_1) \Gamma(1 + r_2)} < 1,
\]

where

\[ k'_p = \sup_{(t, x) \in [0, p] \times [0, b]} k'(t, x), \quad l'_p = \sup_{(t, x) \in [0, p] \times [0, b]} l'(t, x); \quad p \in \mathbb{N}, \]

then the equation (2) has at least one solution in the space \(C(J)\). Moreover, solutions of equation (2) are uniformly globally attractive.

**Proof.** Let us define the operators \(A', B' : C(J) \to C(J)\) by formulas

\[
(A'u)(t, x) =
\int_0^t \int_0^x (t - s)^{r_1 - 1}(x - y)^{r_2 - 1} g(t, x, s, y, u(s, y)) dy ds; \quad (t, x) \in J,
\]

\[
(B'u)(t, x) = \mu(t, x) + f(t, x, \int_0^p u(t, x), u(t, x)); \quad (t, x) \in J.
\]
We shall show that operators $A'$ and $B'$ satisfy all conditions of Theorem 2. The proof will be given in several steps.

**Step 1. $A'$ is compact.**
The proof will be given in several claims.

**Claim 1. $A'$ is continuous.**
Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence such that $u_n \to u$ in $C(J)$. Then, for each $(t, x) \in J$, we have

\[
| (A' u_n)(t, x) - (A' u)(t, x) | 
\leq \frac{1}{\Gamma(r_1) \Gamma(r_2)} \int_0^t \int_0^x (t - s)^{r_1 - 1} (x - y)^{r_2 - 1} 
\times | g(t, x, s, y, u_n(s, y)) - g(t, x, s, y, u(s, y)) | dy ds
\]

\[
\leq \frac{1}{\Gamma(r_1) \Gamma(r_2)} \int_0^t \int_0^x (t - s)^{r_1 - 1} (x - y)^{r_2 - 1} 
\times | g(t, x, s, y, u_n(s, y)) - g(t, x, s, y, u(s, y)) | dy ds.
\]

If $(t, x) \in [0, p] \times [0, b]$; $p \in \mathbb{N}$, then, since $u_n \to u$ as $n \to \infty$ and $g$ is continuous, (15) gives

\[
\| A'(u_n) - A'(u) \|_p \to 0 \quad \text{as } n \to \infty.
\]

**Claim 2. $A'$ maps bounded sets into bounded sets in $C(J)$.**
Let $M$ be a bounded set in $C(J)$, then, for each $p \in \mathbb{N}$, there exists $\ell_p > 0$, such that for all $u \in C(J)$ we have $\| u \|_p \leq \ell_p$. Then, for arbitrarily fixed $(t, x) \in [0, p] \times [0, b]$ we have

\[
| (A' u)(t, x) | 
\leq \frac{1}{\Gamma(r_1) \Gamma(r_2)} \int_0^t \int_0^x (t - s)^{r_1 - 1} (x - y)^{r_2 - 1} 
\times | g(t, x, s, y, u(s, y)) | dy ds
\]

\[
\leq \frac{1}{\Gamma(r_1) \Gamma(r_2)} \int_0^t \int_0^x (t - s)^{r_1 - 1} (x - y)^{r_2 - 1} 
\times | g(t, x, s, y, u_n(s, y)) - g(t, x, s, y, u(s, y)) | dy ds
\]

\[
\leq \frac{1}{\Gamma(r_1) \Gamma(r_2)} \int_0^t \int_0^x (t - s)^{r_1 - 1} (x - y)^{r_2 - 1} 
\times (P'(t, x, s, y) + Q'(t, x, s, y)) dy ds
\]

\[
\leq P'_p + Q'_p,
\]
where
\[ P'_p = \sup_{(t,x) \in [0,p] \times [0,b]} \int_0^t \int_0^x \frac{(t-s)^{r_1-1}(x-y)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} P'(t,x,s,y)dyds \]
and
\[ Q'_p = \sup_{(t,x) \in [0,p] \times [0,b]} \int_0^t \int_0^x \frac{(t-s)^{r_1-1}(x-y)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} Q'(t,x,s,y)dyds. \]

Thus
\[ \|A'(u)\|_p \leq P'_p + Q'_p := \ell'_p. \]

**Claim 3.** \( A' \) maps bounded sets into equicontinuous sets in \( C(J) \).

Let \((t_1, x_1), (t_2, x_2) \in [0,p] \times [0,b] \), \( t_1 < t_2 \), \( x_1 < x_2 \) and let \( u \in M \). Thus we have
\[
\begin{align*}
|A'(u)(t_2, x_2) - A'(u)(t_1, x_1)| & \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \left| \int_0^{t_2} \int_0^{x_2} (t_2-s)^{r_1-1}(x_2-y)^{r_2-1} \right| \\
& \times |g(t_2, x_2, s, y, u(s, y)) - g(t_1, x_1, s, y, u(s, y))| dyds \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \left| \int_0^{t_2} \int_0^{x_2} (t_2-s)^{r_1-1}(x_2-y)^{r_2-1} g(t_1, x_1, s, y, u(s, y)) dyds \\
& - \int_0^{\beta(t_1)} \int_0^{x_2} (t_2-s)^{r_1-1}(x_2-y)^{r_2-1} g(t_1, x_1, s, y, u(s, y)) dyds \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \left| \int_0^{t_1} \int_0^{x_2} (t_2-s)^{r_1-1}(x_2-y)^{r_2-1} g(t_1, x_1, s, y, u(s, y)) dyds \\
& - \int_0^{\beta(t_1)} \int_0^{x_2} (t_1-s)^{r_1-1}(x_1-y)^{r_2-1} g(t_1, x_1, s, y, u(s, y)) dyds \\
& \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \left| \int_0^{t_2} \int_0^{x_2} (t_2-s)^{r_1-1}(x_2-y)^{r_2-1} \right| \\
& \times |g(t_2, x_2, s, y, u(s, y)) - g(t_1, x_1, s, y, u(s, y))| dyds \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \left| \int_0^{t_2} \int_0^{x_2} (t_2-s)^{r_1-1}(x_2-y)^{r_2-1} \right| g(t_1, x_1, s, y, u(s, y)) dyds \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \left| \int_0^{t_1} \int_0^{x_1} (t_2-s)^{r_1-1}(x_2-y)^{r_2-1} - (t_1-s)^{r_1-1}(x_1-y)^{r_2-1} \right| \\
& \times |g(t_1, x_1, s, y, u(s, y))| dyds.
\end{align*}
\]
+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1} \int_{x_1}^{x_2} (t_2 - s)^{r_1-1} (x_2 - y)^{r_2-1} \left| g(t_1, x_1, s, y, u(s, y)) \right| dy \, ds \\
\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_2} \int_0^{x_2} (t_2 - s)^{r_1-1} (x_2 - y)^{r_2-1} \\
\times \left| g(t_2, x_2, s, y, u(s, y)) - g(t_1, x_1, s, y, u(s, y)) \right| dy \, ds \\
+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_0^{x_2} (t_2 - s)^{r_1-1} (x_2 - y)^{r_2-1} \left( P'(t_1, x_1, s, y) + Q'(t_1, x_1, s, y) \right) dy \, ds \\
+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1} \int_0^{x_1} \left| (t_2 - s)^{r_1-1} (x_2 - y)^{r_2-1} - (t_1 - s)^{r_1-1} (x_1 - y)^{r_2-1} \right| \\
\times \left( P'(t_1, x_1, s, y) + Q'(t_1, x_1, s, y) \right) dy \, ds \\
+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_{x_1}^{x_2} (t_2 - s)^{r_1-1} (x_2 - y)^{r_2-1} \left( P'(t_1, x_1, s, y) + Q'(t_1, x_1, s, y) \right) dy \, ds.

From continuity of $g, P', Q'$ the right-hand side of the above inequality tends to zero as $t_1 \to t_2$ and $x_1 \to x_2$. As a consequence of claims 1 to 3 together with the Arzelà-Ascoli theorem, we conclude that $A'$ is continuous and compact.

**Step 2.** $B'$ is a contraction.

Consider $v, w \in X$. Then, by $(H'_2)$, for any $p \in \mathbb{N}$ and each $(t, x) \in [0, p] \times [0, b]$, we have

$$|(B'v)(t, x) - (B'w)(t, x)| \leq l'(t, x)|I'_b(v - w)(t, x)| + k'(t, x)|(v - w)(t, x)|$$

$$\leq \left( k'(t, x) + \frac{l'(t, x)p^{r_1}b^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} \right) \|v - w\|.$$ 

Thus,

$$\|(B'v) - B'(w)\|_p \leq \left( k'_p + \frac{l'_p p^{r_1}b^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} \right) \|v - w\|_p.$$ 

By (12), we conclude that $B'$ is a contraction.

**Step 3.** The set $\mathcal{E}' := \{ u \in C(J) : u = \lambda A'(u) + \lambda B'(\frac{u}{\lambda}), \lambda \in (0, 1) \}$ is bounded.

Let $u \in C(J)$ be such that $u = \lambda A'(u) + \lambda B'(\frac{u}{\lambda})$ for some $\lambda \in (0, 1)$. Then, for any $p \in \mathbb{N}$ and each $(t, x) \in [0, p] \times [0, b]$, we have
\[ |u(t, x)| \leq \lambda |A'(u)| + \lambda \left| B' \left( \frac{u}{\lambda} \right) \right| \]
\[ \leq |\mu(t, x)| + f^* + k'(t, x) + l'(t, x) \]
\[ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \]
\[ \times (P'(t, x, s, y) + Q'(t, x, s, y))dyds \]
\[ \leq \mu^* + f^* + k_p' + l_p' + P_p + Q_p'. \]

Thus,
\[ \|u\|_p \leq \mu^* + f^* + k_p' + l_p' + P_p + Q_p'. =: \ell_p^* . \]

Hence, the set \( E' \) is bounded.

Finally, we show the uniform global attractivity of solutions of the equation (2). Let \( u \) and \( v \) be any two solutions of equation (2). Then for each \((t, x) \in J \) we have
\[ |u(t, x) - v(t, x)| \leq |f(t, x, I_0^u u(t, x), u(t, x)) - f(t, x, I_0^u v(t, x), v(t, x))| \]
\[ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \]
\[ \times |g(t, x, s, y, u(s, y)) - g(t, x, s, y, v(s, y))|dyds \]
\[ \leq k'(t, x) + l'(t, x) \]
\[ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \]
\[ \times (P'(t, x, s, y) + Q'(t, x, s, y))dyds \]
\[ \leq k'(t, x) + l'(t, x) \]
\[ + \int_0^x \frac{(x-y)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \left( \int_0^t \frac{P'(t, x, s, y) + Q'(t, x, s, y)}{(t-s)^{1-r_i}}ds \right) dy. \]

Thus, by (10) and (11), we deduce that
\[ \lim_{t \to \infty} (u(t, x) - v(t, x)) = 0. \]

Consequently, equation (2) has a solution and all solutions are uniformly globally attractive.
4. Examples

Example 4.1. Consider the following fractional order integral equation of the form

\begin{equation}
\begin{aligned}
\int_{1+t+x}^e \norm{u(t, x)}^{|x|} \
\int_{1+t+x}^{e^{2-t+x}|u(t, x)|} \Gamma(r_1) \Gamma(r_2) g(t, x, s, y, u(s, y)) dy ds; \quad (t, x) \in \mathbb{R}_+ \times [0, 1],
\end{aligned}
\end{equation}

We have \( \mu^* = e^4 \), then condition \((H_1)\) is satisfied. Also, the function \( h \) is continuous and satisfies assumption \((H_2)\), with \( l(t, x) = e^{-1-t-x} \). Finally, the function \( f \) is continuous and satisfies assumption \((H_3)\), with \( P(t, x) = e^{-1-t-x} \) and \( Q(t, x) = e^{2-t+x} \). Hence by Theorem 3.2, the equation (16) has a solution defined on \( \mathbb{R}_+ \times [0, 1] \) and all solutions are uniformly globally attractive.

Example 4.2. Consider now the following fractional order integral equation of the form

\begin{equation}
\begin{aligned}
\int_{1+t+x}^{e^{2-t+x}} \norm{u(t, x)}^{|x|} \
\int_{1+t+x}^{e^{2-t+x}|u(t, x)|} \Gamma(r_1) \Gamma(r_2) g(t, x, s, y, u(s, y)) dy ds; \quad (t, x) \in \mathbb{R}_+ \times [0, 1],
\end{aligned}
\end{equation}

where \( c_p = e^{-p}, \quad p \in \mathbb{N}, \quad r = (r_1, r_2) \in (0, \infty) \times (0, \infty), \)

\begin{equation}
\begin{cases}
\frac{x s \sin \sqrt{1+|u|} \sin \sqrt{1+|u|}}{\sqrt{1+|u|}^2}; \\
\text{if} \quad (t, x, s, y) \in J', \ s \neq 0, \ y \in [0, 1] \text{ and } u \in \mathbb{R}, \\
g(t, x, 0, u) = 0; \quad \text{if} \quad (t, x) \in J, \ y \in [0, 1] \text{ and } u \in \mathbb{R},
\end{cases}
\end{equation}

\( g(t, x, s, y, u) = 0; \quad \text{if} \quad (t, x, s, y) \in J', \ s \neq 0, \ y \in [0, 1] \text{ and } u \in \mathbb{R}, \)
\[ J' = \{ (t, x, s, y) \in J^2 : s \leq t \text{ and } x \leq y \} \]

Set
\[ \mu(t, x) = \frac{e^{3-2t+x}}{1 + t + x^2}, \quad f(t, x, u, v) = \frac{e^{x-t-2}}{c_p(1 + e^{-2p|u|} + e^{-p|v|})}; \quad p \in \mathbb{N}. \]

We have \( \mu^* = e^4 \), then condition \( (H'_1) \) is satisfied. The function \( f \) is continuous and satisfies assumption \( (H'_2) \), with \( k'(t, x) = \frac{e^{x-t-2-p}}{c_p} \), \( l'(t, x) = \frac{e^{x-t-2-2p}}{c_p} \), \( k'_p = \frac{e^{-1-p}}{c_p} \) and \( l'_p = \frac{e^{-1-2p}}{c_p} \). Also, the function \( g \) is continuous and satisfies assumption \( (H'_3) \),

\[
\begin{cases}
    P'(t, x, s, y) = Q'(t, x, s, y) = \frac{x s^{\frac{3}{4}} \sin \sqrt{t} \sin s}{1 + y^2 + t^2}; & (t, x, s, y) \in J', \; y \in [0, 1], \; s \neq 0, \\
    P'(t, x, 0, y) = Q'(t, x, 0, y) = 0; & (t, x) \in J, \; y \in [0, 1].
\end{cases}
\]

Then,
\[
\left| \int_0^t (t-s)^{r-1} P'(t, x, s, y) ds \right| \leq \int_0^t (t-s)\frac{3}{4} x s^{\frac{3}{4}} \sin \sqrt{t} \sin s ds \\
\leq x \left| \sin \sqrt{t} \right| \int_0^t (t-s)^{\frac{3}{4}} s^{\frac{3}{4}} ds \\
\leq \frac{\Gamma^2(\frac{1}{4})}{\sqrt{\pi}} \left| \sin \sqrt{t} \right| \\
\leq \frac{\Gamma^2(\frac{1}{4})}{\sqrt{\pi_t}} \rightarrow 0 \text{ as } t \rightarrow \infty.
\]

Finally, We shall show that condition \( (12) \) holds with \( b = 1 \). Indeed, for each \( p \in \mathbb{N} \), we get

\[
k'_p + \frac{l'_p p^{r_1} b^{r_2}}{\Gamma(1 + r_1) \Gamma(1 + r_2)} = \frac{1}{c_p} \left( e^{-1-p} + \frac{e^{-1-2p} p^{r_1}}{\Gamma(1 + r_1) \Gamma(1 + r_2)} \right) = e^{-1} < 1.
\]

Hence by Theorem 3.3, the equation (17) has a solution defined on \( \mathbb{R}_+ \times [0, 1] \) and all the solutions are uniformly globally attractive.

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Received 21 August 2012